EXOTIC SYMPLECTIC STRUCTURES ON $S^3 \times \mathbb{R}$

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Abstract. We construct exotic symplectic structures on $S^3 \times \mathbb{R}$ which is obtained by the symplectic sum of two smooth symplectic four-manifolds with exotic symplectic structures, each of which is diffeomorphic to $\mathbb{R}^4$.

1. Introduction

Let $\omega_0$ be the standard symplectic structure on $\mathbb{R}^{2n}$ and $L \subset \mathbb{R}^{2n}$ be a closed Lagrangian submanifold. In [3], Gromov have shown the following theorem:

Theorem (Gromov). As a cohomology class $[\omega_0]$ is non-zero in $H^2(\mathbb{R}^{2n}, L; \mathbb{R})$, the form $\omega_0$ has a potential $\psi$ on $\mathbb{R}^{2n}$, i.e., $\omega_0 = d\psi$. Furthermore, $[\psi|_L] \neq 0$ in $H^1(L; \mathbb{R})$.

The Lagrangian submanifold $L$ in a $2n$-dimensional symplectic manifold $M$ is called exact (non-exact) if the restriction to the Lagrangian $L$ of the potential is exact (non-exact). Thus, in the above Theorem, $L$ is a non-exact Lagrangian in $\mathbb{R}^{2n}$.

Gromov have also proved that there are no exact Lagrangian subvarieties in $\mathbb{R}^{2n}$, for the standard symplectic structure. Recently, Bates and Peschke [1] have explicitly endowed a manifold $M$ diffeomorphic to $\mathbb{R}^4$ with a symplectic form $\omega$ admitting a Lagrangian torus $T$ such that $[\omega] = 0$ in $H^2(M, T; \mathbb{R})$. Hence $T$ is an exact Lagrangian. By Gromov's theorem, $(M, \omega)$ does not symplectically embed in $(\mathbb{R}^4, \omega_0)$, such a structure $\omega$ is called an exotic symplectic structure on $M$.

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Let $M_i$ ($i = 1, 2$) be smooth symplectic four-manifolds diffeomorphic to $\mathbb{R}^4$ with symplectic forms admitting Lagrangian tori $(T_i')^i$ ($i = 1, 2$).

In section 2, we introduce the symplectic sum of these two manifolds and construct symplectic forms $\omega_M$ on the sum $M = M_1 \#_{\psi} M_2$ from symplectic forms on the $M_i$ ($i = 1, 2$). We first show that

**Lemma 2.3.** $M = M_1 \#_{\psi} M_2 \cong ((M_1 - S_1) - K) \cup_{\varphi} ((M_2 - S_2) - j_2(D^2)) \cong S^3 \times \mathbb{R}$, where $S_i$ are the interior surfaces of $S_i$ on $(T_i')^i$ with the boundaries $S_i^1 = j_i(\partial D^2)$ ($i = 1, 2$). Hence $H^1(T_2'; \mathbb{R}) \cong H^2(M, T_2'; \mathbb{R})$ is an isomorphism, where $T_2'$ is a Lagrangian surface of genus 2 in $M$.

In section 3, we show the process of constructing symplectic forms $\omega'_M$ on $M = M_1 \#_{\psi} M_2 \cong S^3 \times \mathbb{R}$ from exotic symplectic forms on two smooth symplectic four-manifolds $M_i$ ($i = 1, 2$) diffeomorphic to $\mathbb{R}^4$.

In section 4, we get the following two Lemmas 4.1 and 4.2 from each case of manifolds $(M, \omega_M)$ and $(M, \omega'_M)$:

**Lemma 4.1.** The symplectic forms $\omega_M$ admit a non-exact Lagrangian surface $T'_2$ of genus 2 in $M$ and hence $[\omega_M] \neq 0$ in $H^2(M, T'_2; \mathbb{R})$.

**Lemma 4.2.** The symplectic forms $\omega'_M$ admit an exact Lagrangian surface $T_2$ of genus 2 in $M$ and hence $[\omega'_M] = 0$ in $H^2(M, T_2; \mathbb{R})$.

By the Lemmas 4.1 and 4.2, we can get the following Theorem 4.3.

**Theorem 4.3.** The symplectic forms $\omega_M$ on the symplectic sum $M$ of two smooth symplectic four-manifolds $M_i$ ($i = 1, 2$) diffeomorphic to $\mathbb{R}^4$ with symplectic forms admitting non-exact Lagrangian tori $(T_i')^i$ ($i = 1, 2$) admit a non-exact Lagrangian surface $T_2'$ of genus 2 and $[\omega_M] \neq 0$ in $H^2(M, T'_2; \mathbb{R})$.

