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JACOBI FIELDS AND CONJUGATE POINTS ON HEISENBERG GROUP

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ABSTRACT. Let N be the 3-dimensional Heisenberg group equipped with a left-invariant metric on N. We characterize the Jacobi fields and the conjugate points along a geodesic on N, which points out that Theorem 4 of [1] is not correct.

1. Introduction

Let $\mathcal{N}$ be a 2-step nilpotent Lie algebra with an inner product $\langle , \rangle$ and N be its unique simply connected 2-step nilpotent Lie group with the left invariant metric induced by $\langle , \rangle$ on $\mathcal{N}$. The center of $\mathcal{N}$ is denoted by $\mathcal{Z}$. Then $\mathcal{N}$ can be expressed as the direct sum of $\mathcal{Z}$ and its orthogonal complement $\mathcal{Z}^{\perp}$.

For $Z \in \mathcal{Z}$, a skew symmetric linear transformation $j(Z) : \mathcal{Z}^{\perp} \to \mathcal{Z}^{\perp}$ is defined by $j(Z)X = (\text{ad}X)^* Z$ for $X \in \mathcal{Z}^{\perp}$. Or, equivalently,

$$\langle j(Z)X, Y\rangle = \langle [X, Y], Z\rangle$$

for $X, Y \in \mathcal{Z}^{\perp}$. A 2-step nilpotent Lie group N is said to be *of Heisenberg type* if $j(Z)^2 = -|Z|^2 \,\text{id}$ for all $Z \in \mathcal{Z}$. The classical Heisenberg groups are examples of Heisenberg type. That is, let $n \geq 1$ be any integer and let $\{X_1, \dots, X_n, Y_1, \dots, Y_n\}$ be any basis of $R^{2n} = \mathcal{V}$. Let $\mathcal{Z}$ be an 1-dimensional vector space spanned by $\{Z\}$. Define $[X_i, Y_i] = -[Y_i, X_i] = Z$ for any $i = 1, 2, \dots, n$ with all other brackets are zero. The Lie algebra $\mathcal{N} = \mathcal{V} \oplus \mathcal{Z}$ is called the $(2n+1)$-dimensional Heisenberg algebra, and its unique simply connected Lie group is called the $(2n+1)$-dimensional Heisenberg group.

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In [1], Berndt, Tricerri and Vanhecke got a result about conjugate points along a geodesic on the group of Heisenberg type as follows;

THEOREM 4 OF [1]. *Let N be a group of Heisenberg type with a left invariant metric and $\mathcal{N}$ its Lie algebra. Let $\gamma(t)$ be an unit speed geodesic in N with $\gamma(0) = e$(the identity element of N) and $\gamma'(0) = X_0 + Z_0$ where $X_0 \in \mathcal{Z}^{\perp}$ and $Z_0 \in \mathcal{Z}$.*

(1) *If $Z_0 = 0$, then there are no conjugate points along γ.*
(2) *If $X_0 = 0$, then the conjugate points along γ are at $t \in 2\pi Z^*$.*
(3) *If $X_0 \neq 0 \neq Z_0$, then the conjugate points along γ are at $t \in \frac{2\pi}{|Z_0|}Z^*$ where $Z^* = \{\pm 1, \pm 2, \dots\}$.*

In this paper, we will characterize the Jacobi fields and the conjugate points along a geodesic on 3-dimensional Heisenberg group with a left invariant metric, which point out that case (3) of the above theorem is not correct.

2. Preliminaries

Let $\gamma(t)$ be a curve in N such that $\gamma(0) = e$ (identity element in N) and $\gamma'(0) = X_0 + Z_0$ where $X_0 \in \mathcal{Z}^{\perp}$ and $Z_0 \in \mathcal{Z}$. Since $\exp : \mathcal{N} \to N$ is a diffeomorphism, the curve $\gamma(t)$ can be expressed uniquely by $\gamma(t) = \exp(X(t) + Z(t))$ with

$$X(t) \in \mathcal{Z}^{\perp}, \quad X'(0) = X_0, \quad X(0) = 0$$
$$Z(t) \in \mathcal{Z}, \quad Z'(0) = Z_0, \quad Z(0) = 0.$$

A. Kaplan [3, 4] shows that the curve $\gamma(t)$ is a geodesic in N if and only if

$$X''(t) = j(Z_0)X'(t),$$
$$Z'(t) + \frac{1}{2}[X'(t), X(t)] \equiv Z_0.$$

LEMMA 2.1 [2]. *Let N be a simply connected 2-step nilpotent Lie group with a left invariant metric, and let $\gamma(t)$ be a geodesic of N with $\gamma(0) = e$ and $\gamma'(0) = X_0 + Z_0$ where $X_0 \in \mathcal{Z}^{\perp}$ and $Z_0 \in \mathcal{Z}$. Then*

$$\gamma'(t) = dl_{\gamma(t)}(X'(t) + Z_0), t \in R$$

where $X'(t) = e^{tj(Z_0)}X_0$ and $l_{\gamma(t)}$ is the left translation by $\gamma(t)$.

