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DETERMINATION OF ALL IMAGINARY BICYCLIC BIQUADRATIC NUMBER FIELDS OF CLASS NUMBER 3

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ABSTRACT. Using the list of all imaginary quadratic fields with class number 1, 2, 3 and 6, we determine all imaginary bicyclic biquadratic number fields of class number 3. There are exactly 163 such fields and their conductors are less than or equal to $163 \cdot 883$.

1. Introduction

Brown and Parry [1] have determined all imaginary bicyclic biquadratic number fields of class number 1. There are 47 such fields and their conductors are less than or equal to $67 \cdot 163$(see also [8] and [12]). In 1977, Buell, H. C. Williams and K. S. Williams [2] have determined all imaginary bicyclic biquadratic number fields of class number 2. There are 160 such fields and their conductors are less than or equal to $163 \cdot 31 \cdot 13$. In this paper we prove the following:

THEOREM. *There are exactly* 163 *imaginary bicyclic biquadratic number fields of class number* 3. *These fields are listed in Table* 2, 3, 4 *and* 5. *Their conductors are less than or equal to* $143929 (= 163 \cdot 883)$.

The determination of these fields is done without computing the class numbers of quartic number fields. In fact, Wagner [11] has determined all imaginary quadratic number fields with class numbers 5, 6 and 7. Using the list of all imaginary quadratic number fields with class numbers 1, 2, 3 and 6, a simple observation of the parities of the class

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numbers of the quadratic subfields permits us to determine all imaginary bicyclic biquadratic number fields of class number 3(see Remark of Proposition). We let K be an imaginary bicyclic biquadratic number field and let k_1 and k_2 be the two imaginary quadratic subfields of K and k_+ be the real quadratic subfield of K. We let h, h_1, h_2 and h_+ be the class numbers of K, k_1, k_2 and k_+, respectively. For a number field L, we denote the ring of integers of L by $\mathcal{O}_L$, and the unit group by $\mathcal{O}_L{}^*$. From [5] and [6], we have

$$h = \frac{Q}{2} h_+ h_1 h_2$$

where $Q = [\mathcal{O}_K{}^* : \mathcal{O}_{k_1}{}^* \mathcal{O}_{k_2}{}^* \mathcal{O}_{k_+}{}^*] = 1$ or 2. Since K is a CM-field, h_+ divides h. So, the determination of the desired fields with $h = 3$ is reduced to the following 5 cases:

I. $h_+ = h_1 = 1$, $h_2 = 6$ and $Q = 1$,
II. $h_+ = 1, h_1 = 2, h_2 = 3$ and $Q = 1$,
III. $h_+ = h_1 = 1, h_2 = 3$ and $Q = 2$,
IV. $h_+ = 3, h_1 = h_2 = 1$ and $Q = 2$,
V. $h_+ = 3, h_1 = 1, h_2 = 2$ and $Q = 1$.

The list of the imaginary quadratic number fields with class numbers 1,2,3 and 6 is given in Table 1([7], [9], [10] and [11]).

$h(\mathbb{Q}(\sqrt{-n}))$	n
1	$1, 2, 3, 7, 11, 19, 43, 67, 163$
2	$5, 6, 10, 13, 15, 22, 35, 37, 51, 58, 91, 115, 123, 187, 235, 267, 403, 427$
3	$23, 31, 59, 83, 107, 139, 211, 283, 307, 331, 379, 499, 547, 643, 883, 907$
6	$26, 29, 38, 53, 61, 87, 106, 109, 118, 157, 202, 214, 247, 262, 277, 298, 339, 358, 397, 411, 451, 515, 707, 771, 835, 843, 1059, 1099, 1147, 1203, 1219, 1267, 1315, 1347, 1363, 1563, 1603, 1843, 1915, 1963, 2227, 2283, 2443, 2515, 2563, 2787, 2923, 3235, 3427, 3523, 3763$

Table 1 : $h(Q(\sqrt{-n}))$ is the class number of the field $Q(\sqrt{-n})$

The imaginary bicyclic biquadratic fields with odd class numbers are listed in the following:

PROPOSITION. *The imaginary bicyclic biquadratic fields with odd class numbers are*

1. $\mathbb{Q}(\sqrt{-p}, \sqrt{-pq})$ *with* $\left(\frac{q}{p}\right) = \left(\frac{p}{q}\right) = -1$,
2. $\mathbb{Q}(\sqrt{-1}, \sqrt{-q}), \mathbb{Q}(\sqrt{-2}, \sqrt{-2q})$ *with* $q \equiv 5 \bmod 8$,
3. $\mathbb{Q}(\sqrt{-p}, \sqrt{-2p})$ *with* $p \equiv 3 \bmod 8$,
4. $\mathbb{Q}(\sqrt{-1}, \sqrt{-2})$,
5. $\mathbb{Q}(\sqrt{-1}, \sqrt{-p})$,
6. $\mathbb{Q}(\sqrt{-2}, \sqrt{-p})$,
7. $\mathbb{Q}(\sqrt{-p_1}, \sqrt{-p_2})$.