On the other hand, the symplectic forms $\omega'_M$ on the symplectic sum $M$ of two smooth symplectic four-manifolds $M_i$ ($i = 1, 2$) diffeomorphic to $\mathbb{R}^4$ with symplectic forms admitting exact Lagrangian tori $T_i'$ ($i = 1, 2$) admit an exact Lagrangian surface $T_2$ of genus 2 and $[\omega'_M] = 0$ in $H^2(M, T_2; \mathbb{R})$. Therefore, $(M, \omega'_M)$ does not symplectically diffeomorphic to $(M, \omega_M)$.
2. Symplectic sums

Let $M_i$ ($i = 1, 2$) be smooth symplectic four-manifolds which are diffeomorphic to $\mathbb{R}^4$. Let $\mathbb{R}^4$ be thought of as $\mathbb{R}^2 \times \mathbb{R}^2$ and let $(r, \theta), (s, \phi)$ be polar coordinates on each factor. That is, if $(x_1, x_2)$ and $(y_1, y_2)$ are rectangular coordinates on each factor of $\mathbb{R}^2 \times \mathbb{R}^2$, then $x_1 = r \cos \theta, x_2 = r \sin \theta, y_1 = s \cos \phi, y_2 = s \sin \phi$. Suppose that $\mathbb{R}^4$ has a standard symplectic structure $\omega_{\mathbb{R}^4} = \sum_{i=1}^{2} dx_i \wedge dy_i$.

Let $T_1 = \{(x_1, x_2, y_1, y_2) \in \mathbb{R}^4 | x_1^2 + x_2^2 = \frac{\pi}{2}, y_1^2 + y_2^2 = \frac{\pi}{2}\} = \{(\sqrt{\frac{\pi}{2}} \cos \theta, \sqrt{\frac{\pi}{2}} \sin \theta, \sqrt{\frac{\pi}{2}} \cos \phi, \sqrt{\frac{\pi}{2}} \sin \phi) \in \mathbb{R}^4 | 0 \leq \theta < 2\pi, 0 \leq \phi < 2\pi\}$. Let $j : T_1 \to \mathbb{R}^4$ be an embedding defined by $j(r \cos \theta, r \sin \theta, s \cos \phi, s \sin \phi) = (r \cos \theta, s \cos \phi, r \sin \theta, s \sin \phi)$. Then $T_1 = j(T_1)$ is a torus defined by $x_1^2 + y_1^2 = \frac{\pi}{2}$ and $x_2^2 + y_2^2 = \frac{\pi}{2}$, and a closed Lagrangian in $\mathbb{R}^4$ with respect to $\omega_{\mathbb{R}^4}$ since $\omega_{\mathbb{R}^4}|_{T_1} = j^* \omega_{\mathbb{R}^4}$ and

$$j^* \omega_{\mathbb{R}^4}(m)(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi})$$

$$= \omega_{\mathbb{R}^4}(j(m))(dj(\frac{\partial}{\partial \theta}|_m), dj(\frac{\partial}{\partial \phi}|_m))$$

$$= (dx_1 \wedge dy_1 + dx_2 \wedge dy_2)(j(m))$$

$$\begin{align*}
&= (-r \sin \alpha \cdot \frac{\partial}{\partial x_1} |_{j(m)} - r \cos \alpha \cdot \frac{\partial}{\partial y_1} |_{j(m)}, \\
&\quad - s \sin \beta \cdot \frac{\partial}{\partial x_2} |_{j(m)} + s \cos \beta \cdot \frac{\partial}{\partial y_2} |_{j(m)}) \\
&= -r \sin \alpha \cdot 0 - 0 \cdot r \cos \alpha + 0 \cdot s \cos \beta + s \sin \beta \cdot 0 \\
&= 0
\end{align*}$$

for all $m = (r \cos \theta, r \sin \theta, s \cos \phi, s \sin \phi) \in T_1$.

By Gromov's theorem in section 1, $[\omega_{\mathbb{R}^4}] \neq 0$ in $H^2(\mathbb{R}^4, T_1^* \mathbb{R})$ and $[\sum_{i=1}^{2} x_i dy_i|_{T_1^*}] \neq 0$ in $H^1(T_1^* \mathbb{R})$. If we take $\varphi_i$ as diffeomorphism from $M_i$ to $\mathbb{R}^4$ such that $\varphi_i^{-1}(T_1^i) = (T_1^i)^i$ and if we set $\omega_{M_i} = \varphi_i^* \omega_{\mathbb{R}^4}$ as symplectic structures on $M_i$ ($i = 1, 2$), then $(T_1^i)^i$ are closed Lagrangian tori in $M_i$ since $\omega_{M_i}|_{(T_1^i)^i} = \varphi_i^* \omega_{\mathbb{R}^4}|_{(T_1^i)^i} = \omega_{\mathbb{R}^4}|_{T_1^i} = 0$. Moreover, $(T_1^i)^i$ are non-exact Lagrangian tori in $M_i$ since $[\varphi_i^*(\sum_{i=1}^{2} x_i dy_i)|_{(T_1^i)^i}]$
= \{\sum_{i=1}^{2} x_i dy_i | T_i'\} \neq 0 \text{ in } H^1((T_i')^i; \mathbb{R}). \text{ By isomorphisms } H^1((T_i')^i; \mathbb{R}) \\
\cong H^2(M_i, (T_i')^i; \mathbb{R}), [\omega_{M_i}] \neq 0 \text{ in } H^2(M_i, (T_i')^i; \mathbb{R}).