Throughout this paper, different tangent spaces will be identified with $\mathcal{N}$ via left translation. So, in above lemma, we can consider $\gamma'(t)$ as

$$\gamma'(t) = X'(t) + Z_0 = e^{tj(Z_0)}X_0 + Z_0.$$

Let ∇ be unique Riemannian connection detremined by the left invariant metric on N.

LEMMA 2.2 [2]. *For a 2-step nilpotent Lie group N with a left invariant metric, the followings hold.*

(1) $\nabla_X Y = \frac{1}{2}[X, Y]$ *for* $X, Y \in \mathcal{Z}^{\perp}$
(2) $\nabla_X Z = \nabla_Z X = -\frac{1}{2}j(Z)X$ *for* $X \in \mathcal{Z}^{\perp}$ *and* $Z \in \mathcal{Z}$
(3) $\nabla_Z Z^* = 0$ *for* $Z, Z^* \in \mathcal{Z}$.

The curvature tensor R on $\mathcal{N}$ is defined by

$$R(\xi_1, \xi_2)\xi_3 = -\nabla_{[\xi_1,\xi_2]}\xi_3 + \nabla_{\xi_1}(\nabla_{\xi_2}\xi_3) - \nabla_{\xi_2}(\nabla_{\xi_1}\xi_3)$$

for all $\xi_1, \xi_2, \xi_3 \in \mathcal{N}$.

And recall that the Jacobi operator along a geodesic γ is defined by

$$R_{\gamma'(t)}(\cdot) := R(\cdot, \gamma'(t))\gamma'(t).$$

Using Lemma 2.2, it is easy to show that

LEMMA 2.3 [1]. *If N is of Heisenberg type, then the Jacobi operator $R_{\gamma'(t)}$ is given by*

$$\begin{aligned} &R_{\gamma'(t)}(X + Z) \\ =&\frac{3}{4}j([X, X'(t)])X'(t) + \frac{3}{4}j(Z)j(Z_0)X'(t) + \frac{1}{2}\langle Z, Z_0\rangle X'(t) \\ &+ \frac{1}{4}|Z_0|^2 X - \frac{3}{4}[X, j(Z_0)X'(t)] + \frac{1}{4}|X'(t)|^2 Z + \frac{1}{2}\langle X, X'(t)\rangle Z_0 \end{aligned}$$

where $X \in \mathcal{Z}^{\perp}$ and $Z \in \mathcal{Z}$.

3. Main results

In this section, we will characterize the Jacobi fields and the conjugate points of 3-dimensional Heisenberg group with a left invariant metric.

Throughout this section, we denote $\mathcal{N}$ the 3-dimensional Heisenberg algebra with an inner product $\langle , \rangle$ and N its simply connected 3-dimensional Heisenberg group with the left invariant metric induced by $\langle , \rangle$.

And also, let $\gamma(t)$ be an unit speed geodesic in N with $\gamma(0) = e$ and $\gamma'(0) = X_0 + Z_0$ where $X_0 \in \mathcal{Z}^{\perp}$ and $Z_0 \in \mathcal{Z}$. Assume that $X_0 \neq 0 \neq Z_0$(see Remark 3.4(3) for other cases). Then, we have that

LEMMA 3.1. *Let* $e_0(t) = \gamma'(t) = X'(t) + Z_0 = e^{tj(Z_0)}X_0 + Z_0$, $e_1(t) = X'(t) - \frac{|X_0|^2}{|Z_0|^2} Z_0$ *and* $e_2(t) = j(Z_0)X'(t)$. *Then,* $\{e_0(t), e_1(t), e_2(t)\}$ *is an orthogonal frame along* $\gamma(t)$ *on* N.

Proof. Straightforward. □

Recall that a vector field $J(t)$ along $\gamma(t)$ satisfying the Jacobi equation

$$(\nabla^2_{\gamma'(t)} + R_{\gamma'(t)})J(t) = 0$$

is called a Jacobi field.

