ASSOUAD DIMENSION: ANTIFRACTAL METRIZATION, POROUS SETS, AND HOMOGENEOUS MEASURES

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ABSTRACT. We prove that a non-empty separable metrizable space $X$ admits a totally bounded metric for which the metric dimension of $X$ in Assouad's sense equals the topological dimension of $X$, which leads to a characterization for the latter. We also give a characterization based on this Assouad dimension for the dimension (embedding dimension) of a compact set in a Euclidean space. We discuss Assouad dimension and these results in connection with porous sets and measures with the doubling property. The elementary properties of Assouad dimension are proved in an appendix.

1. Introduction

A fractal might be defined—adapting Mandelbrot—as a non-empty compact metric space $X$ for which at least one of the always valid inequalities

\[(1.1) \quad \dim X \leq \dim_H X \leq \dim_B X \leq \overline{\dim}_B X \leq \dim_A X\]

for the topological, Hausdorff, lower box-counting, upper box-counting, and Assouad dimension, respectively, of $X$ is strict. The original definition of a fractal due to Mandelbrot [45, p. 15] is stronger, requiring $\dim_H X > \dim X$. Here the Assouad dimension $\dim_A X$ of $X$ means the metric dimension of $X$ as defined by Assouad [3]. The definition is recalled in 3.2. For example, $X = \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\} \subset \mathbb{R}$ is a fractal in our sense with $\dim X = \dim_H X = 0$, $\dim_B X = \overline{\dim}_B X = \frac{1}{2}$, and

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dim A X = 1. On the other hand, if the inequalities above are all equalities, that is, if \( \dim A X = \dim X \), then the compact metric space \( X \) and its metric might be called antifractal or flat. The main result of this paper is that flat metrics on \( X \) always exist. Thus, there is no purely topological reason for \( X \) to be fractal. Let \( \text{Dim} \) stand for one of \( \dim H \), \( \dim B \), \( \dim B \), and \( \dim A \). Then it follows that the infimum of the numbers \( \text{Dim} Y \) over all compact metric spaces \( Y \) homeomorphic to \( X \) equals \( \dim X \) and that, moreover, the infimum is attained. This characterization of \( \dim X \) is due to Marczewski [46] if \( \text{Dim} = \dim H \) and essentially due to Pontrjagin and Schnirelmann [53] if \( \text{Dim} = \dim B \) or \( \text{Dim} = \dim B \).

Similarly, we show that Hausdorff dimension can be replaced by Assouad dimension in Väisälä’s [67] characterization of the demension (that is, Š tan’ko’s embedding dimension) of a compact set in a Euclidean space. In particular, such a set is tame if and only if it is ambiently homeomorphic to a flat set.

The existence of a metrization for each non-empty separable metrizable space \( X \) by a totally bounded metric with \( \dim B X = \dim X \) was needed by the author in [41] for a general position argument based on Hausdorff measure. It might be hoped that the similar but stronger totally bounded flat metrization of \( X \) with \( \dim A X = \dim X \) we in fact construct here would also find applications, and indeed we provide a few; see below. Up to now Assouad dimension has proved to be very useful in the problems of characterizing the metric spaces that are bi-Lipschitz embeddable in a Euclidean space [3], [5], [40], [42], [48], [49], [61] and those that carry a nontrivial homogeneous measure, i.e., one with the doubling property [4], [14], [15], [19], [20], [32], [70], [71] (not all of these references deal with Assouad dimension directly).

The inequalities (1.1) challenge to establish upper bounds to the Assouad dimension of a fractal in situations where presently upper bounds are only established to the other, more familiar dimensions. In fact, we managed to strengthen a known result in this way; see below.

The metrization results are presented in Section 4, and they follow from a study of universality properties of certain Menger spaces in Section 2 and a study of the Assouad dimensions of these spaces in Section 3.

In an extensive Appendix A we give detailed proofs for the ele-
mentary properties of Assouad dimension and review the bi-Lipschitz embeddability results just mentioned. As it turned out, Assouad dimension is invariant under inversions of normed spaces in spheres. This fact has been used in the paper, and it should be of interest in analysis, too. It is for the sake of these elementary properties of Assouad dimension, analogous to those of the more familiar dimensions, that Assouad dimension really deserves to be called a dimension.

Section 5 deals with sets in a Euclidean space which are porous in the sense of Väisälä [69]. Väisälä's porosity concept seems to be the strongest one among the various porosity concepts now widely used as an expression of smallness of a set. We prove that a set $X$ in $\mathbb{R}^m$ is porous if and only if the Assouad dimension of $X$ is smaller than $m$, and then show as an application of our characterization of dimension that $X$ is ambiently homeomorphic to a porous set if and only if $X$ is nowhere dense. Salli [58] has studied porous sets a porosity constant of whose is close to its greatest value 1. He established an upper bound of correct type to the Hausdorff dimension and to the upper box-counting dimension of these sets in terms of their porosity constant. We show that this same upper bound is also valid for the Assouad dimension of these sets.

In Section 6 we discuss the conjecture already alluded to above that the Assouad dimension of a complete metric space can be characterized as the infimum of certain exponents associated with the homogeneous measures existing on this space. The conjecture is valid for compact spaces as proved by Volfberg and Konyagin [71]; what they call uniform metric dimension is the same concept that we call Assouad dimension. We correct a slight inaccuracy in the proof of their second main theorem and verify a weak form of the conjecture for noncompact spaces. We also propose that the inversion invariance of Assouad dimension could be used to reduce the conjecture to the compact case. With the aid of our flat metrization results we show that the topological dimension of a locally compact separable metrizable space as well as the dimension of a compact set in a Euclidean space can be characterized in terms of homogeneous measures.

Assouad dimension should certainly become more well-known. It was my recent joint contribution [42] on the bi-Lipschitz embeddability problem that renewed my own interest in Assouad dimension, but
actually the present paper originated already in 1980 at Orsay. I acknowledge with pleasure a scholarship of the French Government that I held then.

2. Universality of Menger spaces

In this section we introduce Menger spaces and establish as an approximation result their universality properties related to topological dimension and demension.

