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EXPONENTIAL FORMULA FOR EXPONENTIALLY BOUNDED C-SEMIGROUPS

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Abstract. In this paper, we establish the exponential formula for C-semigroup. If A is the generator of a C-semigroup $S(t)$, then $S(t)$ can be represented by $\exp(tA)$ in some sense.

1. Introduction

Let X be a Banach space and let A be a linear operator from $D(A) \subset X$ to X. Consider the abstract Cauchy problem $u'(t) = Au(t)$, $t \geq 0$ and $u(0) = x$, where $x \in X$. It is well known that if A is the infinitesimal generator of a C_0-semigroup $\{T(t) : t \geq 0\}$, the abstract Cauchy problem has a unique solution, given by $u(t) = T(t)x$, for every $x \in D(A)$ (see [4] and [5]). In [3], it is also known that if A is the generator of a C-semigroup $\{S(t) : t \geq 0\}$, then the abstract Cauchy problem has a unique solution $u(t)$, given by $u(t) = S(t)C^{-1}x$, for all $x \in C(D(A))$.

If $T(t)$ is a C_0-semigroup and A is the infinitesimal generator of $T(t)$, then $T(t)$ can be represented by $\exp(tA)$ in some sense. It is known that $T(t)$ is represented by the limit of $\exp(tA_r)$, where A_r are the bounded linear operators and the limit of A_r is the infinitesimal generator A of $T(t)$ (see [4] and [5]). Since each A_r is a bounded linear operator, $\exp(tA_r) = \sum_{n=0}^{\infty} t^n/(n!)A_r^n$ is well-defined. In this paper, we establish the similar exponential formula for an exponentially bounded C-semigroup. If A is the generator of an exponentially bounded C-semigroup $S(t)$, then $S(t)$ can be represented by $\exp(tA)C$ in some sense. Like the C_0-semigroup theory, $\exp(tA)C$ is the limit of $\exp(tA(h))C$ and $\exp(tA(h))C = \sum_{n=0}^{\infty} t^n/(n!)A(h)^nC$ is well-defined.

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But $A(h)$ may not be bounded and the limit of $A(h)$ is a densely defined linear operator $G \subset A$.

We denote $D(A)$ by the domain of the operator A and $R(A)$ by the range of the operator A.

Let $C : X \to X$ be an injective bounded linear operator with dense range. The family $\{S(t) : t \geq 0\}$ of bounded linear operators from X to X is said to be an exponentially bounded C-semigroup if

(1) $S(0) = C$,
(2) $S(t)S(s) = CS(t+s)$ for $t,\ s \geq 0$,
(3) for each $x \in X$ $S(t)x$ is continuous in $t \geq 0$,
(4) there exist M and ω such that $\|S(t)\| \leq Me^{\omega t}$ for $t \geq 0$.

If $C = I$, the identity operator on X, then $\{S(t) : t \geq 0\}$ is a C_0-semigroup in the ordinary sense. In this case, (1), (2) and (3) imply (4). But there exist C-semigroups which satisfy (1), (2) and (3) but not (4) (see [1]). By letting $t \to 0$, we obtain $S(t)C = CS(t)$ for all $t \geq 0$.

Let $S(t)$ be an exponentially bounded C-semigroup with $\|S(t)\| \leq Me^{\omega t}$ for $t \geq 0$. For each $r > \omega$, define the bounded linear operator L_r on X by

$$L_r x = \int_0^\infty e^{-rt} S(t)x\,dt, \quad \text{for} \quad x \in X.$$

Then L_r is injective for $r > \omega$ and the closed linear operator A defined by

$$Ax = (r - L_r^{-1}C)x$$

with $D(A) = \{x \in X : Cx \in R(L_r)\}$, is independent of $r > \omega$ (see [1]). The operator A is called the generator of $S(t)$. It is known [3] that

$$Ax = C^{-1}(\lim_{t\to 0}(S(t)x - Cx)/t)$$

with $D(A) = \{x \in X : \lim_{t\to 0}(S(t)x - Cx)/t$ exists and is in $R(C)\}$.

2. The exponential formula

In the following discussion, C will always be an injective bounded linear operator with dense range.

