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ONE-SIDED BEST SIMULTANEOUS L_1-APPROXIMATION FOR A COMPACT SET

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Abstract. In this paper, we discuss the characterizations and uniqueness of a one-sided best simultaneous approximation for a compact subset from a convex subset of a finite-dimensional subspace of a normed linear space $C_1(X)$. The motivation is furnished by the characterizations of the one-sided best simultaneous approximations for a finite subset $\{f_1, \dots, f_\ell\}$ for any $\ell \in \mathbb{N}$.

1. Introduction

Let X be a compact Hausdorff space, let $C(X)$ denotes the Banach space of all real valued continuous functions on X with the supremum norm and let $C_1(X)$ denotes the normed linear space $C(X)$ with the $L_1(X,\mu)$-norm, where μ is an admissible measure defined on X, that is, $\mu(O) > 0$ for every nonempty open set $O \subseteq X$. But $C_1(X)$ is not a Banach space. It is however a dense linear subspace of $L_1(X,\mu)$.

Suppose that F is a compact subset of $C_1(X)$ and S is a finite-dimensional subspace of $C_1(X)$, throughout this chapter. Set

$$S(F) := \bigcap_{f \in F} S(f) := \bigcap_{f \in F} \{\tilde{s} \in S | \tilde{s} \le f\}.$$

If there exists a function $s^* \in S(F)$ such that

$$\max_{f \in F} ||f - s^*||_1 = d(F, S(F)) := \inf_{s \in S(F)} \sup_{f \in F} ||f - s||.$$

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Then it is called a one-sided best simultaneous L_1-approximation for F. Moreover, if F consists of ℓ-elements only, then s^* is called a one-sided best ℓ-simultaneous L_1-approximation for F [1]. Particularly, if $\ell = 1$, then s^* is called a one-sided best L_1-approximation for F [2].

We will study exact conditions on a finite-dimensional subspace S of $C_1(X)$ which imply that there exists a one-sided best simultaneous L_1-approximation for each compact subset $F \subset C_1(X)$ from $S(F)$ and we find the characterizations of the one-sided best simultaneous L_1-approximation. Moreover, we have a necessary and sufficient conditions on a subspace S of $C_1(X)$ in order that for each compact set F, there exists a unique one-sided best simultaneous L_1-approximations from $S(F)$. The motivation is the one-sided best approximation of an element, which has been studied by R. Bojanic, R. DeVore (1966), H. Strauss (1982), G. Nürnberger (1985), A. Pinkus and V. Totik (1986).

For each positive integer n, define the set

$$\bar{F}_n := \{(\bar{\lambda}_n, \bar{f}_n) \mid \bar{\lambda}_n = (\lambda_1, \dots, \lambda_n), \bar{f}_n = (f_1, \dots, f_n),$$
$$\lambda_i \geq 0 \, (i = 1, \dots, n), \sum_{i=1}^{n} \lambda_i = 1, f_i \in F\}.$$

REMARK. By the equality,

$$\max_{f \in F} ||f - s||_1 = \max_{\bar{F}_n} ||\sum_{i=1}^{n} \lambda_i f_i - s||_1$$

and $d(F, S(F)) = d(con(F), S(F))$, so s^* is a one-sided best simultaneous L_1-approximation for F if and only if s^* is a one-sided best simultaneous L_1-approximation for $con(F)$ where $con(F)$ denotes a convex hull of F.

2. Existence and characterizations

By the definition of $S(F)$, $S(F)$ is a closed convex subset, but $S(F)$ may be empty.

LEMMA 2.1 [1]. *Suppose that K is a nonempty closed convex subset of a finite-dimensional subspace of a normed linear space W. For any compact subset $F \subset W$, there exists a best simultaneous approximation for F from K.*

Firstly, if $S(F)$ is nonempty, then there exists a one-sided best simultaneous L_1-approximation, by Lemma 2.1. When is $S(F)$ the nonempty set ? If for all $f \in F$, $f \geq 0$, then $0 \in S(F)$. If S contains a strictly positive function (or equivalently a strictly negative function), then $S(F)$ is nonempty. We have therefore proven that $S(F)$ is nonempty for every compact set F if and only if S contains a strictly positive function.

Now, we will deal with the characterization of the one-sided best simultaneous L_1-approximations. With this notion, we have the following lemma.

