1. Introduction

A near-ring is a nonempty set $R$ with two binary operations $+$ and $\cdot$ such that $(R, +)$ is a group (not necessarily abelian) with identity 0, $(R, \cdot)$ is a semigroup and $a(b + c) = ab + ac$ for all $a, b, c$ in $R$. In general a near-ring $R$ with the extra axiom $0a = 0$ for all $a \in R$ is said to be zero symmetric. An element $d$ in $R$ is called distributive if $(a + b)d = ad + bd$ for all $a$ and $b$ in $R$. Let $(G, +)$ be a group (not necessarily abelian). If we set $M(G) := \{ f \mid f : G \to G \}$, and define the sum $f + g$ of any two mappings $f, g$ in $M(G)$ by the rule $x(f + g) = xf + xg$ for all $x \in G$ and the product $f \cdot g$ by the rule $x(f \cdot g) = (xf)g$ for all $x \in G$ then $(M(G), +, \cdot)$ forms a near-ring. Let $M_0(G) := f \in M(G) \mid 0f = 0$. Then $(M_0(G), +, \cdot)$ is a zero symmetric near-ring. For the remainder results and definitions on near-rings, we refer to G. Pilz [6].

Let $R$ be any near-ring and $G$ an additive group. Then $G$ is called an $R$–group (or module) if there exists a near-ring homomorphism

$$\theta : (R, +, \cdot) \to (M(G), +, \cdot).$$

Such a homomorphism $\theta$ is called a representation of $R$ on $G$, we will write that $xr$ for $x(r\theta)$ for all $x \in G$ and $r \in R$. A representation $\theta$ is called faithful if $\ker \theta = 0$.

The near-ring $R$ is called a distributively generated (briefly, D.G.) near-ring if $(R, +) = gp < S >$ where $S$ is a semigroup of distributive elements in $R$, we denote it $(R, S)$. The distributive elements of $M_0(G)$

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are $End(G)$, the semigroup of all the endomorphisms of the group $G$. Here we denote that $E(G)$ is the D.G. near-ring generated by $End(G)$, and call that $E(G)$ is the endomorphism near-ring of the group $G$.

A homomorphism $\theta : (R, S) \rightarrow (T, U)$ is a D.G. near-ring homomorphism if $\theta$ is a near-ring homomorphism such that $S\theta \subseteq U$. A semigroup homomorphism $\theta : S \rightarrow U$ is a D.G. near-ring homomorphism if it is a group homomorphism from $(R, +)$ to $(T, +)$. See C. G. Lyons and J. D. P. Meldrum([3],[4]).

Let $R$ be a near-ring and let $G$ be an $R$–group. If there exists $x$ in $G$ such that $G = xR$, that is, $G = \{xr | r \in R\}$, then $G$ is called a monogenic $R$–group and the element $x$ is called a generator of $G$. See J. D. P. Meldrum and G. Pilz([5], [6]).

2. Properties of D.G. near-rings $(R, S)$ and D.G. $(R, S)$–modules

Now we may introduce new concepts as follows: Let $(R, S)$ be a D.G. near-ring. Then an additive group $G$ is called a D.G. $(R, S)$–group(or D.G. $(R, S)$–module) if there is a near-ring homomorphism $\theta : (R, S) \rightarrow (E(G), End(G))$

such that $S\theta \subseteq End(G)$. Such a homomorphism is called a D.G. representation of $(R, S)$. This D.G. representation is said to be faithful if $Ker\theta = 0$.


Next, let $R$ be a near-ring and $G$ an additive group. If there is a scalar multiplication $\theta : (R, S) \rightarrow G$

which is defined by $\theta(a, x) = ax$ such that $(ab)x = a(bx)$ and $a(x+y) = ax + ay$ for all $a, b \in R$ and for all $x, y \in G$, Then $G$ is called a $R$–cogroup(or comodule), see Y. U. Cho[2]. If $R$ is a right near-ring,
then every $R$–cogroup is an $R$–group for $R$ as an $R$–group. Similar method of lemma 2.1 shows the following lemma:


**Proposition 2.3.** Let $(R, S)$ be a D.G. near-ring. Then
(1) Every monogenic $R$–group is a D.G. $(R, S)$–group.
(2) Every monogenic $R$–cogroup is a D.G. $(R, S)$–cogroup.