Let $A(h) = (C^{-1}S(h) - I)/h$. Then for each $n \geq 1$ $R(C) \subset D(A(h)^n)$ and $S(t)(X) \subset D(A(h)^n)$ for all $t \geq 0$. Even if $A(h)$ may not be bounded, we can define $\exp(tA(h))C = \sum_{k=0}^{\infty} t^k/k!A(h)^kC$. And $\exp(tA(h))C$ forms a C-semigroup.

THEOREM 2.1. *Let $\{S(t) : t \geq 0\}$ be an exponentially bounded C-semigroup with $\| S(t)\| \leq Me^{\omega t}$ for $t \geq 0$. For each $x \in X$, define*

$$S^h(t)x = \sum_{n=0}^{\infty} \frac{t^n}{n!} A(h)^n Cx = \sum_{n=0}^{\infty} \frac{1}{n!}(\frac{t}{h})^n (C^{-1}S(h) - I)^n Cx.$$

Then $\{S^h(t) : t \geq 0\}$ is an exponentially bounded C-semigroup with $\| S^h(t)\| \leq M\exp(t(e^{\omega h} - 1)/h)$ for $t \geq 0$ and $A(h)$ is a generator of the C-semigroup $\{S^h(t) : t \geq 0\}$.

Before proving Theorem 2.1, we present several lemmas to be used in the proof of Theorem 2.1.

LEMMA 2.2. *For each nonnegative integer k and $x \in X$, we have*

$$(C^{-1}S(h) - I)^k Cx = \sum_{m=0}^{k} (-1)^m \binom{k}{m} S((k-m)h)x.$$

That is, $(C^{-1}S(h) - I)^k C$ is a bounded linear operator and $\| (C^{-1}S(h) - I)^k\| \leq M(e^{\omega h} + 1)^k$.

Proof. For each $x \in X$,

$$\begin{aligned}
(C^{-1}S(h) - I)^n Cx &= (C^{-1}S(h) - I)(C^{-1}S(h) - I)^{n-1}x \\
&= (C^{-1}S(h) - I) \sum_{m=0}^{n-1} (-1)^m \binom{n-1}{m} S((n-1-m)h)x \\
&= \sum_{m=0}^{n-1} (-1)^m \binom{n-1}{m} C^{-1}S(h)S((n-1-m)h)x
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{m=0}^{n-1} (-1)^m \binom{n-1}{m} S((n-1-m)h)x \\
&= \sum_{m=0}^{n-1} (-1)^m \binom{n-1}{m} S((n-m)h)x \\
&+ \sum_{m=0}^{n-1} (-1)^{m+1} \binom{n-1}{m} S((n-(m+1))h)x \\
&= S(nh)x + \sum_{m=1}^{n-1} (-1)^m (\binom{n-1}{m} + \binom{n-1}{m-1}) S((n-m)h)x \\
&+ (-1)^n S(0)x \\
&= \sum_{m=0}^{n} (-1)^m \binom{n}{m} S((n-m)h)x.
\end{aligned}
$$

By induction, the result follows. □

LEMMA 2.3. *For any nonnegative integers k and n, we have*

$$
(C^{-1}S(h) - I)^k C (C^{-1}S(h) - I)^n C = C(C^{-1}S(h) - I)^{k+n} C.
$$

proof. Let $x \in X$. By Lemma 2.2, we have

$$
\begin{aligned}
&(C^{-1}S(h) - I)^k C(Cx) \\
&= \sum_{m=0}^{k} (-1)^m \binom{k}{m} S((k-m)h)Cx \\
&= C \sum_{m=0}^{k} (-1)^m \binom{k}{m} S((k-m)h)x = C(C^{-1}S(h) - I)^k Cx.
\end{aligned}
$$

That is, $(C^{-1}S(h) - I)^k C = C(C^{-1}S(h) - I)^k$ on $R(C)$. By the similar argument in the proof of Lemma 2.2, we obtain

$$
(C^{-1}S(h) - I)^k S(t)x = \sum_{m=0}^{k} (-1)^m \binom{k}{m} S((k-m)h + t)x.
$$