LEMMA 2.2. *The following statements are equivalent:*

(1) *$s^* \in S(F)$ attains the supremum in $\sup\limits_{s \in S(F)} \int_X s d\mu$.*

(2) *s^* is a one-sided best simultaneous L_1-approximation for F.*

Proof. Suppose that $s^* \in S(F)$ and $\int_X s d\mu \leq \int_X s^* d\mu$ for all $s \in S(F)$. Then for any $s \in S(F)$,

$$\begin{aligned}\max_{f\in F} \|f - s\|_1 &= \max_{f\in F} \int_X |f - s| \mathrm{d}\mu \\ &\geq \max_{f\in F} \int_X f \mathrm{d}\mu - \int_X s^* \mathrm{d}\mu \\ &= \max_{f\in F} \int_X |f - s^*| \mathrm{d}\mu \\ &= \max_{f\in F} \|f - s^*\|_1 .\end{aligned}$$

Thus s^* is a one-sided best simultaneous L_1-approximation for F.

Conversely, it is obvious by the above inequality. □

COROLLARY 2.3 [1]. *Suppose that $F = \{f_1, \dots, f_\ell\}$. Then $s^* \in S(F)$ attains the supremum in $\sup\limits_{s \in S(F)} \int_X s d\mu$ if and only if s^* is a one-sided best ℓ-simultaneous L_1-approximation for F.*

THEOREM 2.4. *Suppose that $\int_X s d\mu \neq 0$ for some $s \in S$ and there exists $s_o \in S$ such that $s_o < f$ on X for all $f \in F$. Then s^* is a one-sided best simultaneous L_1-approximation for F if and only if $s^* \in S(F)$ and for any $s \in S$ with $s \leq 0$ on $\bigcup_{f \in F} Z(f - s^*)$, $\int_X s d\mu \leq 0$ where $Z(f - s^*) = \{x \in X | f(x) = s^*(x)\}$.*

Proof. Let s^* be a one-sided best simultaneous L_1-approximation for F. Suppose that there exists $\tilde{s} \in S$ with $\tilde{s} \leq 0$ on $\bigcup_{f \in F} Z(f - s^*)$ and $\int_X \tilde{s} d\mu > 0$. Let $\hat{s} \in S$ with $\int_X \hat{s} d\mu > 0$. If $\bigcup_{f \in F} Z(f - s^*)$ is empty, then there exists $\varepsilon > 0$ such that

$$s^* + \varepsilon \hat{s} \in S(F)$$

and

$$\int_X (s^* + \varepsilon \hat{s}) d\mu > \int_X s^* d\mu.$$

This contradicts Lemma 2.2. Now we assume that $\bigcup_{f \in F} Z(f - s^*)$ is nonempty. By the assumption, there exists $s_o \in S$ such that $s_o < f$ on X for all $f \in F$. This implies that

$$\tilde{g} := s^* - s_o > 0 \qquad \text{on} \qquad \bigcup_{f \in F} Z(f - s^*).$$

Then there exists $\delta > 0$ such that

$$\tilde{h} := \tilde{s} - \delta \tilde{g} < 0 \qquad \text{on} \qquad \bigcup_{f \in F} Z(f - s^*)$$

and

$$\int_X \tilde{h} d\mu = \int_X \tilde{s} d\mu - \delta \int_X \tilde{g} d\mu > 0.$$

For each $f \in C_1(X)$, define $J(f) = \{x \in X : f(x) < 0\}$. Then $\bigcup_{f \in F} Z(f - s^*) \subset J(\tilde{h})$ and $J(\tilde{h})$ is a proper subset of X. Since $X \setminus J(\tilde{h})$ is compact, there exists a constant function $m > 0$ such that $m \leq (f - s^*)$ on $X \setminus$

$J(\tilde{h})$ for all $f \in F$. And let M be a positive constant function such that $\tilde{h} \leq M$ on X. Let $\varepsilon = m/M$. Then $\varepsilon > 0$, $k := s^* + \varepsilon\tilde{h} \in S(F)$ and

$$\int_X k d\mu > \int_X s^* d\mu,$$

which is a contradiction.