Let \( D^2 \) be the standard closed 2-dimensional disk of radius \( \sqrt{\pi} \) with symplectic structure \( \omega_{D^2} = dx_1 \wedge dy_1 \). Let \( h : (D^2, \partial D^2) \to (\mathbb{R}^4, T_i') \) be defined by \( h(x_1, y_1) = (\frac{x_1}{\sqrt{2}}, \frac{y_1}{\sqrt{2}}, \frac{x_2}{\sqrt{2}}, -\frac{x_3}{\sqrt{2}}) \), and let \( j_i = \varphi_i^{-1} \circ h : (D^2, \partial D^2) \to (M_i, (T_i')^i) \). Then \( j_i \) are symplectic embeddings satisfying \( j_i(\partial D^2) \subset (T_i')^i \) and \( (j_i(D^2) - j_i(\partial D^2)) \cap (T_i')^i = \emptyset \) \((i = 1, 2)\) since \( j_i^* \omega_{M_i} = j_i^* \varphi_i^* \omega_{\mathbb{R}^4} = (\varphi_i \circ j_i)^* \omega_{\mathbb{R}^4} = h^* \omega_{D^2} \) and

\[
h^* \omega_{D^2} = h^*(dx_1 \wedge dy_1 + dx_2 \wedge dy_2) \\
= \frac{1}{\sqrt{2}} dx_1 \wedge \frac{1}{\sqrt{2}} dy_1 + \frac{1}{\sqrt{2}} dy_1 \wedge (-\frac{1}{\sqrt{2}})dx_1 \\
= \frac{1}{2} dx_1 \wedge dy_1 - \frac{1}{2} dy_1 \wedge dx_1 \\
= dx_1 \wedge dy_1 \\
= \omega_{D^2}.
\]

We can choose a fiber-orientation reversing bundle isomorphism \( \psi : \nu_1 \to \nu_2 \). We choose fiber metrics on \( \nu_i \) such that \( \psi \) is isometric. Let \( \nu_i^0 \) be disk bundles in \( \nu_i \) \((i = 1, 2)\). Then there is an orientation-preserving diffeomorphism \( \varphi = \iota \circ \psi : \nu_i^0 - j_i(D^2) \to \nu_2^0 - j_2(D^2) \), where the map \( \iota : \nu_2^0 - \{0 - \text{section}\} \to \nu_2^0 - \{0 - \text{section}\} \) is defined by \( \iota(x) = (\frac{1}{\pi ||x||^2} - 1)^{1/2} x \).

Now we construct suitable models for tubular neighborhoods of the submanifolds \( j_i(D^2) \) in \( M_i \) \((i = 1, 2)\). Let \( \nu_i \) denote the \( SO(2) \)-vector bundles over \( D^2 \) and let \( \nu_i^0 \) denote the sub-disk bundles of radius \( \pi^{-1/2} \) \((i = 1, 2)\). Let \( \pi : S \to D^2 \) be the 2-sphere bundle obtained by gluing together \( \nu_1^0 \) and \( \nu_2^0 \) using \( \iota \) defined in the above statement. We may take the sphere bundle \( S \) over \( D^2 \) as \( D^2 \times S^2 \). Let \( i_0, i_\infty : D^2 \to S \) be 0-sections of \( \nu_1^0 \) and \( \nu_2^0 \) with images \( D_0 \) and \( D_\infty \), respectively. Thus, \( \nu_1^0 = S - D_\infty \).

Considering cylindrical polar coordinates \((\theta, x_3)\) on \( S^2 - \{(0, 0, \pm 1)\} \) where \( 0 \leq \theta < 2\pi \) and \(-1 \leq x_3 \leq 1\), we can take a symplectic form \( \omega_{S^2} \) on \( S^2 \) as the area form \( \omega_{S^2} = d\theta \wedge dx_3 \) induced by the Euclidean metric. Hence we may choose a closed 2-form \( \eta \) on the sphere bundle
S \cong D^2 \times S^2 \text{ over } D^2 \text{ as } \omega_{S^2}. \text{ Then } \eta \text{ has the following properties: } i_0^* \eta = \eta|_{i_0(D^2)} = \eta|_{D_0} = 0 \text{ and } \eta|_{S^2} = d\theta \wedge dx_3 \text{ is the symplectic form. By the method of Thurston}[8], \text{ we can thus construct the set of symplectic forms on } S \text{ as } \{ \omega_t = \pi^* \omega_{D^2} + t \cdot \eta \mid 0 < t \leq t_1 \} \text{ for some sufficiently small constant } t_1 > 0.

