NONLINEAR SEMIGROUPS AND DIFFERENTIAL INCLUSIONS IN PROBABILISTIC NORMED SPACES

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ABSTRACT The purpose of this paper is to introduce and study the semigroups of nonlinear contractions in probabilistic normed spaces and to establish the Crandall-Liggett’s exponential formula for some kind of accretive mappings in probabilistic normed spaces. As applications, we utilize these results to study the Cauchy problem for a kind of differential inclusions with accretive mappings in probabilistic normed spaces.

1. Introduction

The concept of accretive mappings is of fundamental importance in the theory of set-valued nonlinear operators, differential equations and partial differential equations in Banach spaces, which was introduced independently by F. E. Browder ([3]) and T. Kato ([11]). On the other hand, many authors have done considerable works on semigroups of nonlinear contractions, differential equations and evolution equations in Banach spaces and Hilbert spaces ([1], [2], [4], [7], [8], [12], [13]).

Recently, the authors introduced the concept of accretive mappings ([5]) and some elementary properties of accretive mappings in probabilistic normed spaces have been deduced by K. S. Ha et al. ([9]).

The purpose of this paper is to introduce and study the semigroups of nonlinear contractions in probabilistic normed spaces and to prove

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that if $A$ is an accretive mapping in probabilistic normed spaces satisfying the range condition, then $A$ generates a semigroup of nonlinear contractions. As applications, we shall use these results to study the Cauchy problem of solutions for a kind of differential inclusions with accretive mappings in probabilistic normed spaces.

For the sake of convenience, we shall recall some definitions and notations ([5], [6], [16]).

Throughout this paper, we denote by $\mathcal{D}$ the set of distribution functions defined on $\mathbb{R}$, i.e., $f \in \mathcal{D}$ if $f$ is nondecreasing left-continuous with $\sup_{t \in \mathbb{R}} f(t) = 1$ and $\inf_{t \in \mathbb{R}} f(t) = 0$.

**Definition 1.1.** A probabilistic normed space (shortly, PN-space) is an ordered pair $(E, \mathcal{F})$, where $E$ is a real linear space and $\mathcal{F}$ is a mapping from $E$ into $\mathcal{D}$ (we denote $\mathcal{F}(x)$ by $F_x$) satisfying the following conditions: For all $x, y \in E$,

- $(\text{PN}-1)$ $F_x(t) = 1$ for all $t > 0$ if and only if $x = 0$;
- $(\text{PN}-2)$ $F_x(0) = 0$;
- $(\text{PN}-3)$ $F_{\alpha x}(t) = F_x(\frac{t}{|\alpha|})$ for any $\alpha \in \mathbb{R}$, $\alpha \neq 0$;
- $(\text{PN}-4)$ If $F_x(t_1) = 1$, $F_y(t_2) = 1$, then $F_{x+y}(t_1 + t_2) = 1$.

**Definition 1.2.** A mapping $\Delta : [0,1] \times [0,1] \to [0,1]$ is called a $t$-norm if it satisfies the following conditions: For any $a, b, c, d \in [0,1]$,

- $(\text{T-1})$ $\Delta(a, 1) = a$;
- $(\text{T-2})$ $\Delta(a, b) = \Delta(b, a)$;
- $(\text{T-3})$ $\Delta(c, d) \geq \Delta(a, b)$ for $c \geq a$ and $d \geq b$;
- $(\text{T-4})$ $\Delta(\Delta(a, b), c) = \Delta(\Delta(a, b), c)$.

A Menger PN-space is a triple $(E, \mathcal{F}, \Delta)$, where $(E, \mathcal{F})$ is a PN-space and $\Delta$ is a $t$-norm satisfying

- $(\text{PN-4}')$ $F_{x+y}(t_1 + t_2) \geq \Delta(F_x(t_1), F_y(t_2))$ for all $x, y \in E$ and $t_1, t_2 \in \mathbb{R}^+ = [0, +\infty)$.

**Definition 1.3** ([5]). Let $(E, \mathcal{F}, \Delta)$ be a Menger PN-space.

(i) $A : D(A) \subset E \to 2^E$ is called an accretive mapping if

$$F_{x-y}(t) \geq F_{x-y+\lambda(u-v)}(t)$$

for all $x, y \in D(A), u \in Ax, v \in Ay$ and $\lambda > 0$. 
(ii) $A$ is called a *maximal accretive mapping* if
\[ F_{x-y_0}(t) \geq F_{x-y_0+\lambda(u-v_0)}(t) \]
for all $x \in D(A)$, $u \in Ax$ and $\lambda > 0$, then $y_0 \in D(A)$ and $v_0 \in Ay_0$.

(iii) $A$ is called a *m-accretive mapping* if $A$ is accretive and $I + A$ is surjective.

(iv) $A$ is called a *strongly accretive mapping* if there exists a $k \in (0, 1)$ such that
\[ F_{(\lambda-k)(x-y)}(t) \geq F_{(\lambda-1)(x-y)+u-v}(t) \]
for all $x, y \in D(A)$, $u \in Ax$, $v \in Ay$ and $\lambda > k$.

(v) $A$ is called a *dissipative mapping* (maximal dissipative, $m$-dissipative, respectively) if $-A$ is accretive (maximal accretive, $m$-accretive, respectively).

2. Semi-inner products in Menger $PN$-spaces

In this section, we always assume that $(E, F, \Delta)$ is a Menger $PN$-space.

For any $\lambda \in (0, 1]$, we define a real nonnegative function $P_\lambda : E \to \mathbb{R}^+$ as follows:
\[ P_\lambda(x) = \inf \{ t : F_x(t) > 1 - \lambda \} \text{ for all } x \in E. \]

From the definition of $P_\lambda(x)$, it is easy to prove the following:

**Proposition 2.1.** Let $(E, F, \Delta)$ be a Menger $PN$-space with $\Delta(t, t) \geq t$ for all $t \in [0, 1]$. Then for any $\lambda \in (0, 1)$
\begin{enumerate}
  \item $P_\lambda(\alpha x) = |\alpha|P_\lambda(x)$ for all $\alpha \in \mathbb{R}$ and $x \in E$;
  \item $P_\lambda(x + y) \leq P_\lambda(x) + P_\lambda(y)$ for all $x, y \in E$;
  \item $(P_\lambda(x + ty) - P_\lambda(x))/t$ is nondecreasing in $t \in (0, +\infty)$ and $x, y \in E$;
  \item $(P_\lambda(x) - P_\lambda(x - ty))/t$ is nonincreasing in $t \in (0, +\infty)$ and $x, y \in E$.
\end{enumerate}

It follows from Proposition 2.1 that the following limits exist:
\[ \lim_{t \to 0^+} (P_\lambda(x + ty) - P_\lambda(x))/t \text{ and } \lim_{t \to 0^+} (P_\lambda(x) - P_\lambda(x - ty))/t. \]
In the sequel, we denote

\[ [x, y]^+_\lambda = \lim_{t \to 0^+} \frac{(P_\lambda(x + ty) - P_\lambda(x))/t}{t} \]

and

\[ [x, y]^-_\lambda = \lim_{t \to 0^+} \frac{(P_\lambda(x) - P_\lambda(x-ty))/t}{t} \]

