STRONG-MAX CYCLIC SUBMODULES

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ABSTRACT. In this paper we define CR(completely reachable), MICR(minimal cyclic refinement) and MACR(maximal cyclic refinement)-Modules. We have obtained equivalent statements for minimal cyclic submodule and maximal cyclic submodule. Also, we have obtained necessary and sufficient conditions for a module M with MICR to be cyclic or strongly cyclic.

1. Introduction

In this paper we characterize the minimal and the maximal cyclic submodules of an arbitrary module $M$. Also we give some characterizations of classes of modules, that is to say, strongly cyclic, CR(completely reachable), strong $CR$. In order to do these we introduce $S(m), C(m), MICR(minimal cyclic refinement)$ and MACR(maximal cyclic refinement) where $S(m)$ is the source set of $m \in M$ and $C(m) = \{0, q \in M : mR = qR\}$.

From now on, we assume that a ring $R$ has an identity 1 and a right $R$-module $M \neq \{0\}$. We have defined strongly cyclic module in Park [1] but we shall restate it here. $\{0\}$ will be denoted 0.

DEFINITION 1. (1) $M$ is strongly cyclic if $M \neq 0$ and $M = mR$ for any $m(\neq 0) \in M$ (or $\forall m(\neq 0), q \in M, q = ma$ for some $a \in R$).
(2) $M$ is cyclic if $M = mR$ for some $m \in M$.
(3) $mR$ is a minimal cyclic submodule if $mR \neq 0$ and $\forall q \in M, 0 \subset qR \subset mR \implies qR = mR$.
(4) $mR$ is a maximal cyclic submodule if $mR \nsubseteq M$ and $\forall q \in M, mR \subset qR \subset M \implies qR = mR$.

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(5) $H \subset M$ is a strongly cyclic subset of $M$ if $\forall m(\neq 0), q \in H, q = ma$ for some $a \in R$.

The proof of the following Lemma is quite straightforward.

**Lemma 1.** $H \leq M$ is minimal submodule of $M \iff H \subset M$ is strongly cyclic submodule.

**Definition 2.** Let $m(\neq 0) \in M$.

1. $S(m) = \{0, q \in M : m = qa \text{ for some } a \in R\}$ is called the source set of $m \in M$.
2. $m \in M$ is completely reachable in $M$ if $M = S(m)$
3. $C(m) = \{0, q \in M : mR = qR\}$
4. $M$ is a CR-module (module with a completely reachable element) if $M = S(n)$ for some $n(\neq 0) \in M$

**Lemma 2.** Let $M$ be a right $R$-module. Then we have the following statements:

1. $S(m) \subset S(ma)$ for any $a \in R$ and $q \in S(m) \implies S(q) \subset S(m)$.
2. $M = S(m) \iff m \in \bigcap_{q(\neq 0) \in M} qR \neq \emptyset$.
3. $\bigcap_{m(\neq 0) \in M} mR$ is a strongly cyclic submodule of $M$ if $M$ is a CR-module.

**Proof.** (1) and (2) are trivial. For (3), we shall show $\bigcap_{m(\neq 0) \in M} mR = qR$ for every $q(\neq 0) \in \bigcap_{m(\neq 0) \in M} mR$. We note that $q \in mR$ for all $m(\neq 0)$ in $M$ and hence $q = mb$ for some $b \in R$. We let $t \in qR$. Then $t = qa$ for some $a \in R$. This implies $t = (mb)a = m(ba) \in mR$ for all $m(\neq 0)$ in $M$. Hence $t \in \bigcap_{m(\neq 0) \in M} mR$. The converse is trivial. \hfill \Box

We define new terminologies.