On the other hand, there is a smooth orientation-preserving embedding } f : \nu^0_1 \to M_1 \text{ (into any preassigned neighborhood of } j_1(D^2)) \text{ with } f \circ i_0 = j_1. \text{ And } f|_{D_0 : (D_0, \omega_{t})} \to (M_1, \omega_{M_1}) \text{ is symplectic, since } i_0^* \omega_t = i_0^* \pi^* \omega_{D^2} + t \cdot i_0^* \eta = (\pi \circ i_0)^* \omega_{D^2} = \omega_{D^2}, f \circ i_0 = j_1 \text{ and } j_1 \text{ is symplectic. Thus we get the following Theorem 2.1 which is the same result as Gompf's.}

**Theorem 2.1.** Let } (\nu^0_1, \omega_t), (M_1, \omega_{M_1}), D_0 \text{ and } f : \nu^0_1 \to M_1 \text{ be the same as above. Then there is a compactly supported isotopy rel } D_0 \text{ from } f \text{ to an embedding } \tilde{f} : \nu^0_1 \to M_1 \text{ that is symplectic in a neighborhood of } D_0.

**Proof.** It can be proved by the same way as the proof of Lemma 2.1 in [2]. \(\square\)

Weinstein's integral operator } I : \Omega^2(\nu^0_1) \to \Omega^1(\nu^0_1) \text{ is defined by } I(\eta) = \int_0^1 \pi_s^* (X_s \cdot \eta) ds, \text{ where } \pi_s : \nu^0_1 \to \nu^0_1 \text{ (0} \leq s \leq 1) \text{ is a multiplication by } s \text{ in this bundle structure, } X_s = \frac{d}{ds} \pi_s \text{ the corresponding vector field, and } \cdot \text{ denotes contraction. The key property of } I \text{ is that if } \eta \text{ satisfies } d\eta = 0 \text{ and } i_0^* \eta = 0, \text{ then } dI(\eta) = \eta. \text{ Set } \varphi = I(\eta), \text{ and define } Y_t \text{ by } Y_t \cdot \omega_t = -\varphi, 0 < t \leq t_1. \text{ Then } Y_t \text{ (0} < t \leq t_1) \text{ is a time-dependent vector field on } \nu^0_1 \text{ that vanishes on } D_0 \text{ and } SO(2)-\text{invariant. For any } SO(2)-\text{invariant compact subset } K \subset \nu^0_1 \text{ and fixed } t_0 \in (0, t_1], Y_t \text{ integrates to an } SO(2)-\text{equivariant flow } F : K \times J \to \nu^0_1 \text{,where } J \text{ is some neighborhood of } t_0 \text{ in } (0, t_1] \text{ and } F_{t_0} = id_K. \text{ Since } \frac{d}{dt}(F_t^* \omega_t) = dF_t^*(Y_t \cdot \omega_t) + F_t^*(\frac{d}{dt} \omega_t) = -F_t^* d\varphi + F_t^* \eta = -F_t^* \eta + F_t^* \omega_t = 0, F_t^* \omega_t \text{ is independent of } t.

For } x \in \nu^0_1 \text{, let } D(x) \text{ be the closed disk in the fiber } \pi^{-1}(\pi(x)) \text{ that } \text{ is bounded by the } SO(2)-\text{orbit of } x. \text{ Let } A(x) = \int_{D(x)} \eta \text{ be the } \eta-\text{area of } D(x). \text{ Then } A : \nu^0_1 \to [0, 1) \text{ is a smooth, } SO(2)-\text{invariant, proper surjection that increases radially. The } \omega_t-\text{area of } D(x) \text{ is given by } \int_{D(x)} \omega_t = \int_{D(x)} (\pi^* \omega_{D^2} + t \cdot \eta) = t \int_{D(x)} \eta = t \cdot A(x). \text{ Fix } x \in \nu^0_1 \text{ and}
$t_0 \in (0, t_1]$, and integrate $Y_t$ as above to obtain a flow of $D(x)$ with $F_{t_0} = id_{D(x)}$. Let $x(t) = F_t(x)$ be the trajectory of $x$, with $x(t_0) = x$. Since $F$ is $SO(2)$-equivariant, $\partial F_{t} D(x) = \partial D(F_{t}(x)) = \partial D(x(t))$. Thus the $\omega_t$-area of $D(x(t))$ is $t \cdot A(x(t)) = \int_{D(x(t))} \omega_t = \int_{F_{t} D(x)} \omega_t = \int_{D(x)} F_{t \omega_0} \omega_t = t_0 \cdot A(x)$, and hence $A(x(t)) = \frac{t_0}{t} \cdot A(x)$, which tells us that all flow lines of $Y_t$ are decreasing in $A$. Since $A : \nu_1^0 \to [0, 1)$ is proper, flow lines cannot escape from $\nu_1^0$ as $t$ increases, and the flow is globally defined as a map $F : \nu_1^0 \times [t_0, t_1] \to \nu_1^0$.