In what follows we give some basic properties of \([x, y]^\pm_\lambda\):

**Lemma 2.2.** Let \((E, \mathcal{F}, \Delta)\) be a Menger PN-space with \(\Delta(t, t) \geq t\) for all \(t \in [0, 1]\). Then we have the following:

(i) \([x, y]^-_\lambda \leq [x, y]^+_\lambda\);
(ii) \(|[x, y]^\pm_\lambda| \leq P_\lambda(y)\) and \([x, \alpha x]^\pm_\lambda = \alpha P_\lambda(x)\) for all \(\alpha \in \mathbb{R}\);
(iii) \([x, y]^\pm_\lambda - [x, z]^\pm_\lambda \leq P_\lambda(y - z)\);
(iv) \([x, y]^\pm_\lambda = -[x, -y]^-_\lambda = -[-x, y]^-_\lambda\);
(v) \([sx, ry]^\pm_\lambda = r[x, y]^\pm_\lambda\) for all \(r, s \geq 0\);
(vi) \([x, y + z]^\pm_\lambda \leq [x, y]^\pm_\lambda + [x, z]^\pm_\lambda\) and \([x, y + z]^-_\lambda \geq [x, y]^-_\lambda + [x, z]^-_\lambda\);
(vii) \([x, y + z]^\pm_\lambda \geq [x, y]^-_\lambda + [x, z]^\pm_\lambda\) and \([x, y + z]^-_\lambda \leq [x, y]^-_\lambda + [x, z]^\pm_\lambda\);
(viii) \([x, y + \alpha x]^\pm_\lambda = [x, y]^\pm_\lambda + \alpha P_\lambda(x)\) for all \(\alpha \in \mathbb{R}\);
(ix) If \(x(t) : [a, b] \rightarrow E\) is differentiable in \(t \in (a, b)\) and \(\varphi_\lambda(t) = P_\lambda(x(t))\), then

\[ D^+ \varphi_\lambda(t) = \lim_{h \to 0^+} \frac{(P_\lambda(x(t + h)) - P_\lambda(x(t)))/h}{h} = [x(t), x'(t)]^+_\lambda; \]

\[ D^- \varphi_\lambda(t) = \lim_{h \to 0^+} \frac{(P_\lambda(x(t)) - P_\lambda(x(t - h)))/h}{h} = [x(t), x'(t)]^-_\lambda; \]

(x) \([x, y]^+_\lambda\) is upper semi-continuous and \([x, y]^-_\lambda\) is lower semi-continuous.

**Proof.** Properties (i)-(v) follow easily and so the details are omitted here.

(vi) Since

\[ (P_\lambda(x + t(y + z)) - P_\lambda(x))/t \]

\[ \leq \frac{1}{2t} \{[P_\lambda(x + 2ty) - P_\lambda(x)] + [P_\lambda(x + 2tz) - P_\lambda(x)]\}, \]

we have

\[ [x, y + z]^+_\lambda \leq [x, y]^+_\lambda + [x, z]^+_\lambda. \]
Similarly, we can prove that $[x, y + z]_\lambda^- \geq [x, y]_\lambda^- + [x, z]_\lambda^-$. 

(vii) Since

$$[x, y]_\lambda^+ = [x, y + z - z]_\lambda^+ \leq [x, y + z]_\lambda^+ + [x, -z]_\lambda^+,$$

from (iv), it follows that $[x, -z]_\lambda^+ = -[x, z]_\lambda^-$ and so we have

$$[x, y + z]_\lambda^+ \geq [x, y]_\lambda^+ + [x, z]_\lambda^-.$$

(viii) By (vi) and (vii), we have

$$[x, y + \alpha x]_\lambda^- \leq [x, y]_\lambda^+ + [x, \alpha x]_\lambda^+ = [x, y]_\lambda^+ + \alpha P_\lambda(x)$$

and

$$[x, y + \alpha x]_\lambda^+ \geq [x, y]_\lambda^+ + [x, \alpha x]_\lambda^- = [x, y]_\lambda^+ + \alpha P_\lambda(x),$$

respectively. Therefore, we have

$$[x, y + \alpha x]_\lambda^+ = [x, y]_\lambda^+ + \alpha P_\lambda(x).$$

Similarly, we can prove that $[x, y + \alpha x]_\lambda^- = [x, y]_\lambda^- + \alpha P_\lambda(x)$. 

(ix) Since

$$|D^+ \varphi_\lambda(t) - [x(t), x'(t)]_\lambda^+|$$

$$= \left| \lim_{h \to 0^+} \frac{1}{h} (P_\lambda(x(t + h)) - P_\lambda(x(t))) \right|$$

$$= \left| \lim_{h \to 0^+} \frac{1}{h} (P_\lambda(x(t + h)) - P_\lambda(x(t) + hx'(t))) \right|$$

$$\leq \left| \lim_{h \to 0^+} \frac{1}{h} (P_\lambda(x(t + h)) - x(t) - hx'(t)) \right|$$

$$= \lim_{h \to 0^+} \left| P_\lambda \left( \frac{x(t + h) - x(t) - hx'(t)}{h} \right) \right| = 0,$$

(ix) is true.

(x) Letting $x_n \to x$ and $y_n \to y$, since

$$[x_n, y_n]_\lambda^+ \leq \frac{1}{t} (P_\lambda(x_n + ty_n) - P_\lambda(x_n))$$

for all $t > 0$,
we have
\[ \lim_{n \to \infty} [x_n, y_n]_\lambda^+ \leq \frac{1}{t}(P_\lambda(x + ty) - P_\lambda(x)). \]

Letting \( t \to 0^+ \), it follows that \( \lim_{n \to \infty} [x_n, y_n]_\lambda^+ \leq [x, y]_\lambda^+ \), which means that \([x, y]_\lambda^+ \) is upper semi-continuous.

Similarly, we can prove that \([x, y]_\lambda^+ \) is lower semi-continuous. This completes the proof.

Next, we define a mapping \( j_\lambda : E \to \mathbb{R}^* \) (\( E^* \) is the dual space of \( E \)) by
\[ j_\lambda(x) = \{ f_\lambda \in E^* : f_\lambda(x) = P_\lambda(x), \ [x, y]_\lambda^+ \leq f_\lambda(y) \leq [x, y]_\lambda^+, \ y \in E \}. \]

Now we claim that for any \( x \in E \), \( j_\lambda(x) \neq \emptyset \). In fact, for any \( y_0 \in E \), we define \( f_\lambda(\alpha y_0) = \alpha[x, y_0]_\lambda^+ \) for all \( \alpha \in \mathbb{R} \).