Conversely, let $s \in S(F)$ be fixed. Then for any $x \in \bigcup_{f \in F} Z(f - s^*)$, there exists $f \in F$ such that $f(x) = s^*(x)$ so $s(x) \leq s^*(x) = f(x)$. Thus $s - s^* \leq 0$ on $\bigcup_{f \in F} Z(f - s^*)$. By the assumption,

$$\int_X (s - s^*) d\mu \leq 0.$$

Hence $\int_X s d\mu \leq \int_X s^* d\mu$ for all $s \in S(F)$. By Lemma 2.2, s^* is a one-sided best simultaneous L_1-approximation for F. □

REMARK. If S contains a strictly positive function, then $\int_X s d\mu \neq 0$ for some $s \in S$ and there exists $s_o \in S$ for which $s_o < f$ on X for all $f \in F$. Thus we get the following corollary.

COROLLARY 2.5. *Suppose that S contains a strictly positive function. Then s^* is a one-sided best simultaneous L_1-approximation for F if and only if $s^* \in S(F)$, and for any $s \in S$ with $s \leq 0$ on $\bigcup_{f \in F} Z(f - s^*)$ it follows that $\int_X s d\mu \leq 0$.*

The following corollaries are the consequence of Theorem 2.4.

COROLLARY 2.6 [1]. *Suppose that $\int_X s d\mu \neq 0$ for some $s \in S$ and that for any ℓ-elements $f_1, \ldots, f_\ell$ in $C_1(X)$ there exists $s_o \in S$ such that $s_o < f_i$ on X, $i = 1, \ldots, \ell$. Then s^* is a one-sided best ℓ-simultaneous L_1-approximation for $\{f_1, \ldots, f_\ell\}$ if and only if $s^* \in \bigcap_{i=1}^{\ell} S(f_i)$ and for any $s \in S$ with $s \leq 0$ on $\bigcup_{i=1}^{\ell} Z(f_i - s^*)$ it follows that $\int_X s d\mu \leq 0$.*

COROLLARY 2.7 [2]. *Suppose that $\int_X s d\mu \neq 0$ for some $s \in S$. Let $\tilde{f} \in C(X)$ be with $s_o < \tilde{f}$ on X for some $s_o \in S$. Then s^* is a one-sided best L_1-approximation for $\tilde{f}$ if and only if $s^* \in S(\tilde{f})$ and for any $s \in S$ satisfying $s \leq 0$ on $Z(\tilde{f} - s^*)$ it follows that $\int_X s d\mu \leq 0$.*

The characterization of Theorem 2.4 is not easy to use. With a little work, we can rewrite Theorem 2.4 as the following more useful form. We begin with a lemma.

LEMMA 2.8 [2]. *Suppose that an n-dimensional subspace S satisfies $\int_X s d\mu \neq 0$ for some $s \in S$. Let K be a closed subset of X with the property that if $s \in S$ satisfies $s(x) \leq 0$ on K then $\int_X s d\mu \leq 0$. Then there exist $x_1, \ldots, x_k$ in K and positive real numbers $\lambda_1, \ldots, \lambda_k$ such that*

$$\int_X s d\mu = \sum_{i=1}^{k} \lambda_i s(x_i)$$

for all $s \in S$, where $1 \leq k \leq n$.

THEOREM 2.9. *Suppose that an n-dimensional subspace S satisfies $\int_X s d\mu \neq 0$ for some $s \in S$, and that there exists $s_o \in S$ for which $s_o < f$ on X for all $f \in F$. Then the following statements are equivalent:*

(1) *s^* is a one-sided best simultaneous L_1-approximation for F.*
(2) *$s^* \in S(F)$ and there exist distinct points $x_1, \ldots, x_k$ in X and positive real numbers $\lambda_1, \ldots, \lambda_k$, $1 \leq k \leq n$ such that*
   (a) $\{x_1, \ldots, x_k\} \subset \bigcup_{f \in F} Z(f - s^*)$
   (b) $\int_X s d\mu = \sum_{i=1}^{k} \lambda_i s(x_i)$ *for all* $s \in S$.

Proof. Suppose that (1) holds, i.e., s^* is a one-sided best simultaneous L_1-approximation for F. Then by Theorem 2.4, $s^* \in S(F)$ and for any $s \in S$ with $s \leq 0$ on $\bigcup_{f \in F} Z(f - s^*)$, it follows that $\int_X s d\mu \leq 0$. Let $K = \bigcup_{f \in F} Z(f - s^*)$. Then K satisfies the hypothesis in Lemma 2.8. By Lemma 2.8, there exist $x_1, \ldots, x_k$ in K and positive real numbers

$\lambda_1, \dots, \lambda_k$ such that for all $s \in S$

$$\int_X s d\mu = \sum_{i=1}^{k} \lambda_i s(x_i)$$

for some $1 \leq k \leq n$. Thus (2) holds.