Let $J(t) = \alpha_0(t)e_0(t) + \alpha_1(t)e_1(t) + \alpha_2(t)e_2(t)$ be a Jacobi field along $\gamma(t)$ with $J(0) = 0$ in N. First, we calculate $\nabla^2_{\gamma'(t)}J(t)$ using Lemma 2.2.

$$\begin{aligned}
&\nabla_{\gamma'(t)}e_1(t) \\
=&\nabla_{\gamma'(t)}(\gamma'(t) - \frac{1}{|Z_0|^2}Z_0) \\
=&-\frac{1}{|Z_0|^2}\nabla_{\gamma'(t)}Z_0 \\
=&\frac{1}{2|Z_0|^2}j(Z_0)X'(t) \\
=&\frac{1}{2|Z_0|^2}e_2(t).
\end{aligned}$$

Note that $e^{tj(Z_0)} = \cos(|Z_0|t)id + \frac{\sin(|Z_0|t)}{|Z_0|}j(Z_0)$ since N is of Heisenberg type.

Since

$$\nabla_{\gamma'(t)}X_0 = -\frac{|X_0|^2}{2|Z_0|^2}\sin(|Z_0|t)Z_0 - \frac{1}{2}j(Z_0)X_0$$

and

$$\nabla_{\gamma'(t)}j(Z_0)X_0 = \frac{1}{2}|X_0|^2\cos(|Z_0|t)Z_0 + \frac{1}{2}|Z_0|^2X_0,$$

we have that

$$\begin{aligned}
&\nabla_{\gamma'(t)}e_2(t)\\
=&\nabla_{\gamma'(t)}e^{tj(Z_0)}j(Z_0)X_0\\
=&\nabla_{\gamma'(t)}(\cos(|Z_0|t)j(Z_0)X_0 + \frac{\sin(|Z_0|t)}{|Z_0|}j(Z_0)^2X_0)\\
=&\nabla_{\gamma'(t)}(\cos(|Z_0|t)j(Z_0)X_0 - |Z_0|\sin(|Z_0|t)X_0)\\
=&-|Z_0|\sin(|Z_0|t)j(Z_0)X_0 + \cos(|Z_0|t)\nabla_{\gamma'(t)}j(Z_0)X_0\\
&-|Z_0|^2\cos(|Z_0|t)X_0 - |Z_0|\sin(|Z_0|t)\nabla_{\gamma'(t)}X_0\\
=&-\frac{1}{2}|Z_0|^2(\cos(|Z_0|t)id + \frac{\sin(|Z_0|t)}{|Z_0|}j(Z_0))X_0 + \frac{1}{2}|X_0|^2Z_0\\
=&-\frac{1}{2}|Z_0|^2e_1(t).
\end{aligned}$$

So,

$$\begin{aligned}
&\nabla_{\gamma'(t)}J(t)\\
=&\alpha_0'(t)e_0(t) + \alpha_1'(t)e_1(t) + \alpha_1(t)\nabla_{\gamma'(t)}e_1(t)\\
&+\alpha_2'(t)e_2(t) + \alpha_2(t)\nabla_{\gamma'(t)}e_2\\
=&\alpha_0'(t)e_0(t) + (\alpha_1'(t) - \frac{1}{2}|Z_0|^2\alpha_2(t))e_1(t)\\
&+(\alpha_2'(t) + \frac{1}{2|Z_0|^2}\alpha_1(t))e_2(t).
\end{aligned}$$

Similar calculations give that

$$\begin{aligned}
&\nabla^2_{\gamma'(t)}J(t)\\
=&\alpha_0''(t)e_0(t) + (\alpha_1''(t) - |Z_0|^2\alpha_2'(t) - \frac{1}{4}\alpha_1(t))e_1(t)\\
&+ (\alpha_2''(t) + \frac{1}{|Z_0|^2}\alpha_1'(t) - \frac{1}{4}\alpha_2(t))e_2(t).
\end{aligned}$$

By using Lemma 2.3, it is easy to show that

$$\begin{aligned}
R_{\gamma'(t)}e_1(t) &= \frac{1}{4}e_1(t)\\
R_{\gamma'(t)}e_2(t) &= \frac{1}{4}(1 - 4|X_0|^2)e_2(t).
\end{aligned}$$

Hence, we have that

$$\begin{aligned}
0 =&(\nabla^2_{\gamma'(t)} + R_{\gamma'(t)})J(t)\\
=&\alpha_0''(t)e_0(t) + (\alpha_1''(t) - |Z_0|^2\alpha_2'(t))e_1(t)\\
&+ (\alpha_2''(t) + \frac{1}{|Z_0|^2}\alpha_1'(t) - |X_0|^2\alpha_2(t))e_2(t),
\end{aligned}$$

which gives the following differential equations:

$$\begin{aligned}
&\alpha_0''(t) = 0\\
&\alpha_1''(t) - |Z_0|^2\alpha_2'(t) = 0\\
&\alpha_2''(t) + \frac{1}{|Z_0|^2}\alpha_1'(t) - |X_0|^2\alpha_2(t) = 0
\end{aligned}$$

with $\alpha_0(0) = \alpha_1(t) = \alpha_2(0) = 0$.