**Definition 3.** Let $m(\neq 0) \in M$.

1. $S(m)$ is minimal set if $\forall q \in M, 0 \subset S(q) \subset S(m) \implies S(q) = S(m)$.
2. $S(m)$ is maximal set if $\forall q \in M, S(m) \subset S(q) \subset M \implies S(q) = S(m)$.
LEMMA 3. Let \(m, n(\neq 0) \in M\). Then the following statements hold:

1. \(mR \subseteq nR \iff S(m) \supseteq S(n)\)
2. \(mR = nR \iff S(m) = S(n)\).
3. \(mR\) is minimal/\(\max\) \(\iff S(m)\) is maximal/\(\min\) set.
4. \(mR\) is strongly cyclic \(\iff mR\) is minimal.
5. \(S(m)\) is strongly cyclic set \(\iff S(m)\) is minimal set.

Proof. For (1), \(\Rightarrow\) we let \(t \in S(n)\). Then \(n = tb\) for some \(b \in R\). But \(m = m1 \in mR \subseteq nR\) and then \(m = nc\) for some \(c \in R\). This implies \(m = nc = (tb)c = t(bc)\). Hence \(t \in S(m)\). (\(\Leftarrow\)) we can prove it in the same way. For (2), we can prove it like (1). (3) comes from (1) and (4) comes from Lemma 1. For (5), \(\Rightarrow\) we let \(0 \subseteq S(q) \subseteq S(m), \forall q(\neq 0) \in M\). To prove \(S(q) \supseteq S(m)\) we let \(p(\neq 0) \in S(m)\). We note that \(q \in S(m)\). Since \(S(m)\) is strongly cyclic, we have \(q = pa\) for some \(a \in R\) and hence \(p \in S(q)\). (\(\Leftarrow\)) also it is trivial. \(\Box\)

LEMMA 4. Let \(M\) be a right \(R\)-module and \(m(\neq 0) \in M\). Then we have the following statements:

1. \(C'(m)\) is a strongly cyclic subset of \(M\).
2. \(C'(m) = C(n) \iff (i) \ ma = n\) and \(nb = m\) for some \(a, b \in R\) \(\iff (ii) \ S(m) = S(n)\)
3. Let \(D_m\) be a strongly cyclic subset of \(M\) with \(m \in D_m\). Then \(D_m \subseteq C'(m)\).
4. \(C'(m) = nR \cap S(n)\) for any \(n(\neq 0) \in M\).
5. \(\bigcap_{m(\neq 0) \in M} C'(m) = \{0\}\)

Proof. For (1), let \(p(\neq 0), t \in C(m)\). Then \(mR = pR\) and \(mR = tR\). From this we have \(pa = t\) for some \(a \in R\).

For (2)(i), \(\Rightarrow\) : trivial. \(\Leftarrow\) : To show \(C(m) \subseteq C(n)\) we let \(t \in C(m)\). Then \(mR = tR\). This implies \(S(m) = S(t)\). But \(n = ma \Rightarrow S(m) \subseteq S(n)\) and \(m = nb \Rightarrow S(n) \subseteq S(m)\). This means \(S(m) = S(n)\). Hence we have \(S(n) = S(t)\) and then \(nR = tR\). i.e., \(t \in C(n)\).

Similarly, we can prove the converse.

For (2)(ii), \(\Rightarrow\) : To show \(S(m) \subseteq S(n)\) we let \(t \in S(m)\). Then \(m = tc\) for some \(c \in R\). But \(ma = n\) for some \(a \in R\). Hence we have \(n = ma = (tc)a = t(ca)\). This means \(t \in S(n)\). Similarly, it is easy to show the converse. \(\Leftarrow\) : It is trivial.
For (3), let \( p(\neq 0) \in D_m \). Then we have \( m = pa \) and \( p = mb \) for some \( a, b \in R \). From (2) we have \( p \in C(p) = C(m) \).

For (4), \( C(n) \subset nR \cap S(n) \): Let \( t \in C(n) \). Then \( nR = tR \). From this we have \( n = ta \) for some \( a \in R \). Hence \( t \in S(n) \) and then \( t \in nR \cap S(n) \). Also, it is easy to check the converse.