Here, p, p_1, p_2 *denote primes congruent to* 3 *mod* 4 *and* q *is a prime congruent to* 1 *mod* 4.

Proof. See Theorem (20.3) in [3] and Satz 43 in [4]. □

REMARK. In 4,5,6 and 7, all three quadratic subfields have odd class numbers, therefore Q = 2 ; in 1,2 and 3 the quadratic subfields $\mathbb{Q}(\sqrt{-pq})$, $\mathbb{Q}(\sqrt{-q})$, $\mathbb{Q}(\sqrt{-2q})$ and $\mathbb{Q}(\sqrt{-2p})$ have class numbers congruent to 2 mod 4 and the other two quadratic subfields have odd class numbers, therefore Q = 1.

The determination of all imaginary bicyclic biquadratic number fields of class number 3 proceeds as follows. We write $K = \mathbb{Q}(\sqrt{-n_1}, \sqrt{-n_2})$, $k_1 = \mathbb{Q}(\sqrt{-n_1}), k_2 = \mathbb{Q}(\sqrt{-n_2})$ and $k_+ = \mathbb{Q}(\sqrt{n_1 n_2})$.

Case I : $h_1 = 1, h_2 = 6, Q = 1$ and $h_+ = 1$.
We determine all pairs of (n_1, n_2) such that the class numbers of $\mathbb{Q}(\sqrt{-n_1})$ and $\mathbb{Q}(\sqrt{-n_2})$ are 1 and 6, respectively, and satisfy the conditions 1, 2 or 3 in Proposition: there are exactly 32 pairs of (n_1, n_2). For these fields $\mathbb{Q}(\sqrt{-n_1}, \sqrt{-n_2})$, we compute the class numbers of $\mathbb{Q}(\sqrt{n_1 n_2})$. If the class number of $\mathbb{Q}(\sqrt{n_1 n_2})$ is 1, then we conclude that the field $\mathbb{Q}(\sqrt{-n_1}, \sqrt{-n_2})$ is of class number 3: there are 28 such quartic fields as listed in Table 2.

Case II : $h_1 = 2, h_2 = 3, Q = 1$ and $h_+ = 1$.

There are 2 pairs of (n_1, n_2) such that the class number of $\mathbb{Q}(\sqrt{-n_1})$ is equal to 2, that of $\mathbb{Q}(\sqrt{-n_2})$ is equal to 3, and satisfy the condition 1 or 3 in Proposition: $(115, 23)$ and $(403, 31)$. We verify that $\mathbb{Q}(\sqrt{115 \cdot 23})$ and $\mathbb{Q}(\sqrt{403 \cdot 31})$ have class number 1. Thus $\mathbb{Q}(\sqrt{-115}, \sqrt{-23})$ and $\mathbb{Q}(\sqrt{-403}, \sqrt{-31})$ have class number 3.

Case III : $h_1 = 1, h_2 = 3, Q = 2$ and $h_+ = 1$.

There are exactly 144 pairs of (n_1, n_2) such that the class number of $\mathbb{Q}(\sqrt{-n_1})$ is 1, that of $\mathbb{Q}(\sqrt{-n_2})$ is 3, and satisfy the conditions 5,6 or 7 in Proposition. For each of these pairs (n_1, n_2), we compute the class number of $\mathbb{Q}(\sqrt{n_1 n_2})$ and select the pairs of (n_1, n_2) such that the class number of $\mathbb{Q}(\sqrt{n_1 n_2})$ is 1: there are exactly 130 pairs of (n_1, n_2) as listed in Table 4.

Case IV : $h_1 = h_2 = 1, Q = 2$ and $h_+ = 3$.

We choose the pairs of (n_1, n_2) such that $\mathbb{Q}(\sqrt{-n_1})$ and $\mathbb{Q}(\sqrt{-n_2})$ have class number 1, and satisfy the conditions 5,6 or 7 in Proposition: there are 35 pairs. For each of these $(n_1, n_2)'s$, we compute the class number of $\mathbb{Q}(\sqrt{n_1 n_2})$ and select the pairs of (n_1, n_2) such that the class number of $\mathbb{Q}(\sqrt{n_1 n_2})$ is 3: there are exactly 3 pairs (n_1, n_2) as listed in Table 5.