Conversely, suppose that (2) holds. Then for any $s \in S(F)$

$$\begin{aligned} \int_X s d\mu &= \sum_{i=1}^{k} \lambda_i s(x_i) \leq \sum_{i=1}^{k} \lambda_i f_i(x_i) \\ &= \sum_{i=1}^{k} \lambda_i s^*(x_i) = \int_X s^* d\mu \end{aligned}$$

where $f_i \in F$ with $f_i(x_i) = s^*(x_i)$. By Lemma 2.2, s^* is a one-sided best simultaneous L_1-approximation for F. Thus (1) holds. □

COROLLARY 2.10 [1]. *Let S be an n-dimensional subspace of $C(X)$, and let any positive integer $\ell \in \mathbb{N}$ be fixed. Suppose that $\int_X s d\mu \neq 0$ for some $s \in S$ and there exists s_o in S such that $s_o < f_i$ on X for all $f_i \in \{f_1, \dots, f_\ell\}$. Then the following statements are equivalent:*

(1) *s^* is a one-sided best ℓ-simultaneous L_1-approximation for $\{f_i\}_{i=1}^{\ell}$.*

(2) *$s^* \in \bigcap_{i=1}^{\ell} S(f_i)$ and there exist distinct points $x_1, \dots, x_k$ in X and positive real numbers $\lambda_1, \dots, \lambda_k$, $1 \leq k \leq n$ such that*

(a) $\{x_1, \dots, x_k\} \subset \bigcup_{i=1}^{\ell} Z(f_i - s^*)$

(b) $\int_X s d\mu = \sum_{i=1}^{k} \lambda_i s(x_i)$ *for all $s \in S$.*

We shall describe several corollaries about the characterization of one-sided best simultaneous L_1-approximations. The first one is the following.

COROLLARY 2.11. *Suppose that the conditions of* Theorem 2.9 *hold and that s^* is a one-sided best simultaneous L_1-approximation for F with $x_1, \dots, x_k$ as in* Theorem 2.9 (2). *Then, for any one-sided best simultaneous L_1-approximation s for F, $s(x_i) = s^*(x_i)$, $i = 1, \dots, k$.*

DEFINITION 2.12. A finite-dimensional subspace S is said to be linearly independent over $\{x_i\}_{i=1}^n$ if $s \in S$ and $s(x_i) = 0$ for $i = 1, \dots, n$ imply $s = 0$.

COROLLARY 2.13. *Let the conditions of* Theorem 2.9 *hold and let s^* be a one-sided best simultaneous L_1-approximation for F with $x_1, \dots, x_k$ as in* Theorem 2.9 (2). *If S is linearly independent over $\{x_1, \dots, x_k\}$, then the one-sided best simultaneous L_1-approximation for F is unique.*

The formula (b) of Theorem 2.9 is called a quadrature formula for S. It has the additional property that all the coefficients $\lambda_1, \dots, \lambda_k$ are positive real numbers. If

$$\int_X s d\mu = \sum_{i=1}^{k} \lambda_i s(x_i) \tag{2.1}$$

for all $s \in S$ where $\lambda_i > 0$, $i = 1, \dots, k$ and $1 \le k < \infty$, then we shall say that (2.1) is a positive quadrature formula with k active points $\{x_i\}_{i=1}^k$ [2].

3. Uniqueness

We now have a question; When does every compact subset F of $C_1(X)$ have a unique one-sided best simultaneous L_1-approximation from $S(F)$? The answer is guaranteed by the following lemma.

LEMMA 3.1 [2]. *Suppose that S is an n-dimensional subspace of $C_1(X)$ and assume that there exists $s \in S$ with $\int_X s d\mu \neq 0$. Let $\{s_1, \dots, s_n\}$ be any basis for S. If for all $s \in S$*

$$\int_X s d\mu = \sum_{i=1}^{k} \lambda_i s(x_i)$$

where $1 \leq k < \infty$, $\lambda_i > 0$, $i = 1, \dots, k$ *and*

$$rank[s_i(x_j)]_{n \times k} < k,$$

then there exists a positive quadrature formula for S *with* r *active points* $\{y_1, \dots, y_r\} \subset \{x_1, \dots, x_k\}$ *where* $1 \leq r < k$.