For (5), let \( p(\neq 0) \in C(m) \cap C(q) \). Then we have \( mR = pR \) and \( qR = pR \). To show \( C(m) \subset C(q) \) we let \( t \in C(m) \). Then \( mR = tR \). From this we have \( tR = qR \). Hence \( t \in C(q) \). Similarly, it is trivial to show \( C(m) \supset C(q) \). This means that we have shown \( p \in C(m) \cap C(q) \implies C(m) = C(q) \).

\[ \square \]

2. Characterizations of minimal and maximal cyclic submodules in a module \( M \)

**Theorem 5.** Suppose that \( M \) is not cyclic and let \( m(\neq 0) \in M \). Then the following assertions are equivalent :

1. \( mR \) is a maximal cyclic submodule of \( M \) ;
2. \( S(m) = C(m) \);
3. \( S(m) \cap S(q) \neq 0, \forall q(\neq 0) \in M \implies S(m) \subset S(q) \);
4. \( S(m) \subset mR \);
5. \( S(m) \) is a strongly cyclic subset of \( M \);
6. \( C(m) \cap qR \neq 0, \forall q(\neq 0) \in M \implies q \in C(m) \);
7. \( m = qa \) for some \( a \in R, \forall q(\neq 0) \in M \implies C(m) = C(q) \);
8. \( S(m) \cap qR \neq 0, \forall q(\neq 0) \in M \implies mR = qR \).

**Proof.** (1) \( \Rightarrow \) (2) : We shall show \( S(m) \subset C(m) \). Let \( q \in S(m) \). Then \( S(q) \subset S(m) \Rightarrow qR \supset mR \). Hence \( qR = mR \) and then \( q \in C(m) \). \( S(m) \supset C(m) \) comes from Lemma 4(4).

(2) \( \Rightarrow \) (3) : Let \( t \in S(m) \cap S(q) \). Then we have \( t \in S(m) \) and \( t \in S(q) \). This implies \( S(t) \subset S(m) \) and \( S(t) \subset S(q) \). But \( t \in C(m) \). This means \( S(m) = S(t) \). Hence \( S(m) = S(t) \subset S(q) \).

(3) \( \Rightarrow \) (4) : Let \( q(\neq 0) \in S(m) \). Then we have \( S(q) \subset S(m) \) and \( S(q) \cap S(m) \neq 0 \). From assumption we have \( S(m) \subset S(q) \). Hence \( S(m) = S(q) \). From this \( q = ma \) for some \( a \in R \) and hence \( q \in mR \).

(4) \( \Rightarrow \) (5) : Let \( p, q(\neq 0) \in S(m) \subset mR \). Then we have \( S(q) \subset S(m) \) and \( S(q) \subset S(m) \). Also \( p = ma \) and \( q = mb \) hold for some \( a, b \in R \). This means that \( m \in S(p) \Rightarrow S(m) \subset S(p) \) and \( m \in S(q) \Rightarrow S(m) \subset S(q) \).
\( S(q) \). Hence \( S(m) = S(p) = S(q) \). This shows that \( p = qa \) for some \( c \in R \).

(5) \( \Rightarrow \) (1) : Let \( mR \subseteq qR \subseteq M \) for \( q \in M \). Then from Lemma 3(1) we have \( S(m) \supseteq S(q) \). Since \( S(m) \) is strongly cyclic, \( q = ma \) holds for some \( a \in R \). This implies \( m \in S(q) \) and then \( S(m) \subseteq S(q) \). Hence we have \( S(m) = S(q) \). This means \( mR = qR \) from Lemma 3(2).

(1) \( \Rightarrow \) (6) : Let \( p \in C(m) \cap qR \neq 0 \). Then we have \( mR = pR \) and \( pR \subseteq qR \). This means \( mR \subseteq qR \) and from assumption \( mR = qR \) holds. Hence \( q \in C(m) \).

(6) \( \Rightarrow \) (1) : From Lemma 4(4) it is trivial.

(1) \( \Leftrightarrow \) (7) : It is trivial.

(1) \( \Rightarrow \) (8) : Let \( p \in S(m) \cap qR \). Then we have \( p \in S(m) \) and \( p \in qR \). This implies that \( m = pa \) for some \( a \in R \) and \( pR \subseteq qR \). Also we have \( mR \subseteq pR \). Since \( mR \) is maximal, we have \( mR = qR \).