**Terminology 2.1.** For a topological space $X$ we let $\dim X$ denote the covering dimension, or topological dimension, of $X$ \cite{23, 31}; $\dim X$ is either $-1$ (for $X = \emptyset$), a nonnegative integer, or $\infty$. For two maps $f, g$ of a set $X$ to a metric space $(Z, d)$ we let $\varrho(f, g) = \sup_{x \in X} d(f(x), g(x))$. By $\text{id}$ we denote various inclusion maps. Let $I = [0, 1]$. An isotopy of a metric space $Z$ is a continuous map $F : Z \times I \to Z$, $(x, t) \mapsto F_t(x)$, such that each $F_t : Z \to Z$ is a homeomorphism and $F_0 = \text{id}$; the support of $F$ is the set $\text{supp} F = \text{cl}\{ x \in Z \mid F_t(x) \neq x \text{ for some } t \}$. If $\varepsilon > 0$ and $\varrho(F_t, \text{id}) < \varepsilon$ for each $t$, then $F$ is called an $\varepsilon$-isotopy. The $\varepsilon$-neighbourhood of a set $A \subset Z$ for $\varepsilon > 0$ is the set $B(A, \varepsilon) = \{ x \in Z \mid d(x, y) < \varepsilon \text{ for some } y \in A \}$. If $A \subset Z$, $\varepsilon > 0$, and $F$ is an $\varepsilon$-isotopy of $Z$ with $\text{supp} F \subset B(A, \varepsilon)$, then $F_1$ is called an $\varepsilon$-push of $(Z, A)$. For integers $0 \leq n \leq m \geq 1$ let $N_n^m$ denote Nöbeling’s $n$-dimensional set in $\mathbb{R}_n^m$ of all points at most $n$ of whose coordinates are rational \cite{23, 1.8.5}. Let $I_n^m = [0, 1]^m$.

**Dimension 2.2.** For a compact subset $X$ of Euclidean $m$-space $\mathbb{R}_n^m$ we let $\text{dem } X$ denote the dimension, or embedding dimension, of $X$ in $\mathbb{R}_n^m$ in the sense of Štan’ko; see \cite{21}. We can define $\text{dem } X$ as the smallest integer $n \in [-1, m]$ such that for each $(m - n - 1)$-dimensional closed polyhedron $P \subset \mathbb{R}_n^m$ and each $\varepsilon > 0$ there is an $\varepsilon$-push $f$ of $(\mathbb{R}_n^m, X \cap P)$ moving $P$ off $X$, that is, with $fP \cap X = \emptyset$. Then $\text{dim } X \leq \text{dem } X$, and the dimension is invariant under self-homeomorphisms of $\mathbb{R}_n^m$. For example, $\text{dem } X = \text{dim } X$ if $X$ is a polyhedron. Although we do not really need it, for background we present the following fundamental theorem of dimension theory relating dimension to topological dimension more generally. A compact set $X \subset \mathbb{R}_n^m$ is called locally homotopically 1-co-connected (1-LCC) in
\( \mathbb{R}^m \) if for each \( x \in X \) and each neighbourhood \( U \) of \( x \) in \( \mathbb{R}^m \) there is a neighbourhood \( V \subset U \) of \( x \) in \( \mathbb{R}^m \) such that each continuous map \( \alpha: \partial I^2 \rightarrow V \setminus X \) can be extended to a continuous map \( \beta: I^2 \rightarrow U \setminus X \).

**Theorem 2.3.** Let \( X \) be a compact set in \( \mathbb{R}^m \).

(a) If \( \dim X \geq m - 2 \), then \( \dem X = \dim X \), with the exception that also the case \( \dim X = 2 \) is realizable when \( m = 3 \) and \( \dim X = 1 \).

(b) If \( \dim X \leq m - 3 \), then \( \dem X = \dim X \) if and only if \( X \) is 1-LCC in \( \mathbb{R}^m \); otherwise \( \dem X = m - 2 \).

For the proof of this theorem, with the sufficiency case of (b) for \( m = 4 \) and \( X \neq \emptyset \) excluded, see [21, 1.4]. Now consider the sufficiency claim in (b) for \( m = 4 \). In any case \( \dim X \in \{ \dim X, 2 \} \) by [21, 1.4], but according to [16, p. 163], R. D. Edwards has announced that this claim holds, and in [57, p. 598] it is said that the claim follows from Freedman [26]. If \( \dim X = 0 \), then, indeed, with the aid of [26, Theorem 1.11] it can be shown as in the case \( m \neq 4 \) that \( X \) can be covered for each \( \varepsilon > 0 \) by a finite disjoint family of open topological 4-balls of diameter smaller than \( \varepsilon \), which shows according to the (equivalent) definition used in [21] that \( \dim X = 0 \).

In view of Theorem 2.3, in this paper we call a compact set \( X \) in \( \mathbb{R}^m \) *tame* if \( \dem X = \dim X \) and *wild* otherwise.

The next lemma recapitulates universal properties of the Nöbeling spaces \( N_n^m \).

**Lemma 2.4.** Let \( 0 \leq n \leq m \geq 1 \) be integers and \( \varepsilon > 0 \).

(a) If \( X \) is a separable metric space, \( \dim X \leq n \), \( m \geq 2n + 1 \), and \( f: X \rightarrow \mathbb{R}^m \) a bounded continuous map, then there is an embedding \( g: X \rightarrow \mathbb{R}^m \) with \( \varrho(g, f) < \varepsilon \) such that \( \cl gX \subset N_n^m \).

(b) If \( X \subset \mathbb{R}^m \) is a compact set and \( \dim X \leq n \), then there is an embedding \( g: X \rightarrow \mathbb{R}^m \) with \( \varrho(g, \text{id}) < \varepsilon \) such that \( gX \subset N_n^m \).

(c) If \( X \subset \mathbb{R}^m \) is a set, \( n \in \{ m - 1, m \} \), and \( \dim X \leq n \), then there is an embedding \( g: X \rightarrow \mathbb{R}^m \) such that \( \cl gX \subset N_n^m \cap I^m \).

