

# MODULI SPACES OF BUNDLES MOD PICARD GROUPS ON SOME ELLIPTIC SURFACES

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ABSTRACT. We determine the possible types of Chern classes of rank 2 stable bundles (mod Picard groups) on the Enriques surfaces and their covering K3 surfaces.

## 1. Introduction

Moduli spaces of stable vector bundles of rank 2 on complex surfaces have been studied by several authors. The structures of the moduli spaces of stable bundles on surfaces such as rational surfaces(Ba, Hu, DP), ruled surfaces(Br, Q1), K3 surfaces(Mu1,2, Ty1,2), elliptic surfaces(FM, F, OV) and some surfaces of general type(Bh, DK) have been described. In this paper, we study the possible types of the moduli spaces of stable vector bundles of rank two on the surfaces with the big Picard group. Good examples with that property are Enriques surfaces. We classify the possible types of them on Enriques surfaces. The universal covering space of an Enriques surface is a K3 surface. So, we apply our methods to these surfaces. Furthermore, Every Enriques surface is elliptic. In principle, these methods can also be applied to any other elliptic surface with a section.

## 2. Preliminaries

An Enriques surface is a projective nonsingular surface  $X$  with  $2K_X \sim 0$  (but  $K_X \not\sim 0$ ), where  $K_X$  is the canonical divisor of  $X$  and  $h^1(X, O_X) =$

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0. The unramified double covering space of  $X$  defined by the torsion class  $K_X$  is an algebraic K3 surface with a fixed point free involution. Every Enriques surface  $X$  admits an elliptic fibration over  $\mathbf{P}^1$  and every elliptic fibration  $f : X \rightarrow \mathbf{P}^1$  has exactly two multiple fibres  $F_A$  and  $F_B$  such that  $2F_A, 2F_B$  are linearly equivalent to a generic fibre  $F$ .  $K_X$  (briefly  $K$ )  $\sim F_A - F_B$ . Here  $F_A$  and  $F_B$  are called half fibres. The map

$$c_1 : PicX \rightarrow H^2(X, \mathbf{Z}) = \mathbf{Z}^{10} \oplus \mathbf{Z}_2$$

is an isomorphism and

$$NumX = H^2(X, \mathbf{Z})/Tor(H^2(X, \mathbf{Z})) = \mathbf{Z}^{10}.$$

So, we identify  $PicX$  with  $H^2(X, \mathbf{Z})$  in this paper. On a K3 surface, the map

$$c_1 : Pic(X) \rightarrow H^2(X, \mathbf{Z}) = \mathbf{Z}^{22}$$

is injective, so that we identify  $PicX$  with its image. On an Enriques surface or a K3 surface,  $L^2 = L \cdot L$  is an even number for any divisor  $L$ . An Enriques surface  $X$  is called nodal if there exists a smooth rational curve  $R$ . Otherwise, it is called unnodal. In the 10 dimensional moduli space of Enriques surfaces, a generic one is unnodal, while the nodal ones form a 9 dimensional subvariety (CD). The moduli space of K3 surfaces is of dimension 20.

DEFINITION. For any divisor  $D > 0$ , with  $D^2 > 0$ , we define

$$\phi(D) = \inf\{D \cdot f | f \in NumX, f^2 = 0, f > 0\}.$$

THEOREM 1. [CD].  $0 < \phi(D)^2 \leq D^2$  for an Enriques surface  $X$ .