For any $x \in \nu_1^0$, $A(x) < 1$, so $A(F_{t_1}(x)) = A(x(t_1)) < \frac{t_0}{t_1}$. Thus, we may arrange for $F_{t_1}(\nu_1^0)$ to lie in any preassigned neighborhood $V$ of $D_0$ by choosing $t_0$ sufficiently small. Since $F_{t_1} : (\nu_1^0, \omega_{t_0}) \to (\nu_1^0, \omega_{t_1})$ is symplectic, we get the following result with the neighborhood $V = \nu_1^0$ of $D_0$: For the neighborhood $\nu_1^0$ of $D_0$ in $(\nu_1^0, \omega_{t_0})$, there is a $t_0$ with $0 < t_0 \leq t_1$ such that, for all positive $t \leq t_0$, $(\nu_1^0, \omega_t)$ embeds symplectically in $\nu_1^0$ rel $D_0$. From the above fact and Theorem 2.1, we can get a symplectic embedding $\hat{f} : (\nu_1^0, \omega_t) \to (M_1, \omega_{M_1})$ with $\hat{f} \circ i_0 = j_1$, for any fixed $t \in (0, t_0]$ with $t_0$ suitably small, and $\hat{f}$ is isotopic rel $D_0$ to $f$.

We would like to find a similar map from a neighborhood of $D_\infty$ in $(S, \omega_t)$ into a neighborhood of $j_2(D_2)$ in $M_2$. By construction, $\nu_2^0 = S - D_0$ canonically identifies the normal bundles $\nu_\infty$ and $\nu_0$ of $D_\infty$ and $D_0$ (reversing fiber-orientation). We also have isomorphisms $f_* : \nu_0 \to \nu_1$ and $\psi : \nu_1 \to \nu_2$ (the latter reversing orientation). Let $\psi'' : \nu_\infty \to \nu_2$ denote the composite of these (which preserves orientation). Then there is a smooth embedding $g : S - D_0 \to M_2$ (independent of $t$) with $g \circ i_\infty = j_2$ and $g_* = \psi''$ on $\nu_\infty$. Clearly, $M = M_1 \# \psi$ $M_2$ could be constructed as a smooth manifold by composing $f^{-1}$ and $g$. However, we cannot perturb $g$ to be symplectic, since we have $i_\infty^* \omega_t = \omega_{D_2} + t \cdot i_\infty^* \eta$. To remedy this, we choose a smooth map $\mu : S \to S$ that radially rescales $\nu_1^0$, fixing a neighborhood of $D_\infty$ and collapsing a neighborhood of $D_0$ onto $D_0$. By composing $g^{-1}$ and $\mu$, we may assume that $g^{-1}$ extends to a smooth map $\lambda : N \to S$ with $\lambda(N - g(S - D_0)) \subset D_0$, where $N$ is a neighborhood of $g(S - D_0)$. Let $\zeta = \lambda^* \eta$. Then $\zeta$ is a closed 2-form that vanishes on $N - g(S - D_0)$, since $i_0^* \eta = 0$. And $\zeta$ can be extended
over $M_2$ as follows:

$$
\zeta = \begin{cases}
\lambda^*\eta & \text{over } g(S - D_0) \\
0 & \text{over } M_2 - g(S - D_0).
\end{cases}
$$

$\zeta$ is determined by $g$ and $\eta$ (so it is independent of $\lambda$ and $t$) and $j_2^*\zeta = i^*\eta$. Let's replace $\omega_{M_2}$ by $\tilde{\omega}_{M_2} = \omega_{M_2} + t \cdot \zeta$. Since non-degeneracy is an open condition, $\tilde{\omega}_{M_2}$ will be symplectic on $M_2$ provided that $0 < t \leq t_0$ for $t_0$ sufficiently small. Furthermore, $g|_{D_\infty} : (D_\infty, \omega_t) \rightarrow (M_2, \tilde{\omega}_{M_2})$ is a symplectic embedding. Hence we can get the same result as Theorem 2.1 for the smooth embedding $g$, and by this result, there is a compactly supported isotopy rel $D_\infty$ from $g$ to $\tilde{g} : (S - D_0, \omega_t) \rightarrow (M_2, \tilde{\omega}_{M_2})$ which is symplectic on a neighborhood $U_\infty$ of $D_\infty$.

Now we perform the symplectic summation. Let $W = \tilde{g}(U_\infty - D_\infty)$ be a neighborhood of one end of the open manifold $(M_2 - \bar{S}_2) - j_2(D^2)$, where $\bar{S}_i$ are the interior surfaces of $S_i$ on $(T'_i)^i$ with the boundaries $S^1_i = j_i(\partial D^2)$ $(i = 1, 2)$. The map $\tilde{g}^{-1} : (W, \tilde{\omega}_{M_2}) \rightarrow (\nu^0_1, \omega_t)$ symplectically identifies the ends of $((M_2 - \bar{S}_1) - j_2(D^2), \tilde{\omega}_{M_2})$ and $(\nu^0_1, \omega_t)$. Let $K = \hat{f}(\nu^0_1 - U_\infty)$ and let $\varphi$ be the inverse of the symplectic embedding $\hat{f} \circ \tilde{g}^{-1} : (W, \tilde{\omega}_{M_2}) \rightarrow (M_1, \omega_{M_1})$. We use $\varphi$ to glue together the two ends of $((M_1 - \bar{S}_1) - K, \omega_{M_1})$ and $((M_2 - \bar{S}_2) - j_2(D^2), \tilde{\omega}_{M_2})$. The resulting symplectic manifold is diffeomorphic to $M$. As in [2], we can get a unique isotopy class of symplectic forms on $M$ as follows:

$$
\omega_M = \begin{cases}
\omega_{M_1} & \text{on } M_1 - \nu^0_1 \\
\{(1 - s)\omega_{M_1} + s \cdot \pi^*\omega_{D^2} : 0 \leq s < 1\} & \text{on } \text{cl}(\nu^0_1) \\
\{\tilde{\omega}_{M_2} + t \cdot \zeta : 0 < t \leq t_0\} & \text{on } M_2 - j_2(D^2).
\end{cases}
$$

**Theorem 2.2.** In the above notation, we have the following results:

(1) The symplectic sum $(M, \omega_M)$ is a smooth symplectic four-manifold with symplectic structures $\omega_M$. 

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(2) $T_2' = (T_1')^1 \sharp (T_1')^2$ is a non-exact Lagrangian surface of genus 2 in $M$ with respect to $\omega_M$.

(3) $[\omega_M] \neq 0$ in $H^2(M, T_2'; \mathbb{R})$.

((2) and (3) will be shown in Lemma 4.1.)

**Lemma 2.3.** $M = M_1 \sharp \psi M_2 \cong \left((M_1 - \mathring{S}_1) - K\right) \cup_\varphi \left((M_2 - \mathring{S}_2) - j_2(D^2)\right) \cong S^3 \times \mathbb{R}$, where $\mathring{S}_i$ are the interior surfaces of $S_i$ on $(T_1')^i$ with the boundaries $S_1^i = j_i(\partial D^2)$ ($i = 1, 2$). Hence $H^1(T_2'; \mathbb{R}) \cong H^2(M, T_2'; \mathbb{R})$ is an isomorphism, where $T_2'$ is a Lagrangian surface of genus 2 in $M$.

*Proof.* We know that $M \cong \left((M_1 - \mathring{S}_1) - K\right) \cup_\varphi \left((M_2 - \mathring{S}_2) - j_2(D^2)\right) \cong S^3 \times (-\infty, 0) \cup_\varphi S^3 \times (0, \infty) \cong S^3 \times (-\infty, 0] \cup_\varphi S^3 \times [0, \infty)$. Since $\varphi = (\tilde{f} \circ \tilde{g}^{-1})^{-1} = \tilde{g} \circ \tilde{f}^{-1}$ glues together the two ends of $((M_1 - \mathring{S}_1) - K, \omega_{M_1})$ and $((M_2 - \mathring{S}_2) - j_2(D^2), \omega_{M_2})$, $M \cong S^3 \times \mathbb{R}$. $\square$

### 3. The construction of an exotic symplectic form

In this section we would like to construct symplectic forms on $S^3 \times \mathbb{R}$ from exotic symplectic forms on two smooth symplectic manifolds $M_i$ ($i = 1, 2$) diffeomorphic to $\mathbb{R}^4$. In section 4 we will prove that the symplectic forms are exotic.

Let $\psi \in \Omega^1(\mathbb{R}^3)$ be such that the pull-back of $\psi$ to the torus vanishes and $d\psi \neq 0$, and let $\chi \in \Omega^1(\mathbb{R}^3)$ be such that $\chi \wedge d\psi$ is a volume on $\mathbb{R}^3$. Let $\rho = \psi + x^4 \cdot \chi \in \Omega^1(\mathbb{R}^4)$. We define $\tau$ to be the smooth one-form on $\mathbb{R}^4$ given by

$$\tau = r^2 \cos r^2 d\theta + s^2 \cos s^2 d\phi,$$

where $\mathbb{R}^4$ may be thought of as $\mathbb{R}^2 \times \mathbb{R}^2$ and $(r, \theta), (s, \phi)$ are polar coordinates on each factor.

