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EXCEPTIONAL BUNDLES OF HIGHER RANK AND RATIONAL CURVES

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ABSTRACT. We relate the existence of rational curves with the existence of rigid bundles of any even rank on Enriques surfaces and compare with the case of K3 surfaces.

1. Introduction

Exceptional bundles (for the definition, see chapter 3) have been widely studied by several people. The condition of the existence of exceptional bundles on rational surfaces ([DP], [Ru]), K3 surfaces ([Ku]), Enriques surfaces ([DR]) and surfaces of general type ([Ty]) have been determined. Exceptional bundles are very important, in particular on K3 surfaces and Enriques surfaces. Kuleshov proved that there exists an exceptional bundle for every possible Chern classes on K3 surfaces. However, there is some difference between K3 and Enriques surfaces. We showed that there exists an exceptional bundle of rank 2 if and only if the surface has a smooth rational curve. We also related the exceptional bundles with a nodal cycle (a tree of rational curves) ([Ki1]). In this paper we extend this result to higher rank.

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2. Preliminaries

1. An Enriques surface X is a minimal algebraic surface whose canonical divisor $K_X \not\sim 0$, but $2K_X \sim 0$, where $\sim$ denotes the linear equivalence.

2. Every Enriques surface has an elliptic structure over $\mathbf{P}^1$. It has exactly two multiple fibres of multiplicity 2, say them F_A, F_B. Then the canonical divisor can be expressed as a difference of two multiple fibres, that is, $K_X \sim F_A - F_B$.

3. The fundamental group of any Enriques surafce is $\mathbf{Z_2}$ so that the universal covering space is a K3 surface. Let the quotient map be π. That is an etale covering with respect to K_X. So, $\pi_*(O_{\overline{X}}) \cong O_X \oplus K_X$, $\pi^*(K_X) \cong O_{\overline{X}}$.

4. An Enriques surface X is called nodal if there exists a smooth rational curve R. (In this case $R^2 = -2$.) Otherwise, it is called unnodal. In the 10 dimensional moduli space of Enriques surfaces, a generic one is unnodal, while the nodal ones form a 9 dimensional subvariety ([CD]).

5. A nodal cycle N on an Enriques, or a K3 surface is a positive 1-cycle such that $h^1(O_N) = 0$. This is a tree of smooth rational curves and $N^2 = -2$. ([Ar]).

6. We define the slope of E with respect to some ample divisor H, denoted by $\mu_H(E)$, as $\frac{c_1(E)\cdot H}{rank(E)}$. A vector bundle E is called H(-semi)-stable if for every subsheaf F, with $0 < rank(F) < rank(E)$,

$$\mu_H(F) < (\leq)\mu_H(E).$$

We define a bundle E to be simple if $h^0(EndE) = 1$. Then, there exists a moduli space of stable or simple vector bundles which is a quasi-projective algebraic variety.

7. Let X be a K3 surface or an Enriques surface. Then, the map $c_1 : PicX \to H^2(X, \mathbf{Z})$ is injective. So, we identify $PicX$ with its image in $H^2(X, \mathbf{Z})$

Now we fix the notations.

X is an Enriques surface and its universal covering space, which is a K3 surface, is denoted by $\overline{X}$ and the quotient map from $\overline{X}$ to X is π. Let $M_{X,H}(r, c_1, c_2)$(resp.$M_{\overline{X},\pi^*H}(r, c_1, c_2)$) be the moduli space of stable or simple vector bundles on X(resp. $\overline{X}$) with respect to H(resp. π^*H), where r is the rank of the bundles and c_i is the assignment of the i-th Chern class.

We denote by i the involution on $\overline{X}$ compatible to π and by i^* the induced involution on $M_{\overline{X}}$.

3. Exceptional bundles

First we introduce our interpretations of the results of Takemoto [Ta] in our Enriques surface X and the covering $K3$ surafce $\overline{X}$.