A finite-dimensional subspace S of $C_1(X)$ is called an existence space for $C_1(X)$ if for each compact set $F \subset C_1(X)$, F has a one-sided best simultaneous L_1-approximation from $S(F)$, that is, there exists $s^* \in S(F)$ for which $\max_{f \in F} \|f - s^*\|_1 \leq \max_{f \in F} \|f - s\|_1$ for any $s \in S(F)$.

DEFINITION 3.2. Suppose that a finite-dimensional subspace S is an existence space for $C_1(X)$. The subspace S of $C_1(X)$ is called a one-sided simultaneous L_1-unicity space if for each compact subset F of $C_1(X)$, there exists a unique one-sided best simultaneous L_1-approximation for F. And if for each F which consists of ℓ-elements only, there exists a unique one-sided best ℓ-simultaneous L_1-approximation for F, then S is called a one-sided ℓ-simultaneous L_1-unicity space. If $\ell = 1$, then S is called a one-sided L_1-unicity space.

Obviously, if S is a one-sided simultaneous L_1-unicity space, then S is a one-sided ℓ-simultaneous L_1-unicity space, and then S is a one-sided L_1-unicity space. Obviously, if $\int_X s d\mu = 0$ for all $s \in S$, then S is not a one-sided simultaneous L_1-unicity space for $C_1(X)$. Equally obviously, if $\dim S = 1$, $\int_X s d\mu \neq 0$ for some $s \in S$ and S is an existence space, then S is a one-sided simultaneous L_1-unicity space for $C_1(X)$.

THEOREM 3.3. *Suppose that an* n*-dimensional subspace* S $(n \geq 2)$ *satisfies* $\int_X s d\mu \neq 0$ *for some* $s \in S$ *and that for any compact set* F *there exists* $s_o \in S$ *which* $s_o < f$ *on* X *for all* $f \in F$. *Then* S *is a one-sided simultaneous* L_1*-unicity space if and only if each positive quadrature formula for* S *contains at least* n *active points.*

Proof. Suppose that there exist $x_1, \dots, x_k$ in X and positive real numbers $\lambda_1, \dots, \lambda_k$ such that for all $s \in S$

$$\int_X s d\mu = \sum_{i=1}^{k} \lambda_i s(x_i)$$

where $1 \le k \le n-1$. Since $\dim S = n > k$, there exists $s^* \in S \backslash \{0\}$ satisfying $s^*(x_i) = 0,\ i = 1, \ldots, k$. Set $f = |s^*|$. Then $f \in C_1(X),\ \pm s^* \le f$ and

$(a)\ (f \pm s^*)(x_i) = 0, \quad i = 1, \ldots, k$

$$(b) \int_X s d\mu = \sum_{i=1}^{k} \lambda_i s(x_i) \quad \text{for all} \quad s \in S.$$

By Theorem 2.9, $\pm s^*$ are one-sided best L_1-approximations of f. The fact contradicts the assumption.

Conversely, suppose that there exists a compact subset F of $C_1(X)$ such that there exist one-sided best simultaneous L_1-approximations g_1, g_2 for F. Since $dim S = n$, it follows from Lemma 3.1 that we may assume that for every positive quadrature formula for S with n active points $\{x_1, \ldots, x_n\}$, we have $\det[s_i(x_j)]_{n \times n} \neq 0$, where $\{s_1, \ldots, s_n\}$ is any basis for S. Thus S is linearly independent over any n active points. By Theorem 2.9, there exists a positive quadrature formula for S with n active points $\{\dot{x}_1, \ldots, \dot{x}_n\}$. Since S is linearly independent over $\{\dot{x}_i\}_{i=1}^n$ and $g_1(\dot{x}_i) = g_2(\dot{x}_i),\ i = 1, \ldots, n$, and so $g_1 = g_2$. □

By Theorem 3.3 and its proof, we get the following corollaries.