For details, we take $\psi = (p^{-1})^* i^* \tau$, $\chi = (p^{-1})^* i^* \xi$, and $\xi = *(d\tau \wedge d\phi^2)$, where $S^3$ is a three sphere defined by $r^2 + s^2 = \phi^2$, $i : S^3 \to \mathbb{R}^4$ the standard embedding, and $p : S^3 - \{x\} \to \mathbb{R}^3$ the stereographic projection, where $x$ is a point in $S^3 - T_1$. Then there is an open ball $B$ in
$\mathbb{R}^3$ containing $p(T_1)$ and an interval $I$ about $x^4 = 0$ so that $\omega'_{M'}(= d\rho)$ is a symplectic form on a smooth symplectic four-manifold $M' (\cong B \times I)$ diffeomorphic to $\mathbb{R}^4$. We see that $\tau$ vanishes only on the torus $T_1$ defined by $x_1^2 + x_2^2 = r^2 = \pi/2$ and $y_1^2 + y_2^2 = s^2 = \pi/2$. $T_1$ is an exact Lagrangian torus in $M'$, since $\rho |_{T_1} = 0$ and $\omega'_{M'} |_{T_1} = d\rho |_{T_1} = 0$. By an isomorphism $H^1(T_1; \mathbb{R}) \cong H^2(B \times I, T_1; \mathbb{R})$, the relative class $[\omega'_{M'}]$ vanishes in $H^2(B \times I, T_1; \mathbb{R})$. We call this structure $\omega'_{M'}$, an exotic symplectic structure on $M'$. By the same procedure as in the section 2 with $h : (D^2, \partial D^2) \rightarrow (\mathbb{R}^4, T_1)$ defined by $h(x_1, y_1) = \left(\frac{x_1}{\sqrt{2}}, \frac{y_1}{\sqrt{2}}, \frac{x_1}{\sqrt{2}}, \frac{-x_1}{\sqrt{2}}\right)$, we can get a unique isotopy class of symplectic forms on $M = M_1 \sharp_{\psi} M_2$, where $\omega'_{M_i} = d\rho_i$ are exotic symplectic forms on $M_i$ as follows:

$$\omega'_{M} = \begin{cases} \omega'_{M_1} = d\rho_1 & \text{on } M_1 - \nu^0_1 \\
(1 - s)\omega'_{M_1} + s \cdot \pi^* \omega_{D^2} & \text{on } cl(\nu^0_1) \\
[\tilde{\omega}'_{M_2} = \omega'_{M_2} + t \cdot \zeta & \text{on } M_2 - j_2(D^2).\end{cases}$$

**Theorem 3.1.** In the above notations, we have the following results:

1. The symplectic sum $(M, \omega'_{M})$ is a smooth symplectic four-manifold with symplectic structures $\omega'_{M}$.
2. $T_2 = T_1 \sharp T_1$ is an exact Lagrangian surface of genus 2 in $M$ with respect to $\omega'_{M}$.
3. $[\omega'_{M}] = 0$ in $H^2(M, T_2; \mathbb{R})$.

(2) and (3) will be shown in Lemma 4.2.)

We also have the following Lemma 3.2 which is similar to Lemma 2.3.

**Lemma 3.2.** $H^1(T_2; \mathbb{R}) \cong H^2(M, T_2; \mathbb{R})$ is an isomorphism, where $T_2$ is a Lagrangian surface of genus 2 in $M$.

### 4. Exotic symplectic structures

Let $(M, \omega_{M})$ be the smooth symplectic four-manifold in Theorem 2.2. (1).
Lemma 4.1. The symplectic forms $\omega_M$ admit a non-exact Lagrangian surface $T'_2$ of genus 2 in $M$ and hence $[\omega_M] \neq 0$ in $H^2(M, T'_2; \mathbb{R})$.

Proof. Let $S^1_i = j_i(\partial D^2) \cap (T'_1)^i$ ($i = 1, 2$). Let's divide the surface $T'_2$ into 3 parts $[T'_2 \cap (M_1 - \nu_1^0)] \cup [T'_2 \cap \text{cl}(\nu_1^0)] \cup [T'_2 \cap (M_2 - j_2(D^2))]$. In the first part, $\omega_M|_{T'_2 \cap (M_1 - \nu_1^0)} = 0$, since $T'_2 \cap (M_1 - \nu_1^0) \subset (T'_1)^1$ and $\omega_M|_{(T'_1)^1} = 0$. In the second part, $\omega_M|_{T'_2 \cap \text{cl}(\nu_1^0)} = 0$, since $T'_2 \cap \text{cl}(\nu_1^0) = S^1_1 \subset (T'_1)^1$. And $\pi^*\omega_{D^2}|_{T'_2 \cap \text{cl}(\nu_1^0)} = \omega_{D^2}|_{S^1_1} = 0$. Thus $(1 - s)\omega_M + s \pi^*\omega_{D^2}|_{T'_2 \cap \text{cl}(\nu_1^0)} = 0$ (0 ≤ $s$ < 1). In the third part, $\omega_M|_{T'_2 \cap (M_2 - j_2(D^2)))} = 0$, since $T'_2 \cap (M_2 - j_2(D^2)) \subset (T'_1)^2$ and $\omega_M|_{(T'_1)^2} = 0$. Also $\zeta$ is zero on $T'_2 \cap (M_2 - j_2(D^2))$, since $\zeta$ is zero on $M_2 - g(S - D_0)$ and $T'_2 \cap (M_2 - j_2(D^2)) \subset M_2 - g(S - D_0)$. Thus $\tilde{\omega}_M|_{T'_2 \cap (M_2 - j_2(D^2))} = 0$ and hence, $T'_2$ is a Lagrangian surface of genus 2 in $M$.