So we have

$$
\begin{aligned}
(C^{-1}S(h)-I)^kCS(t)x &= \sum_{m=0}^{k}(-1)^m\binom{k}{m}S((k-m)h)S(t)x \\
&= \sum_{m=0}^{k}(-1)^m\binom{k}{m}S((k-m)h+t)Cx \\
&= C(C^{-1}S(h)-I)^kS(t)x.
\end{aligned}
$$

This means that $(C^{-1}S(h)-I)^kC = C(C^{-1}S(h)-I)^k$ on $S(t)(X)$ for all $t \geq 0$. Since $(C^{-1}S(h)-I)^nCx$ is a sum of the elements in $R(C)$ or $S(t)(X)$, the result follows. □

LEMMA 2.4. *For each $x \in X$, $(C^{-1}S(h))^kCx = S(kh)x$. That is, $(C^{-1}S(h))^kC$ is a bounded linear operator and $\|(C^{-1}S(h))^kC\| \leq Me^{k\omega h}$.*

Proof. For each $x \in X$,

$$
\begin{aligned}
(C^{-1}S(h))^nCx &= (C^{-1}S(h))(C^{-1}S(h))^{n-1}Cx \\
&= C^{-1}S(h)S((n-1)h)x = S(nh)x.
\end{aligned}
$$

By induction, the result follows. □

Proof of Theorem 2.1. It is clear that $S^h(0) = C$.
By Lemma 2.2 and 2.3, for $t,\ s \geq 0$

$$
\begin{aligned}
&S^h(t)S^h(s) \\
=&(\sum_{n=0}^{\infty}\frac{1}{n!}(\frac{t}{h})^n(C^{-1}S(h)-I)^nC)(\sum_{k=0}^{\infty}\frac{1}{k!}(\frac{t}{h})^k(C^{-1}S(h)-I)^kC) \\
=&\sum_{l=0}^{\infty}\sum_{k+n=l}\frac{1}{n!}\frac{1}{k!}(\frac{t}{h})^n(\frac{s}{h})^k(C^{-1}S(h)-I)^nC(C^{-1}S(h)-I)^kC \\
=&\sum_{l=0}^{\infty}\sum_{n=0}^{l}\frac{1}{(l-n)!}\frac{1}{n!}(\frac{t}{h})^n(\frac{s}{h})^{l-n}C(C^{-1}S(h)-I)^lC \\
=&\sum_{l=0}^{\infty}\frac{1}{l!}(\frac{t+s}{h})^lC(C^{-1}S(h)-I)^lC = CS^h(t+s).
\end{aligned}
$$

Let $S_n^h(t)x = \sum_{k=0}^n 1/k!(t/h)^k(C^{-1}S(h) - I)^k Cx$ for $x \in X$. Then $S_n^h(t)x$ is continuous in $t \geq 0$. Since $\lim_{n\to\infty} S_n^h(t)x = S^h(t)x$, uniformly on any finite t-intervals, $S^h(t)x$ is continuous in $t \geq 0$.

By Lemma 2.2 and 2.4, we have $S^h(t) = e^{-\frac{t}{h}I}(\sum_{k=0}^\infty 1/k!(\frac{t}{h})^k(C^{-1} S(h))^k C)$. Thus

$$\begin{aligned} ||\, S^h(t)x|| &= ||\, e^{-\frac{t}{h}I}(\sum_{k=0}^{\infty}\frac{1}{k!}(\frac{t}{h})^k(C^{-1}S(h))^kC)x|| \\ &\leq ||\, e^{-\frac{t}{h}I}||\,||\sum_{k=0}^{\infty}\frac{1}{k!}(\frac{t}{h})^k(C^{-1}S(h))^kC||\,||\,x|| \\ &\leq e^{-\frac{t}{h}}M\exp(\frac{t}{h}e^{\omega h})||\,x||\,. \end{aligned}$$

Therefore $\{S^h(t) : t \geq 0\}$ is an exponentially bounded C-semigroup $||\, S^h(t)|| \leq M\exp(t(e^{\omega h} - 1)/h)$ for $t \geq 0$.