COROLLARY 3.4. *Suppose that $\int_X s d\mu \neq 0$ for some $s \in S, \dim S = n \ge 2$ and for any compact set F there exists $s_o \in S$ such that $s_o < f$ on X for all $f \in F$. Then S is a one-sided L_1-unicity space of $C_1(X)$ if and only if each positive quadrature formula for S contains at least n active points.*

COROLLARY 3.5 [1]. *Let S be an n-dimensional subspace of $C_1(X)$ and $n \ge 2$. Assume that S contains a strictly positive function. Then S is a one-sided L_1-unicity space of $C_1(X)$ if and only if each positive quadrature formula for S contains at least n active points.*

Thus using Theorem 3.3 and Corollary 3.4, we obtain Theorem 3.6.

THEOREM 3.6. *Suppose that $\int_X s d\mu \neq 0$ for some $s \in S, \dim S = n \ge 2$ and for any compact set F there exists $s_o \in S$ for which $s_o < f$ on X for all $f \in F$. Then the following statements are equivalent:*

(1) *S is a one-sided L_1-unicity space.*

(2) *Each positive quadrature formula for S contains at least n active points.*
(3) *S is a one-sided ℓ-simultaneous L_1-unicity space for some $\ell \in \mathbb{N}$.*
(4) *S is a one-sided simultaneous L_1-unicity space.*

PROPOSITION 3.7 [2]. *An existence subspace S of $C_1(X)$ is a one-sided L_1-unicity space if and only if for every $s \in S \setminus \{0\}$, the zero function is not a one-sided best L_1-approximation of $|s|$.*

COROLLARY 3.8. *Suppose that $\int_X s d\mu \neq 0$ for some $s \in S, \dim S = n \geq 2$, and for any compact set F there exists $s_o \in S$ such that $s_o < f$ on X for all $f \in F$. Then S is a one-sided simultaneous L_1-unicity space if and only if for every $s \in S \setminus \{0\}$, the zero function is not a one-sided best L_1-approximation of $|s|$.*

As an immediate consequence of this corollary, we have two results.

COROLLARY 3.9. *Suppose that $\int_X s d\mu \neq 0$ for some $s \in S$ and that for any compact set F there exists $s_o \in S$ for which $s_o < f$ on X for all $f \in F$. Then the subspace S is a one-sided simultaneous L_1-unicity space if and only if for each $s \in S \setminus \{0\}$ there exists $u \in S$ satisfying*
(a) $u \leq |s|$,
(b) $\int_X u d\mu > 0$.

COROLLARY 3.10. *Suppose that S contains a strictly positive function. Then S is a one-sided simultaneous L_1-unicity space if and only if for each $s \in S \setminus \{0\}$, there exists $v \in S$ satisfying*
(a) $v \leq 0 \quad on \quad Z(s)$,
(b) $\int_X v d\mu > 0$.

Note that $int(X)$ represents the interior of X.

THEOREM 3.11 [2]. *Let S be an n-dimensional subspace of $C_1(X)$ and $n \geq 2$. Assume that $int(X)$ is connected and there exists $s^* \in S$ such that $s^* > 0$ on $int(X)$. Then S is not a one-sided L_1-unicity space.*

By Theorem 3.6 and Theorem 3.11, we have the following corollary.

COROLLARY 3.12. *Suppose that $\int_X s d\mu \neq 0$ for some $s \in S, dim S = n \geq 2$, and for any compact set F there exists $s_o \in S$ for which $s_o < f$ on X for all $f \in F$. Assume that $int(X)$ is connected and there* *exists $s^* \in S$ such that $s^* > 0$ on $int(X)$. Then S is not a one-sided simultaneous L_1-unicity space.*

The number of connected components of a compact Hausdorff space X is denoted by $[X]$.

THEOREM 3.13 [2]. *Let S be an n-dimensional subspace of $C_1(X)$ and $n \geq 2$. Assume that S contains a strictly positive function and also that $[X] \leq n-1$. Then S is not a one-sided L_1-unicity space.*

By Theorem 3.6, we have the following result.

COROLLARY 3.14. *Let S be an n-dimensional subspace of $C_1(X)$ and $n \geq 2$. Assume that S contains a strictly positive function and $[X] \leq n-1$. Then S is not a one-sided simultaneous L_1-unicity space.*

Henceforth, we ask about the size of the set of $F \subset C(X)$ for which F has a unique one-sided best simultaneous L_1-approximation. We begin with a lemma.