Case V : $h_1 = 1, h_2 = 2, Q = 1$ and $h_+ = 3$.

There are 15 pairs of (n_1, n_2) such that the class number of $\mathbb{Q}(\sqrt{-n_1})$ is 1, that of $\mathbb{Q}(\sqrt{-n_2})$ is 2, and satisfy the condition 1,2 or 3 in Proposition. For each of these $(n_1, n_2)'s$, we compute the class number of $\mathbb{Q}(\sqrt{n_1 n_2})$: there is no pair (n_1, n_2) such that the class number of $\mathbb{Q}(\sqrt{n_1 n_2})$ is 3.

This completes the proof of Theorem.

n_1	n_2
1	29, 53, 61, 109, 157, 277, 397
2	26, 106, 202, 298
3	87, 339, 411, 843, 1059, 1347, 1563, 2787
7	707, 1099, 1267, 2443
11	451, 2563
19	38, 247, 1843

Table 2 : Biquadratic fields $\mathbb{Q}(\sqrt{-n_1}, \sqrt{-n_2})$ with $h_+ = 1, h_1 = 1$ and $h_2 = 6$

n_1	n_2
115	23
403	31

Table 3 : Biquadratic fields $\mathbb{Q}(\sqrt{-n_1}, \sqrt{-n_2})$ with $h_+ = 1, h_1 = 2$ and $h_2 = 3$

n_1	n_2
1	23, 31, 59, 83, 107, 139, 211, 283, 307, 331, 379, 547, 643, 883, 907
2	23, 31, 59, 83, 107, 139, 211, 283, 307, 331, 379, 499, 547, 643, 907
3	23, 31, 59, 83, 139, 211, 283, 307, 379, 499, 883, 907
7	23, 31, 59, 83, 107, 139, 211, 283, 307, 331, 379, 499, 547, 643, 883, 907
11	23, 31, 59, 83, 107, 139, 211, 283, 307, 331, 379, 499, 547, 643, 883, 907
19	23, 31, 59, 83, 107, 139, 283, 307, 379, 499, 547, 643, 907
43	23, 31, 59, 107, 139, 283, 307, 331, 379, 499, 547, 643, 883, 907
67	23, 31, 59, 83, 107, 139, 211, 283, 307, 331, 379, 499, 547, 643, 883, 907
163	23, 31, 59, 83, 107, 139, 211, 331, 379, 499, 547, 643, 883

Table 4 : Biquadratic fields $\mathbb{Q}(\sqrt{-n_1}, \sqrt{-n_2})$ with $h_+ = h_1 = 1$ and $h_2 = 3$

n_1	n_2
2	163
7	67
11	43

Table 5 : Biquadratic fields $\mathbb{Q}(\sqrt{-n_1}, \sqrt{-n_2})$ with $h_+ = 3, h_1 = 1$ and $h_2 = 1$

References

[1] E. Brown and C. J. Parry, *The imaginary bicyclic biquadratic fields with class number 1*, J. Reine Angew. Math. **266** (1974), 118–120.

[2] D. A. Buell, H. C. Williams and K. S. Williams, *On the imaginary bicyclic biquadratic fields with class number 2*, Math. Comp. **31** (1977), 1034–1042.

[3] P. E. Conner and J. Hurrelbrink, *Class Number Parity*, vol. 8, World Scientific Series in Pure Math., 1988.

[4] H. Hasse, *Über die Klassenzahl abelscher Zahlkörper*, Springer-Verlag, 1985.

[5] S. Kuroda, *Über den Dirichletschen Körper*, J. Fac. Sci. Imp. Univ. Tokyo Sect. I **4** (1943), 383–406.

[6] F. Lemmermeyer, *Kuroda's class number formula*, Acta Arith. **LXVI. 3** (1994), 245–260.

[7] H. L. Montgomery and P. J. Weinberger, *Notes on small class numbers*, Acta Arith. **24** (1974), 529–542.

[8] B. Setzer, *The determination of all imaginary, quartic, abelian number fields with class number 1*, Math. Comp. **35** (1980), 1383–1386.

[9] H. M. Stark, *A complete determination of the comlex quadratic fields of class number one*, Michigan Math. J. **14** (1967), 1–27.

[10] ———, *On complex quadratic fields with class number two*, Math. Comp. **29** (1975), 289–302.

[11] C. Wagner, *Class number 5,6 and 7*, Math. Comp. **65** (1996), 785–800.

[12] K. Yamamura, *The determination of the imaginary abelian number fields with class number one*, Math. Comp. **62** (1994), 899–921.

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