(d) If \( X \subset \mathbb{R}^m \) is a compact set and \( \dem X \leq n \), then there is an \( \varepsilon \)-push \( f \) of \((\mathbb{R}^m, X)\) such that \( fX \subset N_n^m \).

**Proof.** (a) This is a version of a classical result [31, Theorem V 5].

(c) For \( n = m \) this is obvious as \( N_n^m = \mathbb{R}^m \); for \( n = m - 1 \) see [23, p. 126].
(d) The property is in fact a characteristic one for the condition \( \text{dem } X \leq n \); see [21, 1.2].

(b) If \( n \geq m - 2 \) and \((m,n) \neq (3,1)\), then \( \text{cem } X \leq n \) by 2.3, and the claim follows from (d). If \((m,n) = (3,1)\), the claim follows from (a). Finally, suppose that \( n \leq m - 3 \) (by (a) we could assume that, in addition, \( n \geq 3 \) and \( m \geq 6 \)). Then by Štančiško’s approximation theorem [64], a new proof of which is given by Edwards [22], there is an embedding \( f : X \to \mathbb{R}^m \) with \( g(f, \text{id}) < \varepsilon \) such that \( \text{dim } fX = \text{dim } X \) (for this formulation [21, 1.5] of the theorem see [64, Theorem 2] and [63, Propositions 1 and 2] or, respectively, 2.3, but also [22, Remark 4 on p. 97]). The claim now follows from (d). □

**Menger spaces 2.5.** We first recall the construction of Menger compacta. For each pair \((m,n)\) of integers with \(0 \leq n \leq m \geq 1\) and for each sequence \( P = (p_1, p_2, \ldots) \) of integers \( \geq 3 \) we call \((m,n,P)\) a *Menger triple* and define an \( n \)-dimensional compact set \( M^m_n(P) \), called a *Menger compactum*, in \( \mathbb{R}^m \) as follows; cf. [23, pp. 121–122]. Call the unit cube \( I^m \) the cube of rank zero, and let \( t_0 = 1 \). Proceeding inductively, assume that \( Q \) is a cube of rank \( i - 1 \). Subdivide \( Q \) into \( p_i^m \) axes-parallel closed \( m \)-cubes of side length \( l_i = (p_1 \ldots p_i)^{-1} \) by subdividing its sides into \( p_i \) intervals of equal length. Of these cubes those meeting an \( n \)-face of \( Q \) are called cubes of rank \( i \). Let \( \Gamma_i \) be the union of all cubes of rank \( i \). Then \( \Gamma_0 \supset \Gamma_1 \supset \cdots \). Now let \( M^m_n(P) = \bigcap_{i>0} \Gamma_i \). Since \( M^m_n(P) \) contains the \( n \)-faces of \( I^m \) and since obviously for each \( \varepsilon > 0 \) there is a continuous map of \( M^m_n(P) \) into an \( n \)-dimensional polyhedron in \( I^m \) moving no point more than \( \varepsilon \), we conclude that \( \text{dim } M^m_n(P) = n \). Clearly, \( M^m_m(P) = I^m \). The case where \( P \) is the sequence \( P_M = (3, 3, \ldots) \) is due to Menger; \( M_0^m(P_M) \) is the usual Cantor set. The general case has been considered by Bothe [9].

Consider a Menger triple \((m,n,P)\) and an integer \(q \geq 1\). Call \((m,n,P,q)\) a *Menger quadruple*. Let \( K_q \) be the decomposition of \( \mathbb{R}^m \) into axes-parallel closed \( m \)-cubes of side length \( q^{-1} \) and with vertexes in \( q^{-1} \mathbb{Z}^m \). For a cube \( Q \in K_q \), let \( x_Q \) be the vertex of \( Q \) with smallest coordinates, let \( h_Q : \mathbb{R}^m \to \mathbb{R}^m \) be the similarity map \( x \mapsto q(x - x_Q) \), and let \( M^m_n(P,Q) = h_Q^{-1} M^m_n(P) \); then \( h_Q Q = I^m \) and \( h_Q N^m_n = N^m_n \). We define a closed set \( M^m_n(P,q) = \bigcup \{ M^m_n(P,Q) \mid Q \in K_q \} \) in \( \mathbb{R}^m \),
the union of the copies of the Menger compactum $M_n^m(P)$ in all cubes of $\mathcal{K}_q$, and call it a Menger space. Obviously the copies are compatible in the sense that $M_n^m(P, Q_1) \cap Q_2 = Q_1 \cap M_n^m(P, Q_2)$ if $Q_1, Q_2 \in \mathcal{K}_q$.

The next lemma is based on a construction of Bothe [9], and it makes it possible to replace Nöbeling spaces in 2.4 by Menger spaces in 2.9.

**Lemma 2.6.** Let $(m, n, P, q)$ be a Menger quadruple and $Y \subset N_n^m$ a compact set. Then there is an isotopy $H$ of $\mathbb{R}^m$ with $H_t Q = Q$ for all $Q \in \mathcal{K}_q$ and $t \in I$ moving $Y$ into the Menger space $M_n^m(P, q)$.