LEMMA 3.15 [2]. *Let S be a finite-dimensional subspace of $C_1(X)$. Then there exist points $x_1, \dots, x_m$ such that if $s \in S$ satisfies $s(x_i) \leq 0$, $i = 1, \dots, m$ and $\int_X s d\mu \geq 0$ then $s = 0$.*

DEFINITION 3.16. Let W be a normed linear space and let $H(W)$ denotes the family of all nonempty, bounded and closed subsets of W. Define $H : H(W) \times H(W) \longrightarrow \mathbb{R}$ by

$$H(A, B) = \max\{h(A, B), h(B, A)\}$$

where $h(A, B) = \sup_{a \in A} \inf_{b \in B} ||a - b||$. Then H is a metric on $H(W)$ and is called the Hausdorff metric on $H(W)$.

THEOREM 3.17. *If S is a finite-dimensional subspace of $C_1(X)$, then the set*

$$\{F \mid F \text{ has a unique one-sided best simultaneous } L_1\text{-approximation}\}$$

is dense in $\{F \mid S(F)$ is nonempty$\}$ with the Hausdorff metric.

Proof. Assume that F is a compact set with $S(F) \neq \phi$. By Lemma 3.15, there exist points $x_1, \ldots, x_m$ such that if $s \in S$ satisfies $s(x_i) \leq 0$, $i = 1, \ldots, m$ and $\int_X s d\mu \geq 0$ then $s = 0$. Choose any $s^* \in S(F)$, and let $\varepsilon > 0$ be given. For $f \in F$, let $B_{\varepsilon/2}(f) = \{g \in C_1(X) : ||f - g|| < \varepsilon/2\}$. Then $\{B_{\varepsilon/2}(f) : f \in F\}$ is an open cover of F. Since F is compact, there exist $\{f_i\}_{i=1}^{\ell}$ such that $\{B_{\varepsilon/2}(f_i) : i = 1, \ldots, \ell\}$ is an open cover of F. For each $i = 1, \ldots, \ell$, let $g_i \in C_1(X)$ satisfy $s^* \leq g_i, s^*(x_j) = g_i(x_j)$, $j = 1, \ldots, m$, and

$$||f_i - g_i||_1 < \varepsilon/2.$$

Let $G = \{g_1, \ldots, g_\ell\}$. Then by Lemma 1.1, G has a one-sided best ℓ-simultaneous L_1-approximation. Moreover, G has a unique one-sided best ℓ-simultaneous L_1-approximation s^*. Indeed, suppose that s is a one-sided best simultaneous L_1-approximation for G. Since $s^* \in S(G)$, $\int_X s d\mu \geq \int_X s^* d\mu$, i.e., $\int_X (s - s^*) d\mu \geq 0$. And $(s - s^*)(x_j) = s(x_j) - g_i(x_j) \leq 0$, for any $i = 1, \ldots, \ell$, $j = 1, \ldots, m$. By Lemma 3.15, $s = s^*$. Finally, we show that G is closed to F with the Hausdorff metric. For any $g_i \in G, ||f_i - g_i||_1 < \varepsilon/2$ and so $h(G, F) < \varepsilon$. Similarly, for any $f \in F$, there exists $f_i \in F$ such that $f \in B_{\varepsilon/2}(f_i)$. Thus $||f - g_i||_1 \leq ||f - f_i||_1 + ||f_i - g_i||_1 < \varepsilon$ and so $h(F, G) < \varepsilon$. Hence, $H(F, G) < \varepsilon$. So the proof is complete. □

From Theorem 3.17 and its proof, we also obtain the following characterizations.

COROLLARY 3.18 [1]. *If S is a finite-dimensional subspace of $C_1(X)$, then the set $\{\{f_i\}_{i=1}^{\ell} | \{f_i\}_{i=1}^{\ell}$ has a unique one-sided best ℓ-simultaneous L_1-approximation$\}$ is dense in $\{\{f_i\}_{i=1}^{\ell} | \bigcap_{i=1}^{\ell} S(f_i) \neq \phi\}$ with the Hausdorff metric.*

COROLLARY 3.19 [2]. *If S is a finite-dimensional subspace of $C_1(X)$, then the set $\{f \mid f$ has a unique one-sided best L_1- approximation$\}$ is dense in $\{f \mid S(f) \neq \phi\}$ with L_1-norm.*

References

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