THEOREM 2. [CD]. Let  $D$  be an effective divisor on an Enriques surface  $X$  with  $D^2 \geq 0$ . Then

$$D \sim D' + \sum m_i R_i, m_i \geq 0,$$

where  $R_i$  is a smooth rational curve and one of the following cases occurs:

- i)  $D'$  is an irreducible curve with  $D'^2 > 0$  and is ample;
- ii)  $D'$  is a divisor of canonical type, that is,  $D' \sim \sum n_i D_i$  is an effective divisor with irreducible components  $D_i$  such that  $K \cdot D_i = D' \cdot D_i = 0$  for all  $i$ ;
- iii)  $D' \sim 2E + R$ , where  $|2E|$  is a genus 1 pencil and  $R$  is a smooth rational curve. with  $E \cdot R = 1$ .

iv) *There exists  $w$ , an element of the Weyl group of  $X$  generated by the reflections with the smooth rational curves, such that  $D' = w(D)$ .*

**THEOREM 3.** [BPV]. *Let  $D$  be a divisor with  $D^2 \geq 0$  and  $D \not\cong 0, K$ . Then  $D$  is effective or  $-D$  is effective. If  $D$  is effective, then  $D + K$  is also effective.*

**THEOREM 4.** [CD]. *For every elliptic pencil  $|2E|$  on an Enriques surface  $X$ , there exists an elliptic pencil  $|2F|$  such that  $E \cdot F = 1$ .*

**DEFINITION** A vector bundle  $E$  is called (semi-)stable with respect to an ample divisor  $H$ , if for any subsheaf  $F$ , where  $0 < \text{rank}(F) < \text{rank}(E)$ ,

$$\frac{c_1(F) \cdot H}{\text{rank}(F)} (\leq) < \frac{c_1(E) \cdot H}{\text{rank}(E)}.$$

Let us fix the notations.

$X$  is an Enriques surface and its universal covering space, which is a K3 surface, is denoted by  $\overline{X}$  and the quotient map from  $\overline{X}$  to  $X$  is  $\pi$ . Let  $M_{X,H}(r, D, c_2)$  be the moduli space of stable vector bundles with respect to an ample divisor  $H$ , where  $r$  is the rank of the bundle,  $D$  is the determinant bundle and  $c_2$  is the second Chern class. Let  $M_{\overline{X}, \pi^*H}(r, \pi^*D, 2c_2)$  be the corresponding moduli space of stable bundles with respect to  $\pi^*H$  on  $\overline{X}$ .

### 3. Classification

By the theorem 1 in chapter 2, for any effective divisor  $D$  with  $D^2 > 0$  on an Enriques surface  $X$ , there exists a non-trivial effective divisor  $f$  with  $f^2 = 0$  such that

$$0 < D \cdot f \leq \sqrt{D^2}.$$

That theorem gives the following lemmas which are crucial to the proof of the main theorem.

**LEMMA 1.** *For any divisor  $L$  on an Enriques surface  $X$ , we can find a divisor  $S$  such that  $0 \leq (L - 2S)^2 \leq 10$  and  $L - 2S$  is effective if  $(L - 2S)^2 > 0$ .*

*Proof.* Since  $(L + 2H)^2 > 0$  for some ample divisor  $H$ , we can assume that  $L$  is effective and  $L^2 > 0$  after tensoring by ample line bundles. Since  $0 < L \cdot f \leq \sqrt{L^2}$  for some  $f$  with  $f^2 = 0$  and  $f > 0$ , we have

$$4L \cdot f \leq 4\sqrt{L^2} \leq L^2,$$

if  $L^2 \geq 16$ . From this we can conclude that for  $L^2 \geq 16$ , we can find an  $f$  with  $f^2 = 0$  and  $f > 0$  such that

$$0 \leq (L - 2f)^2 < L^2.$$

Here if  $(L - 2f)^2 > 0$ ,  $L - 2f$  is effective by the theorems 3 in chapter 2 and the fact that  $f$  is numerically effective. If  $L^2=14$  and  $L$  is effective, there exists an  $f(f^2 = 0, f > 0)$  such that  $0 < L \cdot f \leq 3$ , which implies  $0 < (L - 2f)^2 < 14$  and  $L - 2f$  is effective.