**Proof.** Let $Z = \bigcup \{ h_Q[Y \cap Q] \mid Q \in \mathcal{K}_q \}$; then $Z$ is a compact set in $N_n^m \cap I^m$. If $m = 2n + 1$, by [9, §4] there is a homeomorphism $g : I^m \to I^m$ of the form $g = \psi \times \cdots \times \psi$ with $\psi : I \to I$ an increasing homeomorphism such that $gZ \subset M_n^m(P)$; moreover, the assumption $m = 2n + 1$ can be omitted by replacing every occurrence of the expression $m + 1$ in [9, §3 and §4], except the first one in [9, p. 214], by $n - m$. Now define an isotopy $\Psi$ of $I$ by $\Psi_t(x) = (1 - t)x + t\psi(x)$. Then setting $G_t = \Psi_t \times \cdots \times \Psi_t$ we get an isotopy $G$ of $I^m$ with $G_1 = g$. We establish that if $Q, R \in \mathcal{K}_q$, $x \in Q \cap R$, and $t \in I$, then $h_Q^{-1}G_t h_Q(x) = h_R^{-1}G_t h_R(x)$. Noting that $h_Q h_R^{-1}(y) = y + e$ for each $y \in \mathbb{R}^m$ with $e = q(x_R - x_Q)$, this reduces to showing that $G_t(z + e) = G_t(z) + e$ for $z = h_R(x)$, which is easy as $e \in \{-1, 0, 1\}^m$. Hence, we can define an isotopy $H$ of $\mathbb{R}^m$ by setting $H_t(x) = h_Q^{-1}G_t h_Q(x)$ whenever $x \in Q \in \mathcal{K}_q$, and $H$ is the desired one as $H_1[Y \cap Q] \subset h_Q^{-1}gZ \subset M_n^m(P, Q)$ for each $Q \in \mathcal{K}_q$, implying $H_1 Y \subset M_n^m(P, q)$. \qed

**Regular neighbourhoods 2.7.** The next lemma about regular neighbourhoods in the piecewise-linear (PL) sense needed for 2.9 is well-known (see, e.g., [28, 3.11], in whose proof replace “2.6” by “2.5, 2.6” and “three” by “four”), but we supply a proof. For terminology and notation used in the lemma and its proof we refer to [56]. We remark that the assumption “$|N(L, K)|$ is the frontier of $|N(L, K)|$ in $Y$” that is added to the definition of a regular neighbourhood of a (compact) subpolyhedron $X$ of a polyhedron $Y$ in $Y$ on p. 33 of the 1982 printing of [56] can be generalized to the form “$|\tilde{N}(L, K)| \sim |\tilde{N}(L, K)|$ is open in $Y$” (an observation of Jussi Väisälä).
LEMMA 2.8. Let \( m, p \geq 1 \) be integers, let \( \mathcal{K}_p^* \supseteq \mathcal{K}_p \) be the cell complex in \( \mathbb{R}^m \) consisting of all faces of the cubes in \( \mathcal{K}_p \), let \( \mathcal{L} \) be a subcomplex of \( \mathcal{K}_p^* \), let \( L = \bigcup \mathcal{L} \), and let \( N = \bigcup \{ Q \in \mathcal{K}_{3p} \mid Q \cap L \neq \emptyset \} \). Then \( N \) is a regular neighbourhood of \( L \) in \( \mathbb{R}^m \) and hence a PL \( m \)-manifold.

Proof. Let \( \mathcal{K}_p' \) be the barycentric subdivision of \( \mathcal{K}_p^* \) [56, p. 20]. Then the induced subdivision \( \mathcal{L}' \) of \( \mathcal{L} \) is a full subcomplex of \( \mathcal{K}_p' \) [56, 3.2]. Let \( N_{\frac{2}{3}}(\mathcal{L}', \mathcal{K}_p') \) be the \( \varepsilon \)-neighbourhood of \( \mathcal{L}' \) in \( \mathcal{K}_p' \) with \( \varepsilon = \frac{2}{3} \), and let \( \mathcal{K}_p^+ \) be a derived of \( \mathcal{K}_p' \) near \( \mathcal{L}' \) such that the derived neighbourhood \( N(\mathcal{L}', \mathcal{K}_p') \) of \( \mathcal{L}' \) in \( \mathcal{K}_p' \) subdivides the cell complex \( N_{\frac{2}{3}}(\mathcal{L}', \mathcal{K}_p') \) [56, p. 32]. Then \( N \) is the underlying polyhedron of these cellular neighbourhoods: \( N = |N_{\frac{2}{3}}(\mathcal{L}', \mathcal{K}_p')| = |V(\mathcal{L}', \mathcal{K}_p^+)| \) (note that \( \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3} \)!

Observe also that \( |N(\mathcal{L}', \mathcal{K}_p')| \sim |\tilde{N}(\mathcal{L}', \mathcal{K}_p')| \) is open in \( \mathbb{R}^m \). Thus, \( N \) is a regular neighbourhood of \( L \) in \( \mathbb{R}^m \) [56, p. 33]. Hence, \( N \) is a PL \( m \)-manifold by [56, 3.10].

□

LEMMA 2.9. Let \( (m, n, P) \) be a Menger triple and \( \varepsilon > 0 \).

(a) If \( X \) is a separable metric space, \( \dim X \leq n, m \geq 2n + 1 \), and \( f: X \to \mathbb{R}^m \) a bounded continuous map, then there is an embedding \( g: X \to \mathbb{R}^m \) with \( \varrho(g, f) < \varepsilon \) such that \( gX \subset M_{n}^{m}(P, q) \) for some \( q \geq 1 \).

(b) If \( X \subset \mathbb{R}^m \) is a compact set and \( \dim X \leq n \), then there is an embedding \( g: X \to \mathbb{R}^m \) with \( \varrho(g, \text{id}) < \varepsilon \) such that \( gX \subset M_{n}^{m}(P, q) \) for some \( q \geq 1 \).

(c) If \( X \subset \mathbb{R}^m \) is a set, \( n \in \{ m - 1, m \} \), and \( \dim X \leq n \), then there is an embedding \( g: X \to \mathbb{R}^m \) such that \( gX \subset M_{n}^{m}(P) \).

(d) If \( X \subset \mathbb{R}^m \) is a compact set and \( \dim X \leq n \), then there is an \( \varepsilon \)-push \( f \) of \( (\mathbb{R}^m, X) \) such that \( fX \subset M_{n}^{m}(P, q) \) for some \( q \geq 1 \).

Proof. (a) Let \( g_0 \) in place of \( g \) be the embedding given by 2.4(a) with \( \varepsilon \) replaced by \( \varepsilon /2 \); then \( Y = \text{cl} g_0 X \) is compact. Let \( H \) be the isotopy given by 2.6 with this \( Y \) and with \( q \) so large that \( q^{-1} \sqrt{m} < \varepsilon /2 \). Then \( g = H_1 g_0 \) is the desired embedding.

