Bull. Korean Math. Soc. **35** (1998), No. 1, pp. 141–148

TOTALLY COMPLEX SUBMANIFOLDS IN CaP^2

LIU XIMIN

ABSTRACT. In the present paper, some pinching theorems for the curvatures of the totally complex submanifolds of the Cayley projective plane CaP^2 are obtained.

1. Introduction

Let M be an n-dimensional compact Kaehler submanifold of the complex projective space $CP^m(1)$. Denote by h the second fundamental form of M and UM the unit tangent bundle over M. Ros in [6] showed that if $f(u) = |h(u,u)|^2 < \frac{1}{4}$ for any $u \in UM$, then M is totally geodesic. Moreover in [7], Ros gave a complete list of compact Kaehler submanifolds of $CP^m(1)$ satisfying the condition $\max_{u\in UM} f(u) = \frac{1}{4}$. The same type result for totally complex submanifolds of the quaternion projective space $HP^m(1)$ was obtained by Coulton and Gauchman [3]. In [4], Coulton and Glazebrook proved the analogous result in the case of totally complex submanifolds of the Cayley projective plane CaP^2. In [5], we proved a pinching theorem for the square of the norm of the second fundamental form. In the present paper, we proved some pinching theorems for the curvatures of M.

THEOREM 1. *Let M be a compact totally complex submanifold of complex dimension 2, immersed in Cayley projective plane CaP^2. If the sectional curvature of M satisfies $K_M \geq \frac{1}{8}$, then M is totally geodesic and M is $CP^2(1)$.*

Received August 28, 1997.
1991 Mathematics Subject Classification: 53C42, 53C40.
Key words and phrases: totally complex submanifold, Cayley projective plane, curvature.

THEOREM 2. *Let M be a compact totally complex submanifold of complex dimension n, immersed in Cayley projective plane CaP^2. If the holomorphic sectional curvature H of M satisfies $H \geq \frac{1}{2}$, then either*

(i) $|h|^2 = 0$, *M is totally geodesic in CaP^2, and M is $CP^1(1)$ or $CP^2(1)$, or*

(ii) $|h|^2 = n$ *and M is $CP^1(\frac{1}{2})$.*

THEOREM 2. *Let M be a complete totally complex submanifold of complex dimension n, immersed in Cayley projective plane CaP^2. If the Ricci curvature Q of M satisfies $Q \geq \frac{n}{2}$, then either*

(i) $|h|^2 = 0$, *M is totally geodesic in CaP^2, and M is $CP^1(1)$ or $CP^2(1)$, or*

(ii) $|h|^2 = n$ *and M is $CP^1(\frac{1}{2})$.*

2. Cayley projective plane

In this section, we review the fundamental results about the Cayley projective plane, for details see [4].

Let us denote by Ca the set of Cayley numbers. It possesses a multiplicative identity 1 and a positive definite bilinear form $\langle , \rangle$ with norm $\|a\| = \langle a, a\rangle$, satisfying $\|ab\| = \|a\| \cdot \|b\|$, for $a, b \in Ca$. Every element $a \in Ca$ can be expressed in the form $a = a_0 1 + a_1$ for $a_0 \in R$ and $\langle a_1, 1\rangle = 0$. The conjugation map $a \longrightarrow a^* = a_0 1 - a_1$ is an anti-automorphism $(ab)^* = b^* a^*$.

A canonical basis for Ca is any basis of the form $\{1, e_0, e_1, \ldots, e_6\}$ satisfying: (i) $\langle e_1, 1\rangle = 0$; (ii) $\langle e_i, e_j\rangle = \{0$ for $i \neq j$, and 1 otherwise$\}$; (iii) $e_i^2 = -1; e_i e_j + e_j e_i = 0 (i \neq j)$; (iv) $e_i e_{i+1} = e_{i+3}$ for $i \in Z_7$.