(8) \( \Rightarrow \) (1) : Let \( mR \subseteq qR \). Then from Lemma 4(4) we have \( S(m) \cap qR \neq 0 \). Hence \( mR = qR \) \( \square \)

**Theorem 6.** Let \( m(\neq 0) \in M \). Then the following conditions are equivalent :

1. \( mR \) is minimal ;
2. \( C(m) = mR \);
3. \( C(m) \) is a submodule of \( M \);
4. \( C(m) \cap S(q) \neq 0, \forall q(\neq 0) \in M \implies S(m) = S(q) \);
5. \( mR \subseteq S(m) \);
6. \( \forall a \in R \exists b \in R : mab = m \);
7. \( mR \cap qR \neq 0, \forall q(\neq 0) \in M \implies mR \subseteq qR \);

**Proof.** (2) \( \Leftrightarrow \) (3) : It is trivial. (1) \( \Rightarrow \) (3) : (i) let \( q \in C(m) \) and \( a \in R \). Then we have \( mR = qR \). But \( qaR \subseteq qR = mR \) holds. Hence \( qa \in C(m) \). (ii) to show \( (C(m), +) \) is a subgroup of \( M \) we let \( p, q \in C(m) \). Then \( mR = pR \) and \( mR = qR \) hold. From this for every \( a \in R \) we have \( mb = pa \) and \( mc = qa \) for some \( b, c \in R \). This implies \( (p - q)a = m(b - c) \in mR \). Hence \( (p - q)R \subseteq mR \) holds. Since \( mR \) is minimal, we have \( (p - q)R = mR \). This means \( (p - q) \in C(m) \).

(3) \( \Rightarrow \) (4) : Let \( p \in C(m) \cap S(q) \). Then \( p \in C(m) \) and \( p \in S(m) \). From this we have \( q = pa \in C(m) \) for some \( a \in R \). Hence \( mR = qR \) holds.
(4) \( \Rightarrow \) (5) : \( S(m) \subset S(ma) \) holds for \( a \in R \). From Lemma 4(4) we have \( C(m) \cap S(ma) \neq 0 \). Hence \( m \alpha \in S(m) \).

(5) \( \Rightarrow \) (6) : It is trivial. (6) \( \Rightarrow \) (1) : Let \( m \alpha, mc \in mR \). For \( a \in R \) \( \exists b \in R : mab = m \). From this \( mc = (mab)c = m \alpha (bc) \).

(1) \( \Rightarrow \) (7) : Let \( p \in mR \cap qR \). Then \( pR \subset mR \) and \( pR \subset qR \). Since \( mR \) is minimal, we have \( mR \subset qR \).

(7) \( \Rightarrow \) (1) : It is trivial. \( \square \)

The following Theorem comes from Theorem 5 and Theorem 6. Therefore, we shall omit its proof.

**Theorem 7.** Suppose that \( M \) is not cyclic and let \( m(\neq 0) \in M \). Then the following conditions are equivalent:

1. \( mR \) is minimal and maximal;
2. \( S(m) = C(m) \) is a submodule of \( M \);
3. \( S(m) \cap S(q) \neq 0, \forall q(\neq 0) \in M \implies S(m) = S(q) ; \)
4. \( mR = S(m) ; \)
5. \( S(m) \) is strongly cyclic submodule of \( M \);
6. \( mR \cap qR \neq 0, \forall q(\neq 0) \in M \implies mR = qR \)

3. **MACR-modules and MICR-modules**

We introduce new terminologies.

**Definition 4.** Let \( M \) be a right \( R \)-module.

1. \( R^{-1}(\text{min}) = \{ 0, m \in M : mR \text{ is minimal} \} \).
2. \( R^{-1}(\text{max}) = \{ 0, m \in M : mR \text{ is maximal} \} \).
3. \( M \) is a MACR(maximal cyclic refinement) - module if \( \forall m(\neq 0) \in M \exists q(\neq 0) \in R^{-1}(\text{max}) : mR \subset qR \).
4. \( M \) is a MICR(minimal cyclic refinement) - module if \( \forall m(\neq 0) \in M \exists q(\neq 0) \in R^{-1}(\text{min}) : qR \subset mR \).
5. \( M \) is a strong CR-module if \( M = S(m) \) for every \( m(\neq 0) \in M \).