(b) As for (a), with 2.4(a) replaced by 2.4(b).

(c) As for (a), with 2.4(a) replaced by 2.4(c) and with \( q = 1 \).

(d) Let \( N = B(X, \varepsilon) \). By 2.4(d) there is an \((\varepsilon /2)\)-isotopy \( F^* \) of \( \mathbb{R}^m \) with \( \text{supp} F^* \subset N \) such that \( Y = F_1^* X \subset N_{n}^{m} \). Choose an integer \( p \geq 1 \) such that if \( S_0 = \bigcup \{ Q \in \mathcal{K}_p \mid Q \cap Y \neq \emptyset \} \), \( q = 3p \), and
$S = \bigcup \{ Q \in K_q \mid Q \cap S_0 \neq \emptyset \}$, then $S \subset N$ and $q^{-1}\sqrt{m} < \varepsilon/6$. Now $Y \subset S$. Lemma 2.8 implies that $S$ is a PL $m$-manifold. Then $T = \text{cl}(R^m \setminus S)$ is a PL $m$-manifold by [56, 3.14]. Hence, the compact boundary $\partial T$ of $T$ has a collar in $T$ by [56, 2.26], that is, there is an embedding $c: \partial T \times I \to T$ such that $c(x, 0) = x$ for each $x \in \partial T$ and such that $U = c[\partial T \times [0, 1)]$ is an open neighbourhood of $\partial T$ in $T$. We can choose $c$ such that $\text{cl}U \subset N$ and such that $|c(x, s) - x| < \varepsilon/6$ if $(x, s) \in \partial T \times I$. Now let $H$ be the isotopy of $R^m$ given by 2.6. Then $H$ has a restriction to an $(\varepsilon/6)$-isotopy $H'$ of $S$ with $H'_t Y \subset M^m_n (P, q)$. We can extend $H'$ to an $(\varepsilon/2)$-isotopy $H^*$ of $R^m$ with supp $H^* \subset N$ by setting $H_t^* (c(x, s)) = c(H_t^*(1-s)(x), s)$ and $H_t^* \mid T \setminus U = \text{id}$. Then defining $F_t = H_t^* F_t^*$ gives the desired isotopy $F$. □

REMARKS 2.10. 1. From 2.4 and 2.6 we could have deduced, and more easily, 2.9 entirely in the form where, as now in (c), the Menger compactum $M^m_n (P)$ is used in place of the Menger spaces $M^m_n (P, q)$ and all the approximation aspects but the compactness of the support of the isotopy in (d) are omitted i.e., we let $f = 0$, for example, in (a) and $\varepsilon = \infty$. For 4.3 we need this form of 2.9(a) with $(m, n) = (1, 0)$.

2. The form of 2.9 described in 2.10.1 was known: (a) is due to Bothe [9, §4]; (b) (and partly (d)) with $n \geq m - 2$ to Bothe [10, Satz 2 and Satz 3], (b) with $n \leq m - 3$ to Štančko [63, Theorem 1]; for (c) see [23, p. 126]; and (d) is due to Štančko [63, Theorem 2], [21, 1.2] (these proofs for (b)-(d) are given with $P = P_M$ but they are valid with an arbitrary $P$). For the earlier history of these results see [23, p. 129], completed with [6, p. 98] The case $(m, n) = (1, 0)$ is classical [23, 1.3.15 and 1.3.F]. It is apparently not known whether 2.9(c) holds with $n \geq 2$ and $n - 2 \leq m \leq 2n$.

3. The same idea as in the proof of the approximation result 2.9(d) has also been used by Geoghegan and Summerhill [28, 3.B], Edwards [21, pp. 208–209], and Väisälä [67].

We need the maximum norm $\|x\| = \max_{1 \leq i \leq n} |x_i|$ for $x \in R^m$.

**An ultrametric 2.11.** A metric $d$ on a set $X$ with $d(x, y) \leq \max(d(x, z), d(y, z))$ for all $x, y, z \in X$ is called an ultrametric. Consider $M^m_n (P)$ with $n = 0$. Note that different cubes of the same rank are now disjoint. This allows us to define an ultrametric $d$ on $M^m_0 (P)$ by setting
\[ d(x, y) = l_i \] for two distinct points \( x, y \in M_0^m(P) \) if \( i \geq 0 \) is the greatest integer for which \( x \) and \( y \) are in the same cube of rank \( i \). If \( d(x, y) = l_i \), then \( \|x - y\| \leq l_i \) and \( \|x - y\| \geq (p_{i+1} - 2)l_{i+1} = (1 - 2/p_{i+1})l_i \geq l_i/3 \). Thus, \( d(x, y)/3 \leq \|x - y\| \leq d(x, y) \) for all \( x, y \in M_0^m(P) \), saying that \( d \) is bi-Lipschitz equivalent to the maximum metric.

3. Assouad dimension

In this section, after first stating notation for the familiar dimension concepts used to measure fractals, we introduce Assouad dimension and discuss it generally. Then we prove a lemma about the Assouad dimension of a Menger compactum and give an estimate for dimension in terms of Assouad dimension.