(a) If \( \alpha \geq 0 \), then \( f_\lambda(\alpha y_0) = [x, \alpha y_0]_\lambda^+ \);

(b) If \( \alpha < 0 \), then
\[
\alpha[x, y_0]_\lambda^+ = -|\alpha|[x, y_0]_\lambda^+ = -[x, |\alpha|y_0]_\lambda^+ \\
= [x, -|\alpha|y_0]_\lambda^- = [x, \alpha y_0]_\lambda^- \\
\leq [x, \alpha y_0]_\lambda^+.
\]

Therefore, we have \( f_\lambda(\alpha y_0) \leq [x, \alpha y_0]_\lambda^+ \) for all \( \alpha \in \mathbb{R} \). By (v) and (vi) of Lemma 2.2, \([x, y]_\lambda^+ \) is subadditive in \( y \in E \). By using the Hahn-Banach Theorem ([15]), there exists a linear functional \( \widetilde{f}_\lambda : E \to \mathbb{R} \) such that \( \widetilde{f}_\lambda(\alpha y_0) = f_\lambda(\alpha y_0) \) and \(-[x, -y]_\lambda^+ \leq \widetilde{f}_\lambda(y) \leq [x, y]_\lambda^+ \) for all \( y \in E \), i.e.,
\[ [x, y]_\lambda^- \leq \widetilde{f}_\lambda(y) \leq [x, y]_\lambda^+ . \]

Especially, we have \( \widetilde{f}_\lambda(x) = [x, x]_\lambda^+ = P_\lambda(x) \).

The continuity of \( \widetilde{f}_\lambda \) follows from \( [f_\lambda(x)] \leq [x, y]_\lambda^+ \leq P_\lambda(y) \) immediately. Therefore, we know \( \widetilde{f}_\lambda \in j_\lambda(x) \). This completes the proof.

Moreover, we can also prove that \( j_\lambda(x) \) is convex. Hence, by the Banach-Alaoglu Theorem, we have the following:
Proposition 2.3. For each \( x \in E \) and \( \lambda \in (0, 1] \), \( j_\lambda(x) \) is a nonempty convex weak* compact subset of \( E^* \).

In view of the above argument and Proposition 2.3, we have the following:

**Proposition 2.4.** \([x, y]^+ = \max\{f_\lambda(y) : f_\lambda \in j_\lambda(x)\}\) and

\[ [x, y]^* = \min\{f_\lambda(y) : f_\lambda \in j_\lambda(x)\}. \]

**Definition 2.1.** (i) \((x, y)^+ = P_\lambda(x) \cdot [x, y]^+ \) is called the upper semi-inner product with respect to \( \lambda \in (0, 1] \),

(ii) \((x, y)^* = P_\lambda(x) \cdot [x, y]^* \) is called the lower semi-inner product with respect to \( \lambda \in (0, 1] \).

For some properties of the semi-inner products, refer to [14].

**Definition 2.2.** The mapping \( \mathcal{S}_\lambda : E \to 2^{E^*} \) defined by

\[ \mathcal{S}_\lambda(x) = \{P_\lambda(x) \cdot f_\lambda : f_\lambda \in j_\lambda(x)\} \] for all \( x \in E \)

is called the duality mapping with respect to \( \lambda \in (0, 1] \).

It follows from Lemma 2.2 that the following corollary holds:

**Corollary 2.5.** (i) \((x, y)^+_\lambda \leq (x, y)^+\);

(ii) \(|(x, y)^+_\lambda| \leq P_\lambda(x) \cdot P_\lambda(y) \) and \((x, \alpha x)^+_\lambda \leq \alpha P_\lambda^2(x) \) for all \( \alpha \in \mathbb{R} \);

(iii) \(|(x, y)^+_\lambda - (x, z)^+_\lambda| \leq P_\lambda(x) \cdot P_\lambda(y - z);

(iv) \((x, y)^+ = -x, -y\) \(= -(x, y)^*_\lambda\);

(v) \((sx, sy)^* = s \cdot r \cdot (x, y)^*\) for all \( r, s \geq 0\);

(vi) \((x, y + z)^+_\lambda \leq (x, y)^+_\lambda + (x, z)^+_\lambda \) and \((x, y + z)^* \geq (x, y)^*_\lambda + (x, z)^*_\lambda\);

(vii) \((x, y + z)^+_\lambda \geq (x, y)^+_\lambda + (x, z)^+_\lambda \) and \((x, y + z)^* \leq (x, y)^*_\lambda + (x, z)^*_\lambda\);

(viii) \((x, y + \alpha x)^+_\lambda = (x, y)^+_\lambda + \alpha P_\lambda^2(x) \) for all \( \alpha \in \mathbb{R} \);

(ix) If \( x(t) : [a, b] \to E \) is differentiable in \( t \in (a, b) \) and \( \varphi_\lambda(t) = P_\lambda^2(x(t)) \), then

\[ D^+ \varphi_\lambda(t) = 2(x(t), x'(t))^+_\lambda \text{ and } D^- \varphi_\lambda(t) = 2(x(t), x'(t))^*_\lambda; \]

(x) \((x, y)^+_\lambda\) is upper semi-continuous and \((x, y)^*_\lambda\) is lower semi-continuous.
3. Accretive mappings and nonlinear semigroups in $PN$-spaces

In this section, we always assume that $(E, \mathcal{F}, \Delta)$ is a complete Menger $PN$-space with $\Delta(t, t) \geq t$ for all $t \in [0, 1]$.

**Lemma 3.1.** Let $A : D(A) \subseteq E \rightarrow 2^E$ be a mapping. Then the following conclusions are equivalent:

(i) $A$ is accretive;

(ii) $P_\lambda(x - y) \leq P_\lambda(x - y + \epsilon(u - v))$ for all $x, y \in D(A)$, $u \in Ax$, $v \in Ay$ and for all $\epsilon > 0$, $\lambda \in (0, 1]$;

(iii) $[x - y, u - v]_\lambda^+ \geq 0$ for all $x, y \in D(A)$, $u \in Ax$, $v \in Ay$ and $\lambda \in (0, 1]$.

**Proof.** (i) $\iff$ (ii). If $A$ is accretive, then

$$F_{x-y}(t) \geq F_{x-y+\epsilon(u-v)}(t)$$

for all $x, y \in D(A)$, $u \in Ax$, $v \in Ay$ and $\epsilon > 0$. Besides, for given $x, y \in D(A)$, $u \in Ax$, $v \in Ay$ and $\epsilon > 0$, letting

$$P_\lambda(x - y + \epsilon(u - v)) = \inf \{ t : F_{x-y+\epsilon(u-v)}(t) > 1 - \lambda \} = \lim_{n \rightarrow -\infty} \{ t_n : F_{x-y+\epsilon(u-v)}(t_n) > 1 - \lambda \},$$

then we have $F_{x-y}(t_n) > 1 - \lambda$ for all $n \geq 1$ and so

$$P_\lambda(x - y) = \inf \{ t : F_{x-y}(t) > 1 - \lambda \} \leq \lim_{n \rightarrow -\infty} t_n,$$

which implies that the conclusion (ii) is true.