Let V be a vector space of real dimension 16 with automorphism group $Spin(9)$. The splitting

$$V = Ca \bigoplus Ca$$

together with the above canonical basis on each summand, endows V with what we refer to as a Cayley structure. We know that the Cayley projective plane CaP^2 is a 16-dimensional Riemannian symmetric space whose tangent space admits the Cayley structure pointwise. In the following, Let $\{I_0, \ldots, I_6\}$ be the Cayley structure on CaP^2.

The curvature tensor $\bar{R}$ of CaP^2 is given in [2] as follows

$$\begin{aligned} \bar{R}((a,b),(c,d))(e,f) &= \frac{1}{4}(((\langle c,e\rangle a - 4\langle a,e\rangle c + (ed)b^* - (eb)d^* \\ &+ (ad-cb)f^*), (4\langle d,f\rangle b - 4\langle b,f\rangle d \\ &+ a^*(cf) - c^*(af) + e^*(ad-cb))) \end{aligned} \tag{1}$$

On $Ca \bigoplus Ca$ we have the positive definite bilinear form $\langle , \rangle$ given by

$$\langle (a,b),(c,d)\rangle = \langle a,c\rangle + \langle b,d\rangle \tag{2}$$

3. Totally complex submanifolds

Let $V \subset T_xCaP^2$ be a real vector subspace, we say that V is a totally complex subspace if there exists an I such that there easists a basis with $I = I_0$ and (i) $I_0 \subset V$, and (ii) I_kV is perpendicular to V for $1 \leq k \leq 6$. Clearly, if V is a maximal subspace of this kind then $dim_R V = 4$.

Let M be a compact Riemannian manifold isometrically immersed in CaP^2 by $j : M \longrightarrow CaP^2$. Denote by h and A the second fundamental form of j and the Weingarten endomorphism respectively. Then we have

$$\langle h(X,Y), N\rangle = \langle X, A_N Y\rangle \tag{3}$$

where $X, Y \in TM$, $N \in TM^\perp$. We take $\bar{\nabla}$, ∇ and $\nabla^\perp$ to be the Riemannian connection on CaP^2, M and the normal connection on M respectively. The corresponding curvature tensors are denoted by $\bar{R}$, R, and $R^\perp$ respectively. The first and second covariant derivatives of h are given by

$$(\bar{\nabla} h)(X,Y,Z) = \nabla^\perp_Z(h(X,Y) - h(\nabla_Z X, Y) - h(X, \nabla_Z Y) \tag{4}$$

$$\begin{aligned} (\bar{\nabla}^2 h)(X,Y,Z,W) = \nabla^\perp_W(\bar{\nabla} h)(X,Y,Z) - (\bar{\nabla} h)(\nabla_W X, Y, Z) \\ -(\bar{\nabla} h)(X, \nabla_W Y, Z) - (\bar{\nabla} h)(X, Y, \nabla_W Z) \end{aligned} \tag{5}$$

where $X, Y, Z, W \in TM$. The Codazzi equation takes the following form

$$(\bar{\nabla} h)(X_{\tau(1)}, X_{\tau(2)}, X_{\tau(3)}) = (\bar{\nabla} h)(X_1, X_2, X_3) \tag{6}$$

where $\tau \in S_3$, the permutation group, and the arguments are in the tangent space of M. Recalling that h and $\bar{\nabla} h$ are symmetric, we have the Ricci identity

$$(7)\qquad \begin{aligned}&(\bar{\nabla}^2 h)(X,Y,Z,W)-(\bar{\nabla}^2 h)(X,Y,W,Z)\\ &= -R^{\perp}(Z,W)h(X,Y)+h(R(Z,W)X,Y)+h(X,R(Z,W)Y)\end{aligned}$$

We say that $j: M \longrightarrow CaP^2$ is a totally complex immersion if $W = j_*(TM)$ is a totally complex subspace for each point of M. Observe that every totally complex submanifold of CaP^2 has a Kaehler structure. We set $I = I_0$, and consequently we have

$$(8)\qquad \begin{aligned}&(a)\quad \bar{\nabla}_X I = 0\\ &(b)\quad h(IX,Y) = Ih(X,Y)\\ &(c)\quad A_{IN} = IA_N = -A_N I\\ &(d)\quad IR(X,IX)X = R(X,IX)IX\end{aligned}$$

where $X, Y \in T_xM$ and $N \in T_xM^{\perp}$.