**Lemma 8.** Let \( M \) be a right \( R \)-module. Then the following statements hold:

1. Every CR-module is a MICR-module such that \( M = S(q) \) for every \( q(\neq 0) \in R^{-1}(\text{min}) \).
2. \( mR \) is minimal cyclic submodule for every completely reachable element \( m \in M \).
(3) Strongly cyclic module $\implies$ Strong CR-module $\implies$ CR-module.

Proof. For (1), since $M$ is CR-module, we have $M = S(m)$ for some $m(\neq 0) \in M$. To prove $S(q) \supset M$ we let $p \in M = S(m)$. Then $p \in S(p) \subset S(m)$. On the other hand, we have $q \in M = S(m)$. This implies that $S(q) \subset S(m) \iff qR \supset mR$ and hence $qR = mR$ since $qR$ is minimal. Hence it holds. To prove that $M$ is a MICR we let $m(\neq 0) \in M$. Then $S(m) \subset S(p)$ for $p(\neq 0) \in R^{-1}(\text{min})$ and hence $mR \supset pR$.

For (2), we let $0 \varsubsetneq qR \subset mR$ for $q \in R$. Then $M = S(m) \subset S(q)$. This implies $S(q) = S(m)$ and hence $qR = mR$. (3) comes from definitions.

\[ \square \]

THEOREM 9. Let $M$ be a MACR-module. If there is a $m(\neq 0) \in M$ such that $C(m) = R^{-1}(\text{max})$, then $M$ is cyclic.

Proof. Let $q \in M$. Since $M$ is MACR-module, there is a $p \in R^{-1}(\text{max})$ such that $qR \subset pR$. From this we have $q \in qR \subset pR = mR$. Hence $M = mR$. \[ \square \]

THEOREM 10. Let $M$ be a MICR-module. Then we have the following statements:

(1) $M$ is a CR-module $\iff \exists m(\neq 0) \in M$ such that $C(m) = R^{-1}(\text{min})$.

(2) $M$ is strongly cyclic $\iff \exists m(\neq 0) \in M$ such that $S(m) = R^{-1}(\text{min})$.

(3) $M$ is strong CR $\iff M$ is strongly cyclic.

Proof. For (1), ($\iff$) Let $m \in M$ such that $C(m) = R^{-1}(\text{min})$. We let $q(\neq 0) \in M$. Then $\exists p(\neq 0) \in R^{-1}(\text{min})$ such that $pR \subset qR$. From $p \in C(m)$ we have $mR = pR \subset qR$. Hence $q \in S(m)$ and then $M = S(m)$.

($\Rightarrow$) we note that $M = S(m)$ for some $m(\neq 0) \in M$. From Lemma 8(2) we have $m \in R^{-1}(\text{min})$. It is trivial to show $C(m) = R^{-1}(\text{min})$ from Lemma 2(2).

For (2), ($\iff$) Let $m \in M$ such that $S(m) = R^{-1}(\text{min})$. Claim : $C(m) = S(m)$. (proof) since $mR$ is minimal, we have $mR = C(m) \subset S(m)$. To prove $C(m) \supset S(m)$ we let $q \in S(m)$. From assumption
\[ \exists \ p \in R^{-1}(\text{min}) \text{ such that } pR \subseteq qR. \text{ From this we have } S(p) \subseteq S(m) \]
and then \( mR \subseteq pR \subseteq qR. \) Hence we have \( mR = qR \) and \( q \in C(m). \)

Combining the claim and hypothesis, we have \( C(m) = S(m) = R^{-1}(\text{min}). \) Also, from (1) we have \( M = S(m). \) Now we shall show that \( M \) is strongly cyclic. Let \( p, q \in M = C(m). \) Then we have \( mR = pR \) and \( mR = qR. \) This implies \( p = qa \) for some \( a \in R. \)