Conversely, suppose that (ii) is true, but the conclusion (i) is not true. Then there exist $x_0, y_0 \in D(A)$, $\epsilon_0 > 0$, $u_0 \in Ax_0$, $v_0 \in Ay_0$ and $t_0 > 0$ such that

$$F_{x_0-y_0}(t_0) < F_{x_0-y_0+\epsilon_0(u_0-v_0)}(t_0).$$

Therefore, there exists $\lambda_0 \in (0, 1]$ such that $F_{x_0-y_0}(t_0) = 1 - \lambda_0$. This implies that

$$P_{\lambda_0}(x_0 - y_0) = \inf \{ t : F_{x_0-y_0}(t) > 1 - \lambda_0 \} \geq t_0.$$
Since \( F_{x_0-y_0+\varepsilon_0(u_0-v_0)}(t_0) > 1 - \lambda_0 \) and \( F_{x_0-y_0+\varepsilon_0(u_0-v_0)}(t_0) \) is left continuous, there exists \( \delta_0 > 0 \) such that

\[
F_{x_0-y_0+\varepsilon_0(u_0-v_0)}(t_0 - \delta_0) > 1 - \lambda_0.
\]

Hence we have

\[
P_{\lambda_0}(x_0 - y_0 + \varepsilon_0(u_0 - v_0)) \leq t_0 - \delta_0 < t_0 \leq P_{\lambda_0}(x_0 - y_0),
\]

which is a contradiction.

(ii) \( \iff \) (iii) By Proposition 2.1 (iii) and the definition of \([\cdot, \cdot]_\lambda^+\), it is obvious that the conclusions are true. This completes the proof.

**Lemma 3.2.** Let \( A : D(A) \subset E \to 2^E \) be an accretive mapping and \( J_\varepsilon = (I + \varepsilon A)^{-1} \) for all \( \varepsilon > 0 \), then

(i) \( P_\lambda(J_\varepsilon x - J_\varepsilon y) \leq P_\lambda(x - y) \) and \( F_{J_\varepsilon x - J_\varepsilon y}(t) \geq F_{x-y}(t) \) for all \( t > 0, \lambda \in (0, 1], \) and \( x, y \in R(I + \varepsilon A), \) the range of \( I + \varepsilon A; \)

(ii) \( P_\lambda(J_\varepsilon^n x - x) \leq n \cdot P_\lambda(J_\varepsilon x - x) \) for all \( \lambda \in (0, 1], \) an integer \( n > 0 \) and \( x \in R((I + \varepsilon A)^n) \), and

\[
F_{J_\varepsilon^n x - x}(t) \geq F_{J_\varepsilon x - x}(\frac{t}{n}) \text{ for all } t > 0 \text{ and } x \in R((I + \varepsilon A)^n);
\]

(iii) If \( x_j \in R(I + \varepsilon A) \) and \( x_j \to x_0 \in D(A) \cap R(I + \varepsilon A), \) then

\[
\lim_{j \to \infty} P_\lambda(J_\varepsilon x_j - x_j) \leq \varepsilon \cdot \inf_{u \in A x_0} P_\lambda(u) \text{ for all } \lambda \in (0, 1]
\]

and

\[
\lim_{j \to \infty} F_{J_\varepsilon x_j - x_j}(t) \geq \sup_{u \in A x_0} F_u(\frac{t}{\varepsilon}) \text{ for all } t > 0.
\]

**Proof.** (i) is an immediate consequence of Lemma 3.1 and the accretivity of \( A. \)

(ii) can be obtained from (i) immediately.

Next, we prove (iii). For any given \( u \in A x_0, \) letting \( w = x_0 + \varepsilon u, \) then we have

\[
x_0 = (I + \varepsilon A)^{-1} w = J_\varepsilon w
\]
and
\[ P_\lambda(J_\epsilon x_j - x_j) \leq P_\lambda(J_\epsilon x_j - J_\epsilon w) + P_\lambda(J_\epsilon w - x_j). \]

Hence it follows that
\[
\lim_{j \to \infty} P_\lambda(J_\epsilon x_j - x_j) \leq \lim_{j \to \infty} (P_\lambda(x_j - w) + P_\lambda(x_0 - x_j))
\leq \lim_{j \to \infty} P_\lambda(x_j - w)
\leq \lim_{j \to \infty} (P_\lambda(x_j - x_0) + P_\lambda(x_0 - w))
\leq P_\lambda(-\epsilon u) = \epsilon P_\lambda(u).
\]

Therefore, by the arbitrariness of \( u \in Ax_0 \), we have
\[
\lim_{j \to \infty} P_\lambda(J_\epsilon x_j - x_j) \leq \epsilon \cdot \inf_{u \in Ax_0} P_\lambda(u).
\]

On the other hand, since
\[
F_{J_\epsilon x_j - x_j}(t) \geq \Delta(F_{J_\epsilon x_j - J_\epsilon w}(t - \frac{\eta}{2}), F_{J_\epsilon w - x_j}(\frac{\eta}{2}))
\geq \Delta(F_{x_j - w}(t - \frac{\eta}{2}), F_{x_0 - x_j}(\frac{\eta}{2}))
\]
and
\[
F_{x_j - w}(t - \frac{\eta}{2}) \geq \Delta(F_{x_j - x_0}(\frac{\eta}{2}), F_{\epsilon u}(t - \eta))
\]
for all \( \eta < t \), we have
\[
F_{J_\epsilon x_j - x_j}(t) \geq \Delta(F_{\epsilon u}(t - \eta), F_{x_0 - x_j}(\frac{\eta}{2}))
\]
and so
\[
\lim_{j \to \infty} F_{J_\epsilon x_j - x_j}(t) \geq F_u(t - \frac{\eta}{\epsilon}).
\]

Since \( F_u(t) \) is left-continuous, letting \( \eta \to 0^+ \), we have
\[
\lim_{j \to \infty} F_{J_\epsilon x_j - x_j}(t) \geq F_u(t - \frac{\epsilon}{\epsilon}).
\]
which implies that
\[
\lim_{j \to \infty} F_{x_j, x_0}(t) \geq \sup_{u \in A x_0} F_u(\frac{t}{e}).
\]
This completes the proof.

We are now in a position to consider the Cauchy problem of the following differential inclusion with an accretive mapping \( A \):

\[
(E3.1) \quad \begin{cases} 
    u'(t) \in -Au(t), & t > 0, \\
    u(0) = u_0 \in D(A).
\end{cases}
\]

DEFINITION 3.1. A function \( u(\cdot) \in C(\mathbb{R}^+, E) \) is called a strong solution of (E3.1) if it satisfies the following conditions:

(i) \( u(0) = u_0 \);

(ii) There exists \( y \in E \) such that

\[
F_{u(t)-u(s)}(k) \geq F_{(t-s)y}(k) \text{ for all } k > 0 \text{ and } t, s \in \mathbb{R}^+.
\]

(In this case, we also say \( u(\cdot) \) to be Lipschitz continuous);

(iii) The derivative \( u'(t) \) of \( u(\cdot) \) exists and satisfies

\[
u'(t) \in -Au(t) \text{ for almost all } t \in (0, +\infty)\).

Thus, we have the following:

THEOREM 3.3. Let \((E, \mathcal{F}, \Delta) \) be a complete Menger PN-space with \( \Delta(t, t) \geq t \) for all \( t \in [0, 1] \) and \( A : D(A) \subset E \rightarrow 2^E \) be an accretive mapping. Then (E3.1) has at most one strong solution.