THEOREM [Ta]. (1). *If a π^*H-stable bundle F on $\overline{X}$ is not isomorphic to π^*E for any bundle E on X, then $\pi_*(F)$ is H-stable.*

(2) *If a simple bundle E on X is isomorphic to $E(K)$, then there exists a simple bundle F on X such that $\pi_*(F) = E$.*

Next we give a related one to this theorem.

THEOREM [Ki2]. *Let X be an Enriques surface and $\overline{X}$ be the universal covering space of X and F be a simple vector bundle on $\overline{X}$ such that $F \cong i^*F$. Then there exists a bundle E on X such that $\pi^*E = F$.*

Now we can associate a vector

$$v(E) = (r, D, s) \in \mathbf{Z} \times PicX \times \frac{1}{2}\mathbf{Z}$$

to any vector bundle E on an Enriques surface X, where r is the rank of E, $D = det(E)$ and $s = \frac{1}{2}c_1(E)^2 - c_2(E) + \frac{r}{2}$. This is a Mukai vector on an Enriques surface. Then we can define a symmetric bilinear form on that lattice,

$$v_1 \cdot v_2 = (r_1, D_1, s_1) \cdot (r_2, D_2, s_2) = D_1 \cdot D_2 - r_1 s_2 - r_2 s_1.$$

Then

$$\chi(E^* \otimes F) = -v(E) \cdot v(F).$$

So, if E is stable or simple,

$$\dim M = v(E) \cdot v(E) + 1 + h^2(EndE),$$

where M is the moduli space containing E. Here we need a definition of exceptional bundles.

DEFINITION. A bundle E on an Enriques surface X is exceptional if $h^0(EndE) = 1$ and $h^1(EndE) = 0$.

In this chapter we study the conditions of the existence of exceptional bundles on Enriques surfaces. For an exceptional bundle E of even rank, $v(E) \cdot v(E) = -2$ if and only if $h^2(EndE) = 1$. An exceptional bundle E of even rank such that $E \cong E(K_X)$ has the property $h^2(EndE) = 1$. On the other hand, for an exceptional bundle E of odd rank, $v(E) \cdot v(E) = -1$ if and only if $h^2(EndE) = 0$. There is a general theory on exceptional bundles developed by Russian school ([Ty]), where they emphasized the roles played by exceptional bundles. They play very similar roles played by effective curves of self intersection -2 acting on Picard groups by reflections, in particular on K3 or Enriques surfaces. On K3 surfaces, every exceptional bundle E satisfies $v(E) \cdot v(E) = -2$, while on Enriques surfaces exceptional bundles of even rank can satisfy $v(E) \cdot v(E) = -2$, so that only exceptional bundles of even rank play the role of acting on moduli spaces.

Also there is another difference between K3 surfaces and Enriques surfaces. On K3 surfaces for every divisor D with $D^2 = -2$, D or $-D$ is effective, while on Enriques surfaces it is not always true. It holds only if the Enriques surface is nodal (has a smooth rational curve), since the effective divisor with self intersection -2 must have a smooth rational curve as a component.

There is also an anologue of this fact on exceptional bundles. On K3 surfaces, there always exists an exceptional bundle which realizes the exceptional vector $v(v \cdot v = -2$, with the same definition as above, which is a necessary condition for E to be exceptional) More explicitly,

THEOREM [Ku]. *Suppose that S is a complex algebraic K3 surface. A is an arbitrary ample divisor on S, and $v = (r, l, s), r > 0$, is an exceptional vector belonging to the Mukai lattice on S. Then there exists a simple, A-semi-stable bundle E which realizes the vector v.*

However, on Enriques surfaces we showed that there exists an exceptional bundle of rank 2 if and only if X is nodal, generalizing the result of Dolgachev and Reider on exceptional bundles of rank 2 with $c_1^2 = 10, c_2 = 3$ ([DR]). More explicitly,

THEOREM [Ki1]. *If E is an exceptional bundle of rank 2 such that $v(E) \cdot v(E) = -2$ if and only if $E = E_0(D)$, where D is some divisor and E_0 is a nontrivial extension,*

$$0 \to O_X \to E_0 \to O_X(N + K_X) \to 0,$$

where N is a nodal cycle with $N^2 = -2$ and K_X is the canonical divisor on X.