**Notation 3.1.** Consider a metric space \( X \). Let \( \mathcal{H}^s(X) \) denote the \( s \)-dimensional Hausdorff measure of \( X \) for \( s \geq 0 \), and let \( \dim_H X \) denote the Hausdorff dimension of \( X \) (with \( \mathcal{H}^s(X) = \infty \) and \( \dim_H X = \infty \) if \( X \) is nonseparable) [24], [31]. The upper box-counting dimension \( \dim_B X \) of \( X \) can be defined as follows [24]. If \( X \) is totally bounded, denoting by \( N_\varepsilon(X) \) for \( \varepsilon > 0 \) the smallest positive number of sets of diameter at most \( \varepsilon \) needed to cover \( X \), let \( \dim_B X = \limsup_{\varepsilon \to 0} \log N_\varepsilon(X)/\log(1/\varepsilon) \); otherwise let \( \dim_B X = \infty \). The lower box-counting dimension \( \dim_B^* X \) of \( X \) is defined otherwise similarly but with \( \limsup \) replaced by \( \liminf \) [24]. We denote the packing dimension of \( X \) by \( \dim_P X \); it can be defined as the infimum of the numbers \( \sup_{i \geq 1} \dim_B X_i \) over countable covers \( (X_i)_{i \geq 1} \) of \( X \) [24, Sections 3.3 and 3.4]; in the construction of packing measures leading to an equivalent definition, we must now, with \( X \) an arbitrary metric space, define the number \( P^s_\delta(X) \) of [24, (3.22)] for \( s > 0 \) and \( \delta > 0 \) as the supremum of the sums \( \sum_{j \in J} (2r_j)^s \) over countable families \( (x_j)_{j \in J} \) and \( (r_j)_{j \in J} \) with \( x_j \in X \) and \( r_j \in [0, \delta] \) such that \( d(x_i, x_j) > r_i + r_j \) if \( i \neq j \). It is well-known that \( \dim X \leq \dim_H X \leq \dim_B X \leq \dim_B X \) and \( \dim_{\mathcal{H}} X \leq \dim_P X \leq \dim_B X \) [24], [31].

**Definition 3.2.** Let \( X \) be a metric space and \( d \) its metric. Suppose that \( s \geq 0 \) and \( C \geq 0 \) are numbers such that \( \text{card } A \leq C(b/a)^s \) whenever \( a > 0 \) and \( b \geq a \) are numbers and \( A \subseteq X \) a set with \( a \leq d(x, y) \leq b \) if \( x, y \in A \) and \( x \neq y \). Then \( X \) is called \((C, s)\)-homogeneous. We say
that $X$ is $s$-homogeneous if $X$ is $(C, s)$-homogeneous for some $C$ and that $X$ is homogeneous if $X$ is $s$-homogeneous for some $s$. The infimum in $[0, \infty]$ of the numbers $s$ (if any) for which $X$ is $s$-homogeneous is called the Assouad dimension of $X$ and denoted by $\dim_A X$.

Obviously only finite sets $A$ need be tested, and if $X$ is $(C, s)$-homogeneous and non-empty, then $C \geq 1$. The Assouad dimension $\dim_A X$ of a metric space $X$ was introduced by Patrice Assouad [3], [4], [5] under the name of the metric dimension $\dim(X, d)$ of $X$. Obviously, $X$ has a finite Assouad dimension exactly when $X$ is homogeneous. We do not use the latter term in the sequel; we introduced it only in order to be logical. Assouad dimension could be said to measure the size of a metric space in all scales. Note that if $\dim_A X < \infty$, then every bounded subset of $X$ is totally bounded and $X$ is separable.

After presenting some facts and examples we review the history of this and related concepts in 3.6.

**FACTS 3.3.** We recall the following elementary basic properties of Assouad dimension. We prove them in Theorems A.3 and A.5 of Appendix A. First, $s$-homogeneity and $\dim_A$ are bi-Lipschitz invariants.

If $Y \subset X$, then $\dim_A Y \leq \dim_A X$, with equality if $Y$ is dense. If $X = \bigcup_{i=1}^{n} X_i$, then $\dim_A X = \max_{1 \leq i \leq n} \dim_A X_i$. If finite Cartesian products of metric spaces are metrized in any of the standard ways, then $\dim_A (X \times X') \leq \dim_A X + \dim_A X'$, with equality if $X = X'$. If $m \geq 0$, then $\mathbb{R}^m$ is $m$-homogeneous and $\dim_A X = m$ for each set $X \subset \mathbb{R}^m$ with interior points. If $p \in (0, 1)$, then $\dim_A (X, d^p) = (1/p) \dim_A (X, d)$. We have $\dim X \leq \dim_H X \leq \dim_A X$ for every metric space $X$ (see 3.4 below). If $X$ is bounded, then $\dim_B X \leq \dim_A X$ by 3.4. Always $\dim_P X \leq \dim_A X$. Finally, $X$ has a finite Assouad dimension if and only if $X$ has the doubling property that there is $n \in \mathbb{N}$ such that for each $r > 0$, each closed ball of radius $r$ can be covered by less than $n$ closed balls of radius $r/2$.

In A.10 we prove the new useful result that Assouad dimension is invariant under the inversion $u : \mathbb{R}^m \setminus \{0\} \to \mathbb{R}^m \setminus \{0\}$, $x \mapsto x/|x|^2$, of $\mathbb{R}^m$ in the unit sphere. Note that this invariance is shared also by Hausdorff dimension and by packing dimension but neither by upper box-counting dimension nor by lower box-counting dimension. For example, if $X = \{2^{-k} \mid k \in \mathbb{N}\} \subset \mathbb{R}^1 \setminus \{0\}$, then $\overline{\dim_B} X = 0$ while $\dim_B uX = \infty$. 
Lemma 3.4. Let $X$ be a bounded $(C, s)$-homogeneous metric space with $b = \text{diam } X > 0$. Then $N_\varepsilon(X) \varepsilon^s \leq C(2b)^s$ for each $\varepsilon \in (0, b]$, and $\mathcal{H}^s(X) \leq C(2b)^s$.

Proof. Choose a maximal subset $A \subset X$ such that $d(x, y) \geq \varepsilon/2$ if $x, y \in A$, $x \neq y$. Then clearly $N_\varepsilon(X) \leq \text{card } A \leq C(2b/\varepsilon)^s$, and the lemma follows. □