Proof. Let \( u(\cdot) \) and \( v(\cdot) \) be two strong solutions of (E3.1) and denote \( \varphi_\lambda(t) = P_\lambda(u(t) - v(t)) \) for all \( \lambda \in (0, 1] \). Then, by Lemma 2.2 (ix), we have

\[
D^- \varphi_\lambda(t) = [u(t) - v(t), u'(t) - v'(t)]_\lambda^-.
\]

Therefore, there exist \( w(t) \in Au(t) \) and \( z(t) \in Av(t) \) such that

\[
u'(t) = -w(t), \quad v'(t) = -z(t) \text{ for almost all } t \in (0, +\infty)\)
and so we have

\[
D^- \varphi_\lambda(t) = [u(t) - v(t), (w(t) - z(t))]^-_\lambda
= -[u(t) - v(t), w(t) - z(t)]^+_\lambda
\leq 0.
\]

Therefore, we have

\[
P_\lambda(u(t) - v(t)) \leq P_\lambda(u(0) - v(0)) = 0 \text{ for all } \lambda \in (0, 1].
\]

If \(u(t_0) - v(t_0) \neq 0\) for some \(t_0 \in \mathbb{R}^+\), then there exists \(k_0 > 0\) such that

\[
F_{u(t_0) - v(t_0)}(k_0) < 1.
\]

Letting \(F_{u(t_0) - v(t_0)}(k_0) = 1 - \lambda_0\), then \(\lambda_0 \in (0, 1]\) and so

\[
P_{\lambda_0}(u(t_0) - v(t_0)) = \inf\{k : F_{u(t_0) - v(t_0)}(k) > 1 - \lambda_0\} \geq k_0 > 0,
\]

which contradicts \(P_{\lambda_0}(u(t_0) - v(t_0)) = 0\). This implies that \(u(t) = v(t)\) for all \(t \in \mathbb{R}^+\). This completes the proof.

**Definition 3.2.** Let \((E, F, \Delta)\) be a complete Menger \(PN\)-space and \(C\) be a closed subset of \(E\). A family of operators, \(\{T(t) : C \to E : t \geq 0\}\), is called a *semigroup of nonlinear contractions* if it satisfies the following conditions:

(i) \(T(0)x = x\) for all \(x \in C\);
(ii) \(T(t)T(s) = T(t+s)\) for all \(t, s \geq 0\);
(iii) The mapping \(t \mapsto T(t)x\) is continuous for any \(x \in C\);
(iv) \(F_{T(t)x - T(t)y}(k) \geq F_{x-y}(k)\) for all \(x, y \in C\), \(t \geq 0\) and \(k > 0\).

**Theorem 3.4.** Let \(A : D(A) \subset E \to 2^E\) be an accretive mapping satisfying the following conditions:

\((I + \epsilon A)(D(A)) \supset \overline{D(A)}, \text{ the closure of } D(A), \text{ for all } \epsilon > 0.\)

Then for any \(x \in \overline{D(A)}\), the following limit exists

\[
T(t)x = \lim_{\epsilon \to 0^+} (I + \epsilon A)^{-[\frac{t}{\epsilon}]}x \text{ for all } t \geq 0,
\]

where \([\frac{t}{\epsilon}]\) is the largest integer which does not exceed \(\frac{t}{\epsilon}\). Moreover, \(\{T(t) : t \geq 0\}\) is a semigroup of nonlinear contractions.

In order to prove Theorem 3.4, we need the following:
Lemma 3.5. Let $A : D(A) \subset E \to 2^E$ be an accretive mapping and $\overline{D(A)} \subset (I + \epsilon A)(D(A))$ for all $\epsilon > 0$. Then

$$F_{\epsilon}^m x - J_{\mu}^n x(t) \geq \sup_{u \in Ax} F_u(t \cdot ((m\epsilon - n\mu)^2 + m\epsilon^2 + n\mu^2)^{-\frac{1}{2}})$$

for all $x \in D(A)$, $\epsilon, \mu > 0$ and $m, n$ are nonnegative integers.

Proof. We first prove that for any $x \in D(A)$, $\epsilon, \mu > 0$ and $\lambda \in (0, 1]$,

$$(3.1) \quad P_{\lambda}(J_{\epsilon}^m x - J_{\mu}^n x) \leq \{(m\epsilon - n\mu)^2 + m\epsilon^2 + n\mu^2\}^{\frac{1}{2}} \inf_{u \in Ax} P_{\lambda}(u),$$

where $m, n$ are nonnegative integers.

For each $x \in D(A)$, $\epsilon, \mu > 0$ and $\lambda \in (0, 1]$, let

$$P_{m,n} = P_{\lambda}(J_{\epsilon}^m x - J_{\mu}^n x), \quad m, n = 0, 1, 2, \cdots.$$

By (ii) and (iii) of Lemma 3.2, we have

$$P_{m,0} = \inf_{u \in Ax} P_{\lambda}(u), \quad m = 0, 1, 2, \cdots,$$

$$P_{0,n} = \inf_{u \in Ax} P_{\lambda}(u), \quad n = 0, 1, 2, \cdots.$$

These mean that (3.1) holds for $n = 0$ or $m = 0$.

Now we suppose that (3.1) holds for a couple of integers $(m - 1, n)$, $(m, n - 1)$. For $x \in D(J_{\epsilon})$ and $y \in D(J_{\mu})$, setting $\delta = \frac{\epsilon \mu}{\epsilon + \mu}$, we can easily check

$$J_{\delta}(\frac{\mu}{\epsilon + \mu} x + \frac{\epsilon}{\epsilon + \mu} J_{\epsilon} x) = J_{\epsilon} x,$$

$$J_{\delta}(\frac{\epsilon}{\epsilon + \mu} y + \frac{\mu}{\epsilon + \mu} J_{\mu} y) = J_{\mu} y.$$

Therefore, we have

$$P_{m,n}$$

$$= P_{\lambda}(J_{\epsilon} \cdot J_{\epsilon}^{m-1} x - J_{\mu} \cdot J_{\mu}^{n-1} x)$$

$$= P_{\lambda}(J_{\epsilon} \cdot J_{\epsilon}^{m-1} x - J_{\mu} \cdot J_{\mu}^{n-1} x)$$

$$- \frac{\epsilon}{\epsilon + \mu} J_{\mu}^{n-1} x + \frac{\mu}{\epsilon + \mu} J_{\mu}^{n-1} x)

\leq P_{\lambda}(\frac{\mu}{\epsilon + \mu} J_{\epsilon}^{m-1} x + \frac{\epsilon}{\epsilon + \mu} J_{\epsilon}^{m-1} x - \frac{\epsilon}{\epsilon + \mu} J_{\mu}^{n-1} x - \frac{\mu}{\epsilon + \mu} J_{\mu}^{n-1} x)