We want to generalize this result to any exceptional bundle of even rank as follows.

THEOREM 1. (1) *If there exists an exceptional bundle E on an Enriques surface X such that $E \cong E(K_X)$ which realizes a vector $v = (2k, D, s)$, then $D = N + 2L + kK_X$, where N is a nodal cycle and L is some divisor on X. In particular, there exists an exceptional bundle E of even rank such that $E \cong E(K_X)$ only if X is nodal.*

(2) *Suppose that the vector $v = (2k, D, s)$ with $v \cdot v = -2$ satisfies $g.c.d(k, H \cdot D, s) = 1$ and $D = N + 2L + kK$, where N is a nodal cycle, L is some divisor and H is an ample divisor on X. Then the vector v is realized as a vector of an exceptional and H-stable bundle on X.*

Proof. **(1)**. First we can find a bundle F on $\overline{X}$, such that $\pi_*(F) = E$, since $E \cong E(K_X)$ [Ta]. Then $\pi^*(E) = F \oplus i^*F$ and F is simple since

$$h^0(End(F \oplus i^*F)) = h^0(End(\pi^*E)) = h^0(EndE) + h^0((EndE)(K_X)) = 2$$

and $h^0(EndF) \geq 1, h^0(End(i^*F)) \geq 1$. F is also rigid since

$$h^1(End(\pi^*E)) = h^1(EndE) + h^1((EndE)(K_X))$$

and E is rigid with the property $EndE \cong EndE(K_X)$. Let the Chern polynomial of F be $1 + c_1(F)t + c_2(F)t^2$. Then, that of i^*F is $1 + (i^*c_1(F))t + c_2(F)t^2$. So,

$$\pi^*c_1(E) = c_1(F) + i^*c_1(F),$$

$$2c_2(E) = \pi^*c_2(E) = c_1(F) \cdot i^*c_1(F) + 2c_2(F).$$

Here $c_1(F) \cdot \pi^*H = i^*c_1(F) \cdot \pi^*H$ since $\pi^*H = i^*(\pi^*H)$. We have

$$dimM_X(2k, c_1(E), c_2(E)) = 4kc_2(E) - (2k-1)c_1(E)^2 - 4k^2 + 2$$

$$= 2(2kc_2(F) - (k-1)c_1(F)^2 - 2k^2 + 2) + c_1(F) \cdot i^*c_1(F) - c_1(F)^2 - 2$$

$$= 2dimM_{\overline{X}}(k, c_1(F), c_2(F)) + c_1(F) \cdot i^*c_1(F) - c_1(F)^2 - 2.$$

This implies that

$$c_1(F)^2 = c_1(F) \cdot c_1(i^*F) - 2$$

since both E and F are rigid. However, this holds if and only if $det(E) = N + 2L + kK_X$ for some nodal cycle N and some divisor L on X ([Ki1]).

(2). If we consider a vector $w = (k, N_1 + \pi^* L, s)$ on $\overline{X}$, where N_1 is a component of $\pi^* N = N_1 + N_2 (N_2 = i^* N_1, N_1 \not\sim N_2, N_1 \cdot N_2 = 0)$, then this is an exceptional vector. In fact,

$$w \cdot w = -2 + 2N \cdot L + 2L^2 - 2ks = -2$$

since $v \cdot v = -2 + 4L \cdot N + 4L^2 - 4ks = -2$. Then, by the theorem of Kuleshov [Ku], there exists an exceptional and $\pi^* H$- stable bundle F which realizes w. Then, $\pi_* F = E$ is a H-stable bundle which realizes the given vector v by the theorem of Takemoto since $F \not\cong i^* F$. Since E is stable, $h^0(EndE) = 1$. To prove that E is exceptional we have only to show that $h^1(EndE) = 1$. The rigidity of E comes from the fact that $\chi(EndE) = -v(E) \cdot v(E) = 2$ and $h^2(EndE) = 1$, since $E \cong E(K_X)$. □