*Proof.* Let $G$ be a monogenic $R$–group with $x$ as a generator. Then the map $\phi : r| \rightarrow xr$ is an $R$–epimorphism from $R$ to $G$ as $R$–groups. We see that

$$G \cong R/A(x),$$

where $A(x) = (0 : x) = \text{Ker} \phi$. See for this notation Y. U. Cho[2].

From the Lemma 2.1, we obtain that $G$ is a D.G. $(R, S)$–group.

For $G$ is a monogenic $R$–cogroup with $x$ as a generator, the map $\psi : r| \rightarrow rx$ is also an $R$–epimorphism from $R$ to $G$ as an $R$–cogroups. Thus we have that

$$G \cong R/\text{Ann}(x),$$

where $\text{Ann}(x) = [0 : x] = \text{Ker} \psi$. See also for this notation Y. U. Cho[2].

By the Lemma 2.2, we see that $G$ is a D.G. $(R, S)$–cogroup. □

**Theorem 2.4.** Let $(R, S)$ be a D.G. near-ring and $(G, +)$ is an abelian group. Then
(1) If $G$ is a faithful D.G. $(R, S)$–group, then $R$ is a ring.
(2) If $G$ is a faithful D.G. $(R, S)$–cogroup, then $R$ is also a ring.

*Proof.* (1) Let $x \in G$ and $r, s \in R$. Then, since $(G, +)$ is abelian,

$$x(r + s) = xr + xs = xs + xr = x(s + r).$$

Thus we get that $x(r + s) - (s + r) = 0$ for all $x \in G$, that is, $(r + s) - (s + r) \in \text{Ker} \theta = (0 : G) = A(x)$, where $\theta : R \rightarrow M(G)$ is a representation of $R$ on $G$. Since $G$ is faithful, that is, $\theta$ is faithful,
\[ \text{Ker} \theta = (0 : G) = 0. \text{ Hence for all } r, s \in R, r + s = s + r. \text{ Consequently, } (R, +) \text{ is abelian.} \]

Next we must show that \( R \) satisfies the right distributive law. Obviously, we note that for all \( r, r' \in R \) and all \( s \in S, \)

\[ 0s = 0, \ (-r)s = -(rs) = r(-s) \text{ and } (r + r')s = rs + r's. \]

Let \( x \in G \) and \( r, s, t \in R. \) Then the element \( t \) in \( R \) is represented by

\[ t = \delta_1 s_1 + \delta_2 s_2 + \delta_3 s_3 + \cdots + \delta_n s_n, \]

where \( \delta_i = 1, \text{ or } -1 \) and \( s_i \in S \) for \( 1 \leq i \leq n. \) Thus, using the above note and \( (G, +) \) is abelian, we have the following equalities:

\[
x(r + s)t = (xr + xs)t = (xr + xs)(\delta_1 s_1 + \delta_2 s_2 + \cdots + \delta_n s_n) \\
= (xr + xs)\delta_1 s_1 + (xr + xs)\delta_2 s_2 + \cdots + (xr + xs)\delta_n s_n \\
= \delta_1(xr + xs)s_1 + \delta_2(xr + xs)s_2 + \cdots + \delta_n(xr + xs)s_n \\
= \delta_1(xrs_1 + xss_1) + \delta_2(xrs_2 + xss_2) + \cdots + \delta_n(xrs_n + xss_n) \\
= \delta_1 xrs_1 + \delta_2 xrs_2 + \cdots + \delta_n xrs_n + \delta_1 xss_1 + \delta_2 xss_2 + \cdots + \delta_n xss_n \\
= x(r\delta_1 s_1 + xss_1) + x(r\delta_2 s_2 + xss_2) + \cdots + x(r\delta_n s_n + xss_n) \\
= x\delta_1 (rs_1 + \cdots + \delta_n s_n) + x\delta_2 (rs_2 + \cdots + \delta_n s_n) \\
= x(rt + xst) = x(rt + st). 
\]

thus we obtain that \( x(r + s)t - (rt + st) = 0 \) for all \( x \in G, \) namely,

\[ (r + s)t - (rt + st) \in (0 : G) = A(G). \]

Also using \( G \) is faithful, that is, \( A(G) = 0. \) Applying the beginning part of this proof, we see that \( (r + s)t = rt + st \) for all \( r, s, t \in R, \) consequently, \( R \) satisfies the right distributivr law. Hence \( R \) becomes a ring.