Examples 3.5. 1. As an illustration, we compute the Assouad dimension of the field of $p$-adic numbers. We first recall its definition and a few of its basic properties [60]. Thus, let $p$ be a prime number. There exists a field $\mathbb{Q}_p$, called the field of $p$-adic numbers, which contains $\mathbb{Q}$ as a subfield and for which for each real number $\alpha > 1$ there exists a function $| \cdot | = | \cdot |_\alpha : \mathbb{Q}_p \to \mathbb{R}$, called the $p$-adic valuation with base $\alpha$, such that $|0| = 0$, $|x| > 0$ if $x \neq 0$, $|x + y| \leq \max(|x|, |y|)$, $|xy| = |x||y|$, $|p| = \alpha^{-1}$, and $|n| = 1$ if $n \in \mathbb{Z} \setminus p\mathbb{Z}$, such that $\mathbb{Q}_p$ is complete with respect to the ultrametric $d_\alpha(x, y) = |x - y|$, and such that $\mathbb{Q}$ is dense in $\left(\mathbb{Q}_p, d_\alpha\right)$. With these properties $\mathbb{Q}_p$ is unique up to an isomorphism which keeps $\mathbb{Q}$ pointwise fixed and which is isometric with respect to $d_\alpha$ for each $\alpha$; moreover, $d_\beta = d_\alpha^{\log \beta / \log \alpha}$ if $\beta > 1$. Now fix the base $\alpha$ (a standard choice is $\alpha = p$). The closed unit ball $\mathbb{Z}_p = \{x \mid |x| \leq 1\}$ of $\mathbb{Q}_p$ is a compact subring of $\mathbb{Q}_p$, called the ring of $p$-adic integers, and $\mathbb{Z}_p$ contains $\mathbb{Z}$ as a dense subset. We have $\{ |x| \mid x \in \mathbb{Q}_p, x \neq 0 \} = \{ \alpha^n \mid n \in \mathbb{Z} \}$. Finally, if $S_p = \{0, 1, \ldots, p - 1\}$, then $\mathbb{Z}_p$ is the union of the $p$ disjoint closed balls $\{ i + p\mathbb{Z}_p : i \in S_p \}$ of radius $\alpha^{-1}$ and with mutual distances 1.

Now we show that $\text{dim}_A \mathbb{Q}_p = \log p / \log \alpha$. Thus, denote $s = \log p / \log \alpha$, and consider numbers $0 < a \leq b$ and a set $A \subset \mathbb{Q}_p$, card $A \geq 2$, with $a \leq |x - y| \leq b$ if $x, y \in A$ and $x \neq y$. Choose $m, n \in \mathbb{Z}$, $m \geq n$, with $\alpha^{-m-1} \leq a \leq \alpha^{-m}$ and $\alpha^{-n} \leq b < \alpha^{-n+1}$. Then $b/a \geq \alpha^{m-n}$, and it is easy to see that card $A \leq p^{m-n+1}$. Thus, card $A \leq p(b/a)^s$. By choosing $a = \alpha^{-m}$, $b = \alpha^{-n}$, and $A = \{ \sum_{k=1}^m i_k p^k \mid i_k \in S_p \}$ we have that card $A = p(b/a)^s$ and that $b/a \to \infty$ as $m-n \to \infty$. Hence, $\mathbb{Q}_p$ is $(p, s)$-homogeneous, and $\text{dim}_A \mathbb{Q}_p = \text{dim}_A \mathbb{Z}_p = s$. Note that as $\alpha$ varies, $s$ assumes all positive real values. It follows by A.14 below that $\mathbb{Q}_p$ can be bi-Lipschitz embedded in $\mathbb{R}^k$ if and only if $k > s$. By A.14 again, here the “only if” statement also follows from the fact that $0 < \mathcal{H}^s(\mathbb{Z}_p) < \infty$ and $\text{dim}_H \mathbb{Q}_p = \text{dim}_H \mathbb{Z}_p = s$ by the self-similarity
of $\mathbb{Z}_p$, cf. [24, Propositions 9.6 and 9.7]. Note that $\dim Q_p = 0$ by ultrametricity.

2. A familiar space of uniformly locally finite but globally infinite Assouad dimension is the hyperbolic plane $\mathbb{H}^2$. We take $\mathbb{H}^2$ to be the half-plane $\{ x \in \mathbb{R}^2 \mid x_2 > 0 \}$ endowed with the hyperbolic metric $d$ given by the element of length $|dx|/x_2$. Locally $\mathbb{H}^2$ is of Assouad dimension 2 as locally the Euclidean and hyperbolic metrics on $\mathbb{H}^2$ are bi-Lipschitz equivalent. To see that $\dim_A \mathbb{H}^2 = \infty$, let $s \geq 0$, $p \in \mathbb{N}$ with $p \geq 2$, $A = \{1, \ldots, p\} \times \{1\} \subset \mathbb{H}^2$, $a = \log 2$, and $b = 2 \log p$. Then $a \leq d(x,y) \leq b$ for all $x,y \in A$ with $x \neq y$ by [72, 2.41], and card $A/(b/a)^{\alpha} \geq (\log 2/2)^{\alpha}p/(\log p)^{\alpha} \to \infty$ as $p \to \infty$. Thus, $\dim_A \mathbb{H}^2 \geq s$.

REMARK 3.6. The first concept of dimension related to Assouad dimension was the (uniform upper local) dimensional order of a subset of a Euclidean space due to Bouligand [11, §6], which is defined in an external way with the aid of Lebesgue measure. Due to Appert [1], the definition can be made intrinsic. Now, if this dimensional order is suitably specified, it gives the Assouad dimension of the set in question; see [3, p. 734], [4, 3.17.1 and 3.17.2, and [71, p. 630]. We give a proof for this characterization of Assouad dimension in Theorem A.12. However, it seems to me that the earlier approach of [11] and [1] was interested solely in the smaller scales and did not take into account at all the larger scales which are also so essential in the present definition and interest.