Solving this differential equations, we obtain the following.

PROPOSITION 3.2. *Let $\gamma(t)$ be an unit speed geodesic in 3-dimensional Heisenberg group N with $\gamma(0) = e$ and $\gamma'(0) = X_0 + Z_0$ where $X_0 \in \mathcal{Z}^\perp$*

and $Z_0 \in \mathcal{Z}$. Assume that $X_0 \neq 0 \neq Z_0$. If $J(t)$ is a Jacobi field along γ in N with $J(0) = 0$, then

$$\begin{aligned} J(t) =& c_0 t e_0(t) \\ &+ (c_1(\sin(|Z_0|t) - (1 - |Z_0|^2)|Z_0|t) + c_2(1 - \cos(|Z_0|t)))e_1(t) \\ &+ (\frac{c_1}{|Z_0|}(\cos(|Z_0|t) - 1) + \frac{c_2}{|Z_0|}\sin(|Z_0|t))e_2(t) \end{aligned}$$

where $c_k, k = 0, 1, 2$ are arbitrary constants and $e_k(t), k = 0, 1, 2$ are given in Lemma 3.1.

As a corollary, we characterize the conjugate points in 3-dimensional Heisenberg group.

COROLLARY 3.3. *Let $\gamma(t)$ is an unit speed geodesic in 3-dimensional Heisenberg group N with $\gamma(0) = e$ and $\gamma'(0) = X_0 + Z_0$ where $X_0 \in \mathcal{Z}^\perp$ and $Z_0 \in \mathcal{Z}$.*

If $X_0 \neq 0 \neq Z_0$, then the conjugate points along γ are at $t \in \frac{2\pi}{|Z_0|}Z^* \cup A$ where $Z^* = \{\pm 1, \pm 2, \dots\}$ and $A = \{t \in R - \{0\} | (1 - |Z_0|^2)\frac{|Z_0|t}{2} = \tan\frac{|Z_0|t}{2}\}$.

Proof. $\gamma(t)$ is a conjugate point at $t \neq 0$ if and only if there exists $(c_0, c_1, c_2) \neq (0, 0, 0)$ such that $J(t) = 0$. Or, equivalently, at t the determinant of G,

$$G = \begin{bmatrix} t & 0 & 0 \\ 0 & \sin(|Z_0|t) - (1 - |Z_0|^2)|Z_0|t & 1 - \cos(|Z_0|t) \\ 0 & \frac{1}{|Z_0|}(\cos(|Z_0|t) - 1) & \frac{1}{|Z_0|}\sin(|Z_0|t) \end{bmatrix}$$

must be zero by Proposition 3.2. Since

$$det(G) = \frac{4}{|Z_0|} t \sin\frac{|Z_0|t}{2}(\sin\frac{|Z_0|t}{2} - (1 - |Z_0|^2)\frac{|Z_0|t}{2}\cos\frac{|Z_0|t}{2}),$$

we see that the conjugate points along γ are at $t \in \frac{2\pi}{|Z_0|}Z^* \cup A$. □

REMARK 3.4.

(1) Corollary 3.3 shows that Theorem 4 of [1] is not correct.
(2) In Corollary 3.3, let $A = \{\pm t_1, \pm t_2, \dots\}$ with $0 < t_1 < t_2 < \dots$. Then it is easy to show that $\frac{2n\pi}{|Z_0|} < t_n < \frac{2(n+1)\pi}{|Z_0|}$ for $n = 1, 2, \dots$. So, the first conjugate point is at $t = \frac{2\pi}{|Z_0|}$.
(3) We assumed that $X_0 \neq 0 \neq Z_0$ in Lemma 3.1, Proposition 3.2 and Corollary 3.3. Other cases are similar. In case of $Z_0 = 0$, choose one $Z(\neq 0) \in \mathcal{Z}$. And letting $e_0(t) = \gamma'(t) = X_0, e_1(t) = j(Z)X_0$ and $e_2(t) = Z$, we see that there are no conjugate points. In case of $X_0 = 0$, letting $e_0(t) = \gamma'(t) = Z_0, e_1(t) = X_1(\neq 0) \in \mathcal{Z}^{\perp}$ and $e_2(t) = j(Z_0)X_1$, we see that the conjugate points are at $t \in 2\pi Z^*$.

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