By the definition of $S^h(t)$, we have

$$S^h(t)x = \sum_{n=0}^{\infty}\frac{1}{n!}(\frac{t}{h})^n(C^{-1}S(h) - I)^nCx.$$

Thus

$$\begin{aligned} &||\,\frac{S^h(t)x - Cx}{t} - \frac{S(h)x - Cx}{h}|| \\ &\qquad = ||\,\frac{1}{t}\sum_{n=2}^{\infty}\frac{1}{n!}(\frac{t}{h})^n(C^{-1}S(h) - I)^nCx|| \\ &\qquad \leq \frac{1}{t}\sum_{n=2}^{\infty}\frac{1}{n!}(\frac{t}{h})^nM(e^{\omega h} + 1)^n||\,x|| \\ &\qquad = \frac{M}{t}(\exp(\frac{t}{h}(e^{\omega h} + 1)) - 1 - \frac{t}{h}(e^{\omega h} + 1))||\,x||\,. \end{aligned}$$

So $\lim_{t\to 0}(S^h(t)x - Cx)/t = (S(h)x - Cx)/h$.

$$C^{-1}(\lim_{t\to 0}\frac{S^h(t)x - Cx}{t}) = C^{-1}(\frac{S(h)x - Cx}{h}) = A(h)x.$$

That is, $A(h)$ is the generator of the C-semigroup $\{S^h(t) : t \geq 0\}$. □

Before presenting our main theorem, we introduce a linear operator G defined by

$$Gx = \lim_{t \to 0}(C^{-1}S(t)x - x)/t$$

with $D(G) = \{x \in R(C) : \lim_{t \to 0}(C^{-1}S(t)x - x)/t \text{ exists }\}$. Then $D(G)$ is dense in X and $G \subset A$ (see [1]). And $\lim_{h \to 0} A(h)x = Gx$ for $x \in D(G)$. The idea of the proof is due to E. Hille [2].

THEOREM 2.5. *Let A be the generator of an exponentially bounded C-semigroup $\{S(t) : t \geq 0\}$ with $\| S(t) \| \leq Me^{\omega t}$ for $t \geq 0$ and let $\{S^h(t) : t \geq 0\}$ be the C-semigroups given in* Theorem 1. *Then*

$$S(t)x = \lim_{h \to 0} S^h(t)x \qquad \text{for all } x \in X$$

and the convergence is uniform on bounded t-intervals.

Proof. Let $x \in D(G)$. Then $x \in D(A) \cap D(A(h))$ and $A(h)S(s)x = S(s)A(h)x$. By Theorem 2.4 in [3], $S^h(t-s)S(s)$ is differentiable in s and

$$\begin{aligned}\frac{d}{ds}S^h(t-s)S(s)x &= S^h(t-s)(-A(h))S(s)x + S^h(t-s)S(s)Ax \\ &= S^h(t-s)S(s)(Ax - A(h)x).\end{aligned}$$

Integrating to both sides from 0 to t, we have

$$S^h(0)S(t)x - S^h(t)S(0)x = \int_0^t S^h(t-s)S(s)(Ax - A(h)x)ds.$$

So for $x \in D(G)$ we have

$$\begin{aligned}&\| CS(t)x - S^h(t)Cx \| \\ &\qquad \leq \int_0^t \| S^h(t-s)S(s)(Ax - A(h)x) \| \, ds\end{aligned}$$

$$\leq \| Ax - A(h)x\| \int_0^t M \exp(\frac{t-s}{h}(e^{\omega h}-1)) M e^{\omega s} ds$$
$$\leq M^2 \| Ax - A(h)x\| \int_0^t e^{(t-s)(e^{\omega}-1)} e^{\omega s} ds$$
$$\leq M^2 \| Ax - A(h)x\| \int_0^t e^{t(e^{\omega}-1)} e^{\omega t} ds$$
$$= tM^2 \| Ax - A(h)x\| e^{t(e^{\omega}+\omega+1)}.$$

Therefore we have

$$CS(t)x = \lim_{h\to 0} S^h(t)Cx = C(\lim_{h\to 0} S^h(t)x) \quad \text{for} \quad x \in D(G).$$

Since C is injective, $S(t)x = \lim_{h\to 0} S^h(t)x$ for $x \in D(G)$.

Since $\| S(t)\|$ and $\| S^h(t)\|$ are uniformly bounded on any finite t-intervals and $D(G)$ is dense, the result follows. □

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