COROLLARY 2. *Every exceptional bundle E of rank 2 on an Enriques surface X with $E \cong E(K_X)$ is $\pi_*(O_{\overline{X}}(N + \pi^* L))$, where N is some nodal cycle on $\overline{X}$, the universal covering K3 surface of X, and some divisor L on X.*

Next we consider exceptional bundles of odd rank. The simplest case is the case of rank one. We know that every line bundle on Enriques surfaces is exceptional since $h^1(X, O_X) = 0$ (as in $K3$ surfaces) and these can be identified to the line bundles in $K3$ surfaces invariant by involution. We want to generalize this fact to the exceptional bundles of any odd rank.

THEOREM 3. *Let the vector $v = (2k+1, D, s)$ with $v \cdot v = -1$ satisfies $g.c.d(2k+1, 2H \cdot D, 2s) = 1$ on any Enriques surface X, where H is an ample divisor on X. Then there exists an exceptional and H- stable bundle E which realizes the vector v.*

Proof. For the given vector $v = (2k+1, D, s)$ with $v \cdot v = -1$, we consider a vector $w = (2k+1, \pi^* D, 2s)$. Then, this is an exceptional vector on $\overline{X}$ since $w \cdot w = 2v \cdot v$. So, by the theorem of Kuleshov[Ku], there exists an exceptional and $\pi^* H$-stable bundle F realizing the vector w. Since $i^* \pi^* = \pi^*$ we have $c_i(F) = c_i(i^* F)$. Then we have

$$\chi(F^* \otimes i^* F) = \chi(F^* \otimes F) = -w \cdot w = 2,$$

so that there is a non-trivial homomorphism from F to $i^* F$ or from $i^* F$ to F. Since F and $i^* F$ are $\pi^* H$-stable with the same slope, they are isomorphic ([OSS]). So, we can find a vector bundle E such that $\pi^* E = F$ and E is H-stable. E realizes the vector v. So, it suffices to show that

E is exceptional. Since E is stable, E is simple and $h^1(EndE) = 0$ from the fact that $h^1(EndF) = h^1(EndE) + h^1(EndE(K_X)) = 0$. □

REMARK 1. It is interesting to see that exceptional bundles on Enriques surfaces can be used to re-classify the threefolds whose hyperplane sections are Enriques surfaces ([Co]) just as Mukai re-classified Fano threefolds and Fano variety of co-index 3, using exceptional bundles on K3 surfaces. ([Mu2])

REMARK 2. It is also interesting to construct exceptional bundles by extensions or divisions by exceptional bundles of even rank ([DP], [Ru], [Ku], [Ty]). For rank 2 case we could construct them explicitly. ([Ki1])

REMARK 3. We want to see that the conditions (i) E is exceptional, (ii) E is exceptional and $h^2(EndE) = 1$, (iii) E is exceptional and $E \cong E(K_X)$, (iv) E is exceptional and stable, are equivalent for exceptional bundles of even rank. $(iv) \to (iii) \to (ii) \to (i)$ are automatic. We expect that at least $(i), (ii)$ and (iii) are equivalent. We showed that $(ii), (iii)$ and (iv) are equivalent for rank two([Ki1]). In our work, most results are on exceptional bundles with (iii). On exceptional bundles of odd rank, we can also ask the similar question that $(i)E$ is exceptional, $(ii)E$ is exceptional and $h^2(EndE) = 0$, $(iii)E$ is exceptional and stable are equivalent. On K3 surfaces, there are two exceptional bundles with the same numerical invariants, so that they can not be H-stable at the same time for any ample H. ([Ku])

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