If  $L^2 = 12$  and  $L$  is effective, there exists an  $f(f^2 = 0, f > 0)$  with  $0 < L \cdot f \leq 3$ , which implies  $0 \leq (L - 2f)^2 < 12$  and  $L - 2f$  is effective if  $(L - 2f)^2 > 0$ . □

**LEMMA 2.** *Let  $L$  be an effective divisor such that  $L^2 = 10$ . Then we can find a divisor  $f$  with  $f^2 = 0$  such that  $L \cdot f = 1, 2$  or we can find a divisor  $S$  such that  $0 < (L - 2S)^2 \leq 8$  and  $L - 2S$  is effective.*

*Proof.* If  $L^2=10$ , then  $w(L)$  is ample by the theorem 2 in chapter 2, where  $w$  is an element in the Weyl group  $X$ . If we can find an effective divisor  $g(g^2 = 0)$  with  $w(L) \cdot g = 1$  or  $2$ , then  $L \cdot w^{-1}(g) = w(L) \cdot g = 1, 2$  and  $w^{-1}(g)^2 = g^2 = 0$ . Then  $f = w^{-1}(g)$  satisfies the first condition. Otherwise,  $w(L) \cdot f \geq 3$  for every  $f(f^2 = 0, f > 0)$ . So, we can express  $3w(L)$  as a sum of  $g_i(1 \leq i \leq 10)$  with  $g_i \cdot g_j = 1 - \delta_{ij}$ . (CD) Let  $w^{-1}(g_i) = f_i$  for  $i = 1, 2$ . Then,  $(L - 2(f_1 - f_2))^2 = L^2 - 4(L \cdot f_1 - L \cdot f_2) + 4(-2) = 2$  and  $L - 2(f_1 - f_2)$  is effective. □

**LEMMA 3.** *If  $L$  be an effective divisor such that  $0 < L^2 \leq 8$ , then we can find a divisor  $f$  with  $f^2 = 0$  such that  $L \cdot f = 1$  or  $2$ .*

*Proof.* Obvious from the theorem 1 in chapter 2. □

Combining all the previous arguments, we get

**PROPOSITION.** For any divisor  $L$  on an Enriques surface  $X$ , we can find another divisor  $T$  such that

- (a);  $0 \leq (L - 2T) \cdot f = 1$  or  $2$ , where  $f$  is some divisor with  $f^2 = 0$ ,
- or (b);  $(L - 2T)^2 = 0$ .

The following theorem shows how we can transform one moduli space into another by tensoring by line bundles.

**THEOREM.** *We can find a divisor  $D(V, n)$  depending only on  $V \in \text{Pic}X$  and  $n \in \mathbf{Z}$  such that for any rank 2 vector bundle  $E$  with  $c_1(E) = V$  and  $c_2(E) = n$  on an Enriques surface  $X$ , one of the following holds.*

- (A)  $\frac{1}{2}c_1(E(D))^2 = c_2(E(D)) - 1$ ,
- (B)  $\frac{1}{2}c_1(E(D))^2 = c_2(E(D))$ .

*Proof.* By the above proposition, we can find a divisor  $T$  such that  $(c_1(E(-T)) \cdot f = 1$  or  $2$ , where  $f$  is a divisor with  $f^2 = 0$  or  $c_1(E(-T))^2 = 0$ . If  $c_1(E(-T)) \cdot f = 2$ , then  $\frac{1}{2}(c_1(E(-T + nf)))^2 = \frac{1}{2}(c_1(E(-T))^2 + 4n$ , and  $c_2(E(-T + nf)) = c_2(E(-T)) + 2n$ . So, for some  $n$ ,  $\frac{1}{2}(c_1(E(-T + nf)))^2 = c_2(E(-T + nf))$ , or  $c_2(E(-T + nf)) - 1$ . If  $c_1(E(-T)) \cdot f = 1$ , we use  $2f$  instead of  $f$ , so that  $E(-T + 2nf)$  will satisfy one of the conditions for some  $n$ .