\leq \frac{\epsilon}{\epsilon + \mu} P_{\lambda}(J_{\epsilon}^{m-1} x - J_{\mu}^{n-1} x) + \frac{\mu}{\epsilon + \mu} P_{\lambda}(J_{\epsilon}^{m-1} x - J_{\mu}^{n-1} x),$$
i.e.,
\[P_{m,n} \leq \frac{\epsilon}{\epsilon + \mu} P_{m,n-1} + \frac{\mu}{\epsilon + \mu} P_{m-1,n}\]
and thus we have
\[P_{m,n}\]
\[\leq \frac{\epsilon}{\epsilon + \mu} \left\{ (m\epsilon - n\mu)^2 + 2\mu(m\epsilon - n\mu) + m\epsilon^2 + n\mu^2 \right\}^{\frac{1}{2}} \inf_{u \in A\times} P_\lambda(u)\]
\[+ \frac{\mu}{\epsilon + \mu} \left\{ (m\epsilon - n\mu)^2 - 2\epsilon(m\epsilon - n\mu) + m\epsilon^2 + n\mu^2 \right\}^{\frac{1}{2}} \inf_{u \in A\times} P_\lambda(u)\]
\[\leq \left\{ \frac{\epsilon}{\epsilon + \mu} [(m\epsilon - n\mu)^2 + 2\mu(m\epsilon - n\mu) + m\epsilon^2 + n\mu^2] \right.\]
\[+ \left. \frac{\mu}{\epsilon + \mu} [(m\epsilon - n\mu)^2 - 2\epsilon(m\epsilon - n\mu) + m\epsilon^2 + n\mu^2] \right\}^{\frac{1}{2}} \inf_{u \in A\times} P_\lambda(u)\]
\[= \{ (m\epsilon - n\mu)^2 + m\epsilon^2 + n\mu^2 \}^{\frac{1}{2}} \inf_{u \in A\times} P_\lambda(u).\]

Therefore, the conclusion of (3.1) is proved.

Now, suppose that the conclusion of Lemma 3.5 is not true. There exist \(x_0, m_0, n_0, \epsilon_0, \mu_0\) and \(t_0 > 0\) such that
\[F_{j_{\epsilon_0}^{m_0} x_0 - j_{\mu_0}^{n_0} x_0}(t_0) < \sup_{u \in A\times} F_u(t_0) \cdot \{(m_0\epsilon_0 - n_0\mu_0)^2 + m_0\epsilon_0^2 + n_0\mu_0^2\}^{-\frac{1}{2}}\].

Therefore, there exists \(u_0 \in A\times\) such that
\[F_{j_{\epsilon_0}^{m_0} x_0 - j_{\mu_0}^{n_0} x_0}(t_0) < F_{u_0}(t_0) \cdot \{(m_0\epsilon_0 - n_0\mu_0)^2 + m_0\epsilon_0^2 + n_0\mu_0^2\}^{-\frac{1}{2}}\].

Letting \(F_{j_{\epsilon_0}^{m_0} x_0 - j_{\mu_0}^{n_0} x_0}(t_0) = 1 - \lambda_0\), then \(\lambda_0 \in (0,1]\). It is obvious that
\[P_{\lambda_0}(J_{\epsilon_0}^{m_0} x_0 - J_{\mu_0}^{n_0} x_0) = \inf\{t : F_{j_{\epsilon_0}^{m_0} x_0 - J_{\mu_0}^{n_0} x_0}(t) > 1 - \lambda_0\} \geq t_0\]
and
\[P_{\lambda_0}(u_0) = \inf\{t : F_{u_0}(t) > 1 - \lambda_0\} \]
\[< t_0 \cdot \{(m_0\epsilon_0 - n_0\mu_0)^2 + m_0\epsilon_0^2 + n_0\mu_0^2\}^{-\frac{1}{2}}.\]

Hence we have
\[P_{\lambda_0}(J_{\epsilon_0}^{m_0} x_0 - J_{\mu_0}^{n_0} x_0) > \{(m_0\epsilon_0 - n_0\mu_0)^2 + m_0\epsilon_0^2 + n_0\mu_0^2\}^{\frac{1}{2}} \inf_{u \in A\times} P_{\lambda_0}(u),\]
which contradicts (3.1). This completes the proof.
Proof of Theorem 3.4. For each \(x \in D(A)\), by Lemma 3.5, we have

\[
F_{J^{[\frac{t}{\epsilon}]}_x \rightarrow J^{[\frac{1}{\mu}]}_x} (k) \geq \sup_{u \in Ax} F_u(k \cdot \{(\frac{t}{\epsilon} \cdot \epsilon - \frac{t}{\mu} \cdot \mu)^2 + \frac{t}{\epsilon} \cdot \mu^2 + \frac{t}{\mu} \cdot \mu^2\}^{-\frac{1}{2}}.
\]

Since

\[
\{(\frac{t}{\epsilon} \cdot \epsilon - \frac{t}{\mu} \cdot \mu)^2 + \frac{t}{\epsilon} \cdot \mu^2 + \frac{t}{\mu} \cdot \mu^2\}^{-\frac{1}{2}} \leq \{(\epsilon + \mu)^2 + (\epsilon + \mu)t\}^{-\frac{1}{2}},
\]

it follows that

\[
F_{J^{[\frac{t}{\epsilon}]}_x \rightarrow J^{[\frac{1}{\mu}]}_x} (k) \geq \sup_{u \in Ax} F_u(k \cdot \{(\epsilon + \mu)^2 + (\epsilon + \mu)t\}^{-\frac{1}{2}}).
\]

Letting \(\epsilon, \mu \rightarrow 0^+\), we have

\[
\lim_{\epsilon, \mu \rightarrow 0^+} F_{J^{[\frac{t}{\epsilon}]}_x \rightarrow J^{[\frac{1}{\mu}]}_x} (k) = 1 \text{ for all } k > 0.
\]

This implies that \(\{J^{[\frac{t}{\epsilon}]}_x\}\) is a Cauchy sequence in \(E\). Hence the limit

\[
(3.2) \quad T(t)x = \lim_{\epsilon \rightarrow 0^+} J^{[\frac{t}{\epsilon}]}_x x
\]

exists. Since \(J^{[\frac{t}{\epsilon}]}_x\) is contractive, for each \(x \in \overline{D(A)}\) the limit in (3.2) still exists and \(T(t)\) is contractive on \(\overline{D(A)}\) for all \(t \geq 0\).

Next, let \(t, s \geq 0\) and \(x \in D(A)\). Then, by Lemma 3.5, we have

\[
F_{J^{[\frac{t}{\epsilon}]}_x \rightarrow J^{[\frac{1}{\mu}]}_x} (k) \geq \sup_{u \in Ax} F_u(k \cdot \{|\frac{t}{\epsilon} \cdot \epsilon - \frac{s}{\epsilon} \cdot \epsilon\}^2 + \frac{t}{\epsilon} \cdot \epsilon^2 + \frac{s}{\epsilon} \cdot \epsilon^2\}^{-\frac{1}{2}}).
\]