(2) We can prove this as similar method to the proof of (1). \[ \square \]

As an immediate consequence of theorem 2.4, we have the following important corollary.
**Corollary 2.5.** Let \((R, S)\) be an abelian D.G. near-ring. Then \(R\) is a ring.

Finally, we may define a new concept and then characterize D.G. near-ring with this new concept as following.

A near-ring \(R\) is called generalized right bipotent if for all \(a \in R\) there exists a positive integer \(n\) such that

\[ a^n R = a^{n+1} R. \]

There are many examples of generalized right bipotent near-rings, for example, Boolean near-rings.

**Theorem 2.6.** Let \((R, S)\) be a generalized right bipotent D.G. near-ring. If there exists an element in \(R\) which is not a zero divisor, then \(R\) has an identity.

**Proof.** Let \(a \in R\) such that \(a\) is not a zero divisor then also \(a^n\) is not a zero divisor for any positive integer \(n\). Indeed, suppose that \(a^n\) is a zero divisor, then there exists a nonzero element \(x \in R\) such that \(a^n x = 0\), that is, \(a(a^{n-1}x) = 0\), since \(a\) is not a zero divisor, this implies that \(a^{n-1}x = 0\). Continuing this procedure we get that \(x = 0\), this fact is a contradiction. Hence \(a^n\) is not a zero divisor.

Assume that \(a \in R\) is not a zero divisor which is not zero. Since \(R\) is generalized right bipotent, we have the following equation

\[ a^n R = a^{n+1} R \]

for some positive integer \(n\). This implies that \(a^n a = a^{n+1} e\) for some \(e\) in \(R\), that is, \(a^n (a - ae) = 0\). From the above remark of this proof, since \(a^n\) is not a left zero divisor, we obtain that \(a = ae\). Also, from the equation \(a(ea - a) = a(ea) - aa = (ae)a - aa = aa - aa = 0\), we get that \(a = ea\).

Next, let \(r\) be an arbitrary element of \(R\). From the following equation:

\[ a(er - r) = a(er) - ar = (ae)r - ar = ar - ar = 0, \]

since \(a\) is not a left zero divisor, we obtain that \(er = r\), so that \(e\) is the left identity of \(R\).
Finally, let $r$ be any element of $R$. Suppose $a$ is not a zero divisor on $R$. Then since $(R, S)$ is a D.G. near-ring, there exists a positive integer $n$, we can decompose $a$ as follows:

$$a = \delta_1 s_1 + \delta_2 s_2 + \cdots + \delta_n s_n$$

for some $s_i \in S, \delta_i = 1$ or $-1$ for $1 \leq i \leq n$. Then we have the following equalities:

$$(re - r)a = (re - r)(\delta_1 s_1 + \delta_2 s_2 + \cdots + \delta_n s_n)$$
$$= (re - r)\delta_1 s_1 + (re - r)\delta_2 s_2 + \cdots + (re - r)\delta_n s_n$$
$$= \delta_1(re - r)s_1 + \delta_2(re - r)s_2 + \cdots + \delta_n(re - r)s_n$$
$$= \delta_1(res_1 - rs_1) + \delta_2(res_2 - rs_2) + \cdots + \delta_n(res_n - rs_n)$$
$$= \delta_1(rs_1 - rs_1) + \delta_2(rs_2 - rs_2) + \cdots + \delta_n(rs_n - rs_n)$$
$$= 0 + 0 + \cdots + 0 = 0.$$  

This implies that $re = r$, that is, $e$ is the right identity of $R$. Consequently, $e$ is the identity of $R$. □

References


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