Define $f(u) = |h(u,u)|^2$, where $u \in UM$, the unit tangent bundle over M. Assume f attains its maximum at some vector $v \in UM_p$, then ([6]):

$$(9)\qquad A_{h(v,v)}v = |h(v,v)|^2 v$$

LEMMA 3.1. [5]. *Let M^n be a compact totally complex submanifold in CaP^2. Assume f attains its maximum at $v \in UM_p$, then*

$$(10)\qquad \begin{aligned}3|h(v,v)|^2(1-4|h(v,v)|^2) &+ \sum_{i=1}^{6}\langle h(v,v), I_i v\rangle^2\\ &+ 4|\bar{\nabla}h(v,v,v)|^2 \le 0\end{aligned}$$

LEMMA 3.2. [5]. *Let M be a compact totally complex submanifold in CaP^2. Assume f attains its maximum at $v \in UM_p$, then for any $u \in UM_p$ with $\langle u,v\rangle = \langle u, Iv\rangle = 0$, we have*

$$(11)\qquad \begin{aligned}|h(v,v)|^2(1-8|h(u,v)|^2) - |A_{h(v,v)}u|^2 &+ \sum_{i=1}^{6}\langle h(v,v), I_i u\rangle^2\\ &+ 4|(\bar{\nabla}h)(u,v,v)|^2 \le 0\end{aligned}$$

When the complex dimension of M is 2, we can always choose a vector $u \in UM_p$, such that

$$(12)\qquad A_{h(v,v)}u = \langle h(v,v), h(u,u)\rangle u.$$

Define

$$S = \bigcup_{p\in M} S_p,\ S_p = \{(r,t)|r,t \in UM_p, \langle r,t\rangle = \langle r, It\rangle = 0\}$$

and function σ *on* S *by*

$$\sigma(r,t) = |h(r,t)|^2. \tag{13}$$

LEMMA 3.3. *Let* M *be a compact totally complex submanifold of complex dimension* $n \geq 2$ *in* CaP^2. *If* σ *attains its maximum at some* $(u,v) \in UM_p \times UM_p$, *for some* $p \in M$ *and* f *also attains its maximum at* $v \in UM_p$ *then we have*

$$\begin{aligned} 2|h(u,v)|^2(1-4|h(u,v)|^2-2|h(u,u)|^2) &+ \sum_{i=1}^{6}\langle h(u,v), I_i u\rangle^2 \\ &+ 4|(\bar{\nabla} h)(u,u,v)|^2 \leq 0 \end{aligned} \tag{14}$$

Proof. Let $\gamma(t) = \cos tv + \sin vt$, then $(\gamma(t),\gamma'(t)) \in S_p$. Since the function $\sigma(\gamma,\gamma')$ attains its maximum at $t=0$,

$$0 = \frac{d\sigma(\gamma,\gamma')}{dt}(0) = 2\langle h(v,v), h(u,v)\rangle - 2\langle h(u,u), h(u,v)\rangle \tag{15}$$

$$\begin{aligned} 0 &\geq \frac{d^2\sigma(\gamma,\gamma')}{dt^2}(0) \\ &= 2|h(v,v)|^2 + 2|h(u,u)|^2 - 8|h(u,v)|^2 - 4\langle h(u,u), h(v,v)\rangle. \end{aligned} \tag{16}$$

Suppose now that the function f also attains its maximum at $v \in UM_p$. Fix u, for any $\omega \in UM_p$ with $(u,\omega),(\omega,v) \in S_p$, then $c(t) = (u, \cos tv + \sin t\omega) \in S_p$, and $\sigma \cdot c(t)$ attains its maximum at $t=0$, thus

$$0 = \frac{d(\sigma\cdot c)}{dt}(0) = 2\langle h(u,v), h(u,\omega)\rangle. \tag{17}$$