Since

\[
\{|\frac{t}{\epsilon} \cdot \epsilon - \frac{s}{\epsilon} \cdot \epsilon\}^2 + \frac{t}{\epsilon} \cdot \epsilon^2 + \frac{s}{\epsilon} \cdot \epsilon^2 \leq (|t - s| + \epsilon)^2 + (t + s) \cdot \epsilon,
\]

for any \(u \in Ax\) and \(k > 0\) we have

\[
(3.3) \quad F_{J^{[\frac{t}{\epsilon}]}_x \rightarrow J^{[\frac{1}{\mu}]}_x} (k) \geq \sup_{u \in Ax} F_u(k \cdot \{|t - s| + \epsilon\}^2 + (t + s) \cdot \epsilon\}^{-\frac{1}{2}})
\]

\[
\geq F_u(k \cdot \{|t - s| + \epsilon\}^2 + (t + s) \cdot \epsilon\}^{-\frac{1}{2}})
\]
and

\[ F_{T(t)x-T(s)x}(k) \]
\[ \geq \Delta(F_{T(t)x-J^\xi_{\epsilon}x}(\frac{\eta}{3}), F_{J^\xi_{\epsilon}x-T(s)x}(k-\frac{\eta}{3})) \]
\[ \geq \Delta(F_{T(t)x-J^\xi_{\epsilon}x}(\frac{\eta}{3}), \Delta(F_{J^\xi_{\epsilon}x-J^\xi_{\epsilon}x}(k-\frac{2\eta}{3}), F_{J^\xi_{\epsilon}x-T(s)x}(\frac{\eta}{3}))) \]

where \( 0 < \eta < k \). Since

\[ \lim_{\epsilon \to 0^+} F_{T(t)x-J^\xi_{\epsilon}x}(\frac{\eta}{3}) = 1 \text{ and } \lim_{\epsilon \to 0^+} F_{J^\xi_{\epsilon}x-T(s)x}(\frac{\eta}{3}) = 1, \]

letting \( \epsilon \to 0^+ \), we have

\[ (3.4) \quad F_{T(t)x-T(s)x}(k) \geq \lim_{\epsilon \to 0^+} F_{J^\xi_{\epsilon}x-J^\xi_{\epsilon}x}(k-\frac{2\eta}{3}) \]

for all \( 0 < \eta < k \) and \( k > 0 \). By (3.3) and the left-continuity of \( F_u(\cdot) \), we have

\[ (3.5) \quad \lim_{\epsilon \to 0^+} F_{J^\xi_{\epsilon}x-J^\xi_{\epsilon}x}(k-\frac{2\eta}{3}) \geq F_u((k-\frac{2\eta}{3}) \cdot |t-s|^{-1}) \]

for all \( \eta \in (0, k) \) and \( u \in Ax \). By (3.4) and (3.5), we have

\[ F_{T(t)x-T(s)x}(k) \geq F_u((k-\frac{2\eta}{3}) \cdot |t-s|^{-1}) \]

for all \( \eta \in (0, k) \) and \( u \in Ax \). Letting \( \eta \to 0^+ \), by the left-continuity of \( F_u(\cdot) \), we have

\[ F_{T(t)x-T(s)x}(k) \geq F_u(\frac{k}{|t-s|}) \text{ for all } u \in Ax. \]

This shows that \( T(t)x \) is a Lipschitz continuous function in \( t \) for any \( x \in \text{D}(A) \). Since \( T(t) \) is contractive, \( T(t)x \) is a continuous function in \( t \) for any \( x \in \overline{\text{D}(A)} \).
Finally, letting \( x \in D(A) \) and \( t, s \geq 0 \), then

\[
F_{j_t^+ j_t^+} (k) \geq \sup_{u \in Ax} F_u (k \cdot \left\{ \left[ \frac{t+s}{\epsilon} \right] \cdot \epsilon - \left[ \frac{t}{\epsilon} \right] + \left[ \frac{s}{\epsilon} \right] \right\} \cdot \epsilon^2 \\ + \left[ \frac{t+s}{\epsilon} \right] \cdot \epsilon^2 + \left( \left[ \frac{t}{\epsilon} \right] + \left[ \frac{s}{\epsilon} \right] \right) \epsilon^2 \right) ^{-\frac{1}{2}} \\
\geq \sup_{u \in Ax} F_u (k \cdot \left\{ (3\epsilon)^2 + 2(t+s)\epsilon \right\} ^{-\frac{1}{2}})
\]

for all \( k > 0 \). Letting \( \epsilon \to 0^+ \), we have

\[
\lim_{\epsilon \to 0^+} F_{j_t^+ j_t^+} (k) = 1 \text{ for all } k > 0,
\]

which implies that \( T(t+s)x = T(t)T(s)x \) for all \( t, s \geq 0 \) and \( x \in D(A) \). Therefore, since \( T(t) \) is a contraction, it follows that

\[
T(t+s)x = T(t) \cdot T(s)x \text{ for all } x \in \overline{D(A)} \text{ and } t, s \geq 0.
\]

This completes the proof.

**Remark.** Theorem 3.4 is a generalization of the Crandall-Liggett's exponential formula for some kind of accretive mappings in Banach spaces to probabilistic normed spaces.

**Theorem 3.5.** Let \( A : E \to 2^E \) be an accretive mapping satisfying the following conditions:

(i) \( \overline{D(A)} \subset R(I + \epsilon A) \) for all \( \epsilon > 0 \);

(ii) If \( x_n \in D(A) \), \( y_n \in Ax_n \), \( x_n \to x \) and \( y_n \to y \) as \( n \to \infty \), then \( x \in D(A) \) and \( y \in Ax \).

Let \( \{ T(t) : t \geq 0 \} \) be the semigroups generated by \( A \) as given in Theorem 3.4. If \( x \in D(A) \) and \( u(t) = T(t)x \) is strongly differentiable for almost all \( t > 0 \), then \( u(t) \) is the unique strong solution of the Cauchy problem \( (E3.1) \):

To prove Theorem 3.5, we need the following:

**Lemma 3.6.** Let \( A : D(A) \subset E \to 2^E \) be an accretive mapping satisfying \( D(A) \subset R(I + \epsilon A) \) for all \( \epsilon > 0 \) and \( \{ T(t) : t \geq 0 \} \) be the semigroup given in Theorem 3.4. If \( x \in D(A) \), then for any \( x_0 \in D(A) \), \( y_0 \in Ax_0 \), \( t \geq 0 \) and \( \lambda \in (0, 1) \),

\[
P_\lambda (T(t)x - x_0) \leq P_\lambda (x - x_0) + \int_0^t [T(s)x - x_0, y_0]_\lambda^+ ds.
\]
Proof. Let \( x \in D(A) \), \( x_0 \in D(A) \) and \( y_0 \in A x_0 \). For any \( \epsilon > 0 \) and positive integer \( N \), we have

\[
\epsilon^{-1}(J_\epsilon^N x - J_\epsilon^{N-1} x) \in -A J_\epsilon^N x.
\]

Since \( A \) is accretive, by Lemma 3.1, we have

\[
\begin{align*}
[J_\epsilon^N x - x_0, & \frac{1}{\epsilon}(J_\epsilon^N x - J_\epsilon^{N-1} x) + y_0]_\lambda^- \\
= & -[J_\epsilon^N x - x_0, \frac{1}{\epsilon}(J_\epsilon^{N-1} x - J_\epsilon^N x) - y_0]_\lambda^+ \leq 0.
\end{align*}
\]