Let's examine the exactness of the Lagrangian surface $T'_2$ in $M$. $\varphi_1^*(\sum_{i=1}^{2} x_i dy_i)|_{(T'_1)^i} = \sum_{i=1}^{2} x_i dy_i|_{T'_1} = j^*(\sum_{i=1}^{2} x_i dy_i)$ can be locally written by $\frac{\pi}{2}(\sin \theta \cos \phi - \cos \theta \sin \phi) d\phi$. Let $S_0$ be a meridian in the torus $T'_1$ with $\theta = 0$. Then we have

$$\int_{S_0} j^*(\sum_{i=1}^{2} x_i dy_i) = -\frac{\pi}{2} \int_{0}^{2\pi} \sin \phi d\phi$$

$$= \frac{\pi}{2} \cdot 4 \cos \phi|_{0}^{\frac{\pi}{2}}$$

$$\neq 0.$$  

Since $\int_{\varphi_1^{-1}(j(S_0))} \varphi_1^*(\sum_{i=1}^{2} x_1 dy_i)|_{(T'_2)^n(M_1 - \nu_1^0)} = \int_{S_0} j^*(\sum_{i=1}^{2} x_i dy_i) = 0$, $\varphi_1^*(\sum_{i=1}^{2} x_i dy_i)|_{(T'_2)^n(M_1 - \nu_1^0)}$ is not exact. Thus $T'_2$ is a non-exact Lagrangian in $M$. By the isomorphism in Lemma 2.3, $[\omega_M] \neq 0$ in $H^2(M, T'_2; \mathbb{R})$.  

Let $(M, \omega'_M)$ be the smooth symplectic four-manifold in Theorem 3.1.(1).

Lemma 4.2. The symplectic forms $\omega'_M$ admit an exact Lagrangian surface $T_2$ of genus 2 in $M$ and hence $[\omega'_M] = 0$ in $H^2(M, T_2; \mathbb{R})$.  

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Proof. By the same method shown in the first part of the proof of Lemma 4.1, we can easily see that $\omega'_M | T_2 = 0$ and hence $T_2$ is also a Lagrangian surface of genus 2 in $(M = M_1 \#_\psi M_2, \omega'_M)$.

Let’s examine the exactness of the Lagrangian surface $T_2$ in $M$. $\rho_1 | T_2 \cap (M_1 - \nu^0_1) = 0$, since $T_2 \cap (M_1 - \nu^0_1) \subset T^1_1$ and $\rho_1 | T^1_1 = 0$. Moreover $\pi^*(x_1 dy_1)|_{T_2 \cap cl(\nu^0_1)} = x_1 dy_1|_{S^1_1}$ is an exact form. Therefore $(1 - s)\rho_1 + s \cdot \pi^*(x_1 dy_1)|_{T_2 \cap cl(\nu^0_1)}$ is exact. We know that $\omega_{M_2} | T_2 \cap (M_2 - j_2(D^2)) = d\rho_2 | T_2 \cap (M_2 - j_2(D^2))$, since $\zeta$ is zero on $T_2 \cap (M_2 - j_2(D^2)) \subset M_2 - g(S - D_0)$ and that $\rho_2 | T_2 \cap (M_2 - j_2(D^2)) = 0$, since $T_2 \cap (M_2 - j_2(D^2)) \subset T^2_1$ and $\rho_2 | T^2_1 = 0$. Thus $T_2$ is an exact Lagrangian in $M$ and we conclude Lemma 4.2 by the use of Lemma 3.2.

By the Lemmas 4.1, 4.2, we can get the following Theorem 4.3.

**Theorem 4.3.** The symplectic forms $\omega_M$ on the symplectic sum $M$ of two smooth symplectic four-manifolds $M_i$ $(i = 1, 2)$ diffeomorphic to $\mathbb{R}^4$ with symplectic forms admitting non-exact Lagrangian tori $(T^i_1)$ $(i = 1, 2)$ admit a non-exact Lagrangian surface $T'_2$ of genus 2 and $[\omega_M] \neq 0$ in $H^2(M, T'_2; \mathbb{R})$.

On the other hand, the symplectic forms $\omega'_M$ on the symplectic sum $M$ of two smooth symplectic four-manifolds $M_i$ $(i = 1, 2)$ diffeomorphic to $\mathbb{R}^4$ with symplectic forms admitting exact Lagrangian tori $T^i_1$ $(i = 1, 2)$ admit an exact Lagrangian surface $T_2$ of genus 2 and $[\omega'_M] = 0$ in $H^2(M, T_2; \mathbb{R})$. Therefore, $(M, \omega'_M)$ does not symplectically diffeomorphic to $(M, \omega_M)$.

In addition, we can show the exotcities of $\omega_M$ and $\omega'_M$ for any closed 2-form (not necessarily exact) $\eta$ on the sphere bundle $S \cong D^2 \times S^2$ over $D^2$ with $i^*_0 \eta = 0$ and $\eta$ restricting to a symplectic form on each fiber, since $T'_2 \cap ((M_2 - \overset{\circ}{S}_i) - j_2(D^2)) \subset M_2 - g(S - D_0)$.

**References**


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