Now, let $e_1 = v, e_2 = Iv, e_3 = u, e_4 = Iu$ be an orthonormal basis of UM_p, and assume

$$A_{h(u,v)}u = \sum_{i=1}^{2n} a_i e_i,\ A_{h(u,v)}v = \sum_{i=1}^{2n} b_i e_i,$$

then it is easy to see from (9), (15) and (17) that $a_i = 0$ if $i \neq 1$, and $b_i = 0$,when $j \neq 3$, that is

$$A_{h(u,v)}u = |h(u,v)|^2 v,\ A_{h(u,v)}v = |h(u,v)|^2 u. \tag{18}$$

Let C_u be the geodesic in M satisfying the initial conditions $C_u(0) = p, C'_u = u$. Parallel translating u and v along $C_u(t)$ yields vector fields $U_u(t)$ and $V_u(t)$. Let $\sigma_u = \sigma \cdot (U_u, V_u)$, we have

$$\frac{d^2\sigma_u}{dt^2}(0) = 2\langle(\bar{\nabla}^2 h)(u,u,u,v), h(u,v)\rangle + 2|(\bar{\nabla}h)(u,u,v)|^2, \tag{19}$$

$$\frac{d^2\sigma_{Iu}}{dt^2}(0) = 2\langle(\bar{\nabla}^2 h)(Iu,Iu,u,v), h(u,v)\rangle + 2|(\bar{\nabla}h)(u,u,v)|^2. \tag{20}$$

By a simple calculation and using (17) and (18), we get

$$\begin{aligned} &2\langle(\bar{\nabla}^2 h)(Iu,Iu,u,v), h(u,v)\rangle \\ &= 2\langle(\bar{\nabla}^2 h)(Iu,u,Iu,v), h(u,v)\rangle \\ &= 2\langle(\bar{\nabla}^2 h)(u,Iu,Iu,v), h(u,v)\rangle + \langle(R^{\perp}(Iu,u)h(Iu,v), h(u,v)\rangle \\ &\quad -\langle R(Iu,u)Iu, A_{h(u,v)}v\rangle - \langle R(Iu,u)v, A_{h(u,v)}Iu\rangle \\ &= -2\langle(\bar{\nabla}^2 h)(u,u,u,v), h(u,v)\rangle \\ &\quad +\frac{1}{2}\{\sum_{i=1}^{6}\langle h(u,v), I_i u\rangle^2 - |h(u,v)|^2 - 4|A_{h(u,v)}u|^2\} \\ &\quad -\{-|h(u,v)|^2 + 2|h(u,v)|^2|h(u,u)|^2\} \\ &\quad -\{-\frac{1}{2}|h(u,v)|^2 + 2|h(u,v)|^4\}. \end{aligned} \tag{21}$$

Since σ attains its maximum at (u,v), we have

$$0 \geq \frac{d^2\sigma_u}{dt^2} + \frac{d^2\sigma_{Iu}}{dt^2}. \tag{22}$$

Substituting (18)-(21) into (22), we get (14). □

4. Proof of the main theorems

PROOF OF THEOREM 1. Assume f attains its maximum at $v \in UM_p$ for some $p \in M$. If $f(v) = 0$, then M is totally geodesic. If $f(v) \neq 0$, we can assume from Lemma 3.1 that

$$|h(v,v)|^2 \geq \frac{1}{4}. \tag{23}$$

For any $q \in M$, and any $(r,t) \in S_q$, it follows from $\langle R(r,t)t, r\rangle \geq \frac{1}{8}$ that