If  $c_1(E(-T)) = 0$ , then  $E(-T + f + ng)$  for some  $n$  satisfies the case (A) or the case (B) depending on  $c_2(E(-T))$ , where  $|2f|, |2g|$  are elliptic pencils such that  $f \cdot g = 1$ . In fact,  $\frac{1}{2}c_1(E(-T + f + ng))^2 = 4n$  and  $c_2(E(-T + f + ng)) = c_2(E(-T)) + 2n$ . If  $c_1(E(-T)) = J \neq 0$  and  $J^2 = 0$ , then  $w(J) = mh$ , where  $w$  is an element in the Weyl group,  $|2h|$  is an elliptic pencil and  $m$  is a positive integer. Then, we can find an  $f$  with  $f^2 = 0$  and  $f > 0$  such that  $h \cdot f = 1$  by theorem 4 in chapter 1. If  $m$  is an even integer, then  $c_1(E(-T - \frac{m}{2}w^{-1}(h))) = 0$  and we are done by the same argument as for  $c_1(E(-T)) = 0$ . If  $m$  is odd, then  $c_1(E(-T - \frac{m-1}{2}w^{-1}(h))) \cdot w^{-1}f = 1$ , So that, we can apply the same argument as before.  $\square$

**COROLLARY.** *We can find a divisor  $C(V, n)$  depending only on  $V \in \text{Pic}X$  and  $n \in \mathbf{Z}$  such that for any rank 2 vector bundle  $F$  with  $c_1(F) = \pi^*V$  and  $c_2(F) = 2n$  on the universal covering space  $\overline{X}$  of an Enriques surface  $X$ , one of the following holds.*

- (A)  $\frac{1}{2}c_1(F(C))^2 = c_2(F(C)) - 2$ ,
- (B)  $\frac{1}{2}c_1(F(C))^2 = c_2(F(C))$ .

*Proof.* Let  $C = \pi^*D$ , where  $D$  is the divisor in the previous theorem. If we use the fact that  $\pi^*L_1 \cdot \pi^*L_2 = 2L_1 \cdot L_2$  and  $c_2(\pi^*E) = 2c_2(E)$ , where  $L_i$  is a divisor on  $X$ , then we are done.  $\square$

**REMARK 1.** (1) Enriques surfaces

In the case of (A), we have  $2c_2 \geq \dim M_X(2, c_1, c_2) \geq 2c_2 - 1$  and in the case of (B), we have  $2c_2 - 2 \geq \dim M_X(2, c_1, c_2) \geq 2c_2 - 3$ . So, we have only to consider  $c_2 \geq 0$  and  $c_1^2 \geq -2$  in the case of (A) and  $c_2 \geq 1$  and  $c_1^2 \geq 2$  in the case of (B).

## (2) K3 surfaces

In the case of (A), we have  $\dim M_{\bar{X}}(2, c_1, c_2) = 2c_2 - 2$  and in the case of (B)  $\dim M_{\bar{X}}(2, c_1, c_2) = 2c_2 - 6$ . So, we have only to consider  $c_2 \geq 2$  and  $c_1^2 \geq 0$  in the case of (A) and  $c^2 \geq 4$  and  $c_1^2 \geq 8$  in the case of (B). One component of the type of (B) was described by Tyurin (Ty1.) We will describe the structure of the moduli spaces with other types later. This is also related to the conjecture of Mukai that the moduli space of stable bundles on a K3 surface is deformed to the product of some K3 surfaces.

**REMARK 2.** Qin's result(Q2) shows that the birational type of the moduli space of stable bundles of rank 2 on an Enriques surface or a K3 surface is independent of the choice of an ample divisor. In the case  $c_1 = 0$ , the moduli space can be described in two different ways. The first one is to transform to the cases of (A) or (B) we described here and the second one is to follow the description of Friedman's result. (Fr) (He assumed that every fiber is irreducible, so that this corresponds to an unnodal Enriques surface.)

**REMARK 3.** These methods can also be applied to any other elliptic surfaces with sections.

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