\( \Rightarrow \) We note that \( M = S(m) \) for every \( m(\neq 0) \in M \) since \( M \) is strongly cyclic. We shall show \( R^{-1}(\text{min}) = M. \) Let \( q(\neq 0) \in M. \) Then \( \exists \ p \in R^{-1}(\text{min}) \) such that \( pR \subseteq qR. \) Since \( M \) is strongly cyclic, we have \( M = pR. \) Hence \( pR = qR \) holds and \( q \in R^{-1}(\text{min}). \) From this we have \( S(m) = R^{-1}(\text{min}). \) \( \Box \)

From Theorem 6 and the above Theorem we have the following Corollary.

**Corollary 10.1.** If \( M \) is a CR-module, then \( R^{-1}(\text{min}) = C(m) = mR. \)

4. Examples

**Example 1.** Let \( \mathbb{Z}_3 = \{ 0, 1, 2 \}. \) Then

(1) \( \mathbb{Z}_3 \) is strongly cyclic \( \mathbb{Z}_3 \)-module since \( 1\mathbb{Z}_3 = \mathbb{Z}_3 \) and \( 2\mathbb{Z}_3 = \mathbb{Z}_3. \)

(2) \( 1, 2 \in \mathbb{Z}_3 \) are completely reachable elements of \( \mathbb{Z}_3 \) since \( \mathbb{Z}_3 = S(1) \) and \( \mathbb{Z}_3 = S(2). \)

(3) \( 1\mathbb{Z}_3 \) and \( 2\mathbb{Z}_3 \) are minimal submodules of \( \mathbb{Z}_3 \) from (2).

(4) \( R^{-1}(\text{min}) = \{ 0, 1, 2 \} = S(1) = S(2). \) Hence \( \mathbb{Z}_3 \) is strongly cyclic like we have mentioned in (1).

(5) \( \mathbb{Z}_3 \) is a MICR-module since \( 1\mathbb{Z}_3 \subset 1\mathbb{Z}_3 \) and \( 2\mathbb{Z}_3 \subset 2\mathbb{Z}_3. \)

**Example 2.** Let \( \mathbb{Z}_4 = \{ 0, 1, 2, 3 \}. \) Then

(1) \( \mathbb{Z}_4 \) is not strongly cyclic \( \mathbb{Z}_4 \)-module since \( 2\mathbb{Z}_4 \neq \mathbb{Z}_4. \)

(2) \( 2 \in \mathbb{Z}_4 \) is a completely reachable element of \( \mathbb{Z}_4 \) but \( 1, 3 \in \mathbb{Z}_4 \) are not completely reachable elements of \( \mathbb{Z}_4 \) since \( \mathbb{Z}_4 = S(2) \) but \( \mathbb{Z}_4 \neq S(1) \) and \( \mathbb{Z}_4 \neq S(3). \)

(3) \( 2\mathbb{Z}_4 \) is minimal submodule of \( \mathbb{Z}_4 \) from (2) and also a maximal submodule of \( \mathbb{Z}_4 \) since \( 2 \) is a prime dividing \( 4. \)

(4) \( R^{-1}(\text{min}) = \{ 0, 2 \}. \)

(5) \( \mathbb{Z}_4 \) is a MICR-module since \( 2\mathbb{Z}_4 \subset 1\mathbb{Z}_4, 2\mathbb{Z}_4 \subset 2\mathbb{Z}_4 \) and \( 2\mathbb{Z}_4 \subset 3\mathbb{Z}_4. \)
(6) \( \exists m (\neq 0) \in \mathbb{Z}_4 \) such that \( S(m) = R^{-1}(\text{min}) \) since \( S(1) = \{0, 1, 3\}, S(2) = \{0, 1, 2, 3\} \) and \( S(3) = \{0, 1, 3\} \). Hence \( \mathbb{Z}_4 \) is not strongly cyclic like we have mentioned in (1).

References


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