By Lemma 2.2 (vi), we have

\[
J_\epsilon^N x - x_0, \frac{1}{\epsilon}(J_\epsilon^N x - J_\epsilon^{N-1} x) + y_0]_\epsilon^-
\]

\[
\geq [J_\epsilon^N x - x_0, \frac{1}{\epsilon}(J_\epsilon^N x - J_\epsilon^{N-1} x)]_\lambda^- + [J_\epsilon^N x - x_0, y_0]_\lambda^-.
\]

In view of Proposition 2.1 (iv), we have

\[
\begin{align*}
[J_\epsilon^N x - x_0, & \frac{1}{\epsilon}(J_\epsilon^N x - J_\epsilon^{N-1} x) + y_0]_\lambda^- \\
\geq & \frac{1}{\epsilon} (P_\lambda(J_\epsilon^N x - x_0) - P_\lambda(J_\epsilon^{N-1} x - x_0 - (J_\epsilon^N x - J_\epsilon^{N-1} x))) \\
& + [J_\epsilon^N x - x_0, y_0]_\lambda^-.
\end{align*}
\]

By (3.6) and (3.7), we have

\[
(3.8) \quad P_\lambda(J_\epsilon^N x - x_0) \leq P_\lambda(J_\epsilon^{N-1} x - x_0) + \epsilon[J_\epsilon^N x - x_0, -y_0]_\lambda^+.
\]

Adding up the inequalities in (3.8) from \( N = 1 \) to \( N = n \), we have

\[
(3.9) \quad P_\lambda(J_\epsilon^n x - x_0) \leq P_\lambda(x - x_0) + \sum_{N=1}^{n} \epsilon[J_\epsilon^N x - x_0, -y_0]_\lambda^+.
\]

Letting \( t \geq 0 \) and \( n = \lceil \frac{t}{\epsilon} \rceil \), then (3.9) can be written as follows:

\[
P_\lambda(J_\epsilon^{\lceil \frac{t}{\epsilon} \rceil} x - x_0) \leq P_\lambda(x - x_0) + \int_{\epsilon}^{\lceil \frac{t}{\epsilon} \rceil + 1} [J_\epsilon^s x - x_0, -y_0]_\lambda^+ ds.
\]
Since \([J_\varepsilon^{[t]} x - x_0, -y_0]_\lambda^+ \leq P_\lambda(y_0)\), letting \(\varepsilon \to 0^+\), by the Lebesgue’s convergence theorem, it follows from the upper semi-continuity of \([\cdot, \cdot]_\lambda^+\) that
\[
P_\lambda(T(t)x - x_0) \leq P_\lambda(x - x_0) + \int_0^t \lim_{\varepsilon \to 0} [J_\varepsilon^{[s]} x - x_0, -y_0]_\lambda^+ ds
\leq P_\lambda(x - x_0) + \int_0^t [T(s)x - x_0, -y_0]_\lambda^+ ds.
\]
This completes the proof.

**Proof of Theorem 3.5.** For \(x \in D(A)\), if \(T(t)x\) has a derivative \(\frac{d}{dt}T(t)x|_{t=t_0} = y\) at \(t = t_0 > 0\), then, by Lemma 3.6, we have
\[
P_\lambda(T(t_0 + h)x - x_0) \leq P_\lambda(T(t_0)x - x_0)
+ \int_0^h [T(t_0 + s)x - x_0, -y_0]_\lambda^+ ds
\]
for all \(h > 0\). Dividing by \(h > 0\) on both sides and letting \(h \to 0^+\), from Lemma 2.2 (ix), we have
\[
[T(t_0)x - x_0, y]_\lambda^+ \leq [T(t_0)x - x_0, -y_0]_\lambda^+.
\]
It follows from Lemma 2.2 (vii) that
\[
[T(t_0)x - x_0, y + y_0]_\lambda^+
\leq [T(t_0)x - x_0, y]_\lambda^+ + [T(t_0)x - x_0, y_0]_\lambda^+
= [T(t_0)x - x_0, y]_\lambda^+ - [T(t_0)x - x_0, -y_0]_\lambda^+
\leq 0.
\]
By the condition (i), for any \(\varepsilon \in (0, t_0)\), there exist \(x_\varepsilon \in D(A)\) and \(y_\varepsilon \in Ax_\varepsilon\) such that
\[
x_\varepsilon + \varepsilon y_\varepsilon = T(t_0 - \varepsilon)x.
\]
Taking \(x_0 = x_\varepsilon\), \(y_0 = y_\varepsilon = \varepsilon^{-1}(T(t_0 - \varepsilon)x - x_\varepsilon)\) in (3.10), we have
\[
0 \geq [T(t_0)x - x_\varepsilon, y + \varepsilon^{-1}(T(t_0 - \varepsilon)x - x_\varepsilon)]_\lambda^-
= [T(t_0)x - x_\varepsilon, y + \varepsilon^{-1}(T(t_0 - \varepsilon)x - T(t_0)x)]_\lambda^-
+ \varepsilon^{-1}P_\lambda(T(t_0)x - x_\varepsilon)
\geq \varepsilon^{-1}P_\lambda(T(t_0)x - x_\varepsilon) - P_\lambda(y + \varepsilon^{-1}(T(t_0 - \varepsilon)x - T(t_0)x)).
i.e.,

\[ P_\lambda(T(t_0)x - x_\epsilon) \leq P_\lambda(\epsilon y + (T(t_0 - \epsilon)x - T(t_0)x)) \text{ for all } \lambda \in (0, 1]. \]

Therefore, we must have

\[ F_{T(t_0)x - x_\epsilon}(k) \geq F_{\epsilon y + T(t_0 - \epsilon)x - T(t_0)x}(k) \text{ for all } k \geq 0 \]

and so \( x_\epsilon \to T(t_0)x \) as \( \epsilon \to 0^+ \). Since

\[
(3.11) \quad F_{y + y_\epsilon}(k) = F_{y - \epsilon^{-1}(T(t_0)x - T(t_0 - \epsilon)x) + \epsilon^{-1}(T(t_0)x - x_\epsilon)}(k) \\
\geq \Delta(F_{y - \epsilon^{-1}(T(t_0)x - T(t_0 - \epsilon)x)}(\frac{k}{2}), F_{-\epsilon^{-1}(T(t_0)x - x_\epsilon)}(\frac{k}{2}))
\]

from (3.11), (3.12) and \( \lim_{\epsilon \to 0^+} \epsilon^{-1}(T(t_0)x - T(t_0 - \epsilon)x) = y \), it follows that

\[
F_{y + y_\epsilon}(k) \geq F_{y - \epsilon^{-1}(T(t_0)x - T(t_0 - \epsilon)x)}(\frac{k}{2}) \to 1 \text{ as } \epsilon \to 0^+
\]

and so \( y_\epsilon \to -y \) as \( \epsilon \to 0^+ \). By the condition (ii), we have \( T(t_0)x \in D(A) \) and \( y \in -AT(t_0)x \). This completes the proof.

4. An open question

In the end of this paper, we suggest the following open question:

Let \((E,F,\Delta)\) be a complete Menger \(PN\)-space and \(A : E \to 2^E\) be a continuous accretive mapping. Then is \(A\) a \(m\)-accretive mapping?

References


Nonlinear semigroups and differential inclusions


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