$$8\langle h(r,r), h(t,t)\rangle \geq -(1 - 8|h(r,t)|^2). \tag{24}$$

Similarly, from $\langle R(Ir,t)t, Ir\rangle \geq \frac{1}{8}$, we have

$$8\langle h(r,r), h(t,t)\rangle \leq 1 - 8|h(r,t)|^2. \tag{25}$$

Thus,

$$1 - 8|h(r,t)|^2 \geq 8|\langle h(r,r), h(t,t)\rangle|, \tag{26}$$

and so

$$|h(r,t)|^2 \leq \frac{1}{8} \tag{27}$$

for any $(r,t) \in S_q$, and any $q \in M$. Hence, the function σ defined by $\sigma(r,t) = |h(r,t)|^2$ satisfies

$$\sigma \leq \frac{1}{8}. \tag{28}$$

Now, we take a $u \in UM_p$ with $(u,v) \in S_p$ and $A_{h(v,v)}u = \langle h(u,u), h(v,v)\rangle u$, by (26), we have

$$|\langle h(u,u), h(v,v)\rangle| \leq \frac{1}{8}. \tag{29}$$

From (23), (26) and (29), we obtain

$$|h(v,v)|^2(1 - 8|h(u,v)|^2) \geq 16\langle h(u,u), h(v,v)\rangle^2. \tag{30}$$

Substituting (30) and $A_{h(v,v)}u = \langle h(u,u), h(v,v)\rangle u$ into (11), one can easily deduce

$$\langle h(u,u), h(v,v)\rangle = 0,\ |h(u,v)|^2 = \frac{1}{8}. \tag{31}$$

From (28) and (31), we find that σ attains its maximum at $(u,v) \in S_p$. Substituting (31) into (14), we get

$$|h(u,u)|^2 \geq \frac{1}{4}. \tag{32}$$

It follows from (16), (23), (31) and (32) that $|h(u,u)|^2 = |h(v,v)|^2 = \frac{1}{4}$, and so $f \leq \frac{1}{4}$ on UM. Theorem 1 now follows from Theorem 2.2 in [4] and $K_M \geq \frac{1}{8}$, this completes the proof of Theorem 1. □

PROOF OF THEOREM 2. By (1) and Gauss equation we have

$$H(u) = \langle R(u, Iu)Iu, u\rangle = 1 - 2|h|^2$$

for any $u \in UM$. Hence the condition $H(u) \geq \frac{1}{2}$ is equvalent to the condition $|h|^2 \leq \frac{1}{4}$. Then the theorem 2 now follows from Theorem 2.2 in [4]. □

PROOF OF THEOREM 3. By assumption and Myers theorem we know that M is compact. From condition $Q \geq \frac{n}{2}$, we can get $|h|^2 \leq n$. Then the theorem 3 follows from the theorem of [5]. This completes the proof of Theorem 3. □

References

[1] A. L. Besse, Manifolds all of whose geodesics are closed, Springer-Verlag, Berlin, 1978.

[2] R. Brown and A. Gray, Riemannian manifolds with holonomy group Spin(9), in Diff. Geom. in honor of K. Yano, Tokyo, 1972, pp.42-59.

[3] P. Coulton and H. Gauchman, Submanifolds of quaternion projective space with bounded second fundamental form, Kodai Math. J., **12** (1989), 296-307.

[4] P. Coulton and J. Glazebrook, Submanifolds of Cayley projective plane with bounded second fundamental form, Geom. Dedi., **33** (1990), 265-272.

[5] Liu Ximin, Totally complex submanifolds of the Cayley projective plane, Glasgow Math. J., **40** (1998), 161-166.

[6] A. Ros, Positively curved Kaehler submanifolds, Proc. Amer. Math. Soc., **93** (1985), 329-331.

[7] A. Ros, A characterization of seven compact Kaehler submanifolds by holomorphic pinching, Annals of Math., **121** (1985), 377-382.

[8] K. Tsukada, Parallel submanifolds in a quaternion projective space, Osaka J. Math., **22** (1985), 187-241.

DEPARTMENT OF APPLIED MATHEMATICS, DALIAN UNIVERSITY OF TECHNOLOGY, DALIAN 116024, P. R. CHINA