In more recent research, Le Cam [38, p. 39] noted the $k$-homogeneity of $\mathbb{R}^k$ and considered similar conditions in specific situations in statistics; see also below. Vol’berg and Konyagin [70], [71] rediscovered Assouad dimension under the name of uniform metric dimension; see 6.5. Larman [37] and Movahedi-Lankarani [48] have defined two concepts closely related to Assouad dimension, those of the dimensional number $\dim_{-n} X$ and the metric dimension $\dim_{-1} X$, respectively, of a metric space $X$. It is an easily verified observation of Assouad [4] that $\dim_{-n} X$ is the infimum of the numbers $s \geq 0$ for which there is an open cover of $X$ each member of which is $s$-homogeneous. Similarly, it is easily seen that $\dim_{-r} X$ is the infimum of the numbers $s \geq 0$ for which there are numbers $r > 0$ and $C > 0$ such that each closed ball of radius $r$ in $X$ is $(C,s)$-homogeneous. We con-
clude that \( \dim_n X \leq \dim_m X \leq \dim_A X \), with equality in the latter or both places if \( X \) is totally bounded or compact, respectively, and that \( \dim_H X \leq \dim_n X \) if \( X \) is separable. But if, for example, \( X \subset \mathbb{R} \) is either the totally bounded set \( \{1, \frac{1}{2}, \frac{1}{3}, \ldots \} \) or the set \( \mathbb{N} \), then \( \dim_n X = 0 \) and \( \dim_m X = 0 \), respectively, whereas \( \dim_A X = 1 \) in both cases.

It follows that in our main result, Theorem 4.3, where we metrize a non-empty separable metrizable space \( X \) by a totally bounded metric with \( \dim_A X = \dim X \), for this metric it also holds that \( \dim_H X = \dim_B X = \overline{\dim_B} X = \dim_F X = \dim_n X = \dim_m X = \dim X \).

Prior to the definition of Assouad dimension in [3], metric spaces with the doubling property have been considered by Coifman and Weiss [15] in harmonic analysis ("spaces of homogeneous nature"), utilized by Le Cam [39, Assumption 4] in statistics, and studied as such by Assouad himself [2, 1.21]. However, Le Cam [39, Definition 1] is more interested in a weaker form of the doubling property where \( r \geq r_0 \) for a constant \( r_0 > 0 \), allowing thus spaces of infinite topological dimension. The homogeneously totally bounded metric spaces introduced by Tuikia and Väisälä [66] in the theory of quasiconformal maps also coincide with the class of spaces of finite Assouad dimension; see A.3. The class of metric spaces \( X \) with \( \dim_m X < \infty \) has been applied with equivalent definitions by Larman [37] ("\( \beta \)-spaces"), Davies [47, Example 2], and Howroyd [30] ("finite structural dimension").

**Notation 3.7.** We now return to our main theme and next characterize the sequences \( P = (p_1, p_2, \ldots) \) of integers \( \geq 3 \) for which the Assouad dimension of the \( n \)-dimensional Menger compactum \( M_n^m(P) \) equals \( n \). This being always so if \( n = m \), for the case \( n < m \) let \( A(P) \) be the set of the numbers \( i > 0 \) for which there is a constant \( \mu_i > 0 \) such that \( p_{i+1} \ldots p_j \geq \mu_i 2^{(j-i)/t} \) whenever \( 0 \leq i < j \), and let \( a(P) = \inf A(P) \). Then \( a(P) \leq a(P_M) = \log 2/\log 3 \) for each \( P \), and \( a(P) = 0 \) if \( \lim_{i \to \infty} p_i = \infty \). Note that each Menger space \( M_n^m(P, q) \) contains \( \mathbb{Z}^m \) and is thus itself of Assouad dimension \( m \) but that \( \dim_A X \leq \dim_A M_n^m(P) \) for each bounded subset \( X \) of \( M_n^m(P, q) \). I gave a (defective) proof for the special case \( \dim_A M_n^m \left( (2^i+1) \right) = n \) of the following central lemma already in 1980.

**Lemma 3.8.** Let \((m, n, P)\) be a Menger triple with \( n < m \). Then \( \dim_A M_n^m(P) = n \) if and only if \( a(P) = 0 \).
Proof. We are allowed to replace the Euclidean norm on $\mathbb{R}^m$ by the bi-Lipschitz equivalent maximum norm. Let $M = M^m_n(P)$.

Suppose first that $a(P) = 0$. To prove that $\dim_A M = 0$ it suffices to show $M$ to be $(n + \varepsilon)$-homogeneous for each $\varepsilon > 0$. Thus, let $\varepsilon > 0$. Let $r \geq 2$ be the number of all $n$-faces of $I^m$. Then each cube of rank $i \geq 0$ contains at most $r p_{i+1}^n$ cubes of rank $i + 1$ and, hence, at most $\varphi(i, j) = r^{j-i}(p_{i+1} \ldots p_j)^n$ cubes of rank $j$ for each $j > i$. We establish that $\varphi(i, j) \leq \alpha_\varepsilon (l_i/l_j)^{n+\varepsilon}$ for some constant $\alpha_\varepsilon > 0$. Let $\delta = \varepsilon \log 2 / \log r$. Then $\delta \in A(P)$ as $a(P) = 0$. Let $\mu_{\delta}$ be the respective constant. Define $\alpha_\varepsilon = \mu_{\delta}^{-\varepsilon}$. Then

$$\varphi(i, j)(l_i/l_j)^{(n+\varepsilon)} = r^{j-i}(p_{i+1} \ldots p_j)^{-\varepsilon} \leq r^{j-i} \mu_{\delta}^{(j-i)/\delta} = \alpha_\varepsilon.$$

Now consider numbers $0 < a \leq b$ and a set $A \subset M$ with $a \leq \|x - y\| \leq b$ if $x, y \in A$ and $x \neq y$. We must prove that $\text{card} A \leq C_\varepsilon (b/a)^{n+\varepsilon}$ for some constant $C_\varepsilon > 0$. Since $M \subset I^m$, we may assume that $b \leq 1$. Choose integers $0 \leq i \leq j$ with $l_{i+1} < b \leq l_i$ and $l_{j+1} < a \leq l_j$. We show that there are constants $\beta, \gamma > 0$ such that $A$ meets at most $\beta(b/l_{i+1})^n$ cubes of rank $i + 1$ and such that $\text{card}(A \cap Q) \leq \gamma l_j/a^n$ for each cube $Q$ of rank $j$. If $j > i$, this suffices, for then

$$\text{card} A \leq \beta(b/l_{i+1})^n \varphi(i + 1, j) \gamma(l_j/a)^n \leq \beta(b/l_{i+1})^{n+\varepsilon} \alpha_\varepsilon (l_{i+1}/l_j)^{n+\varepsilon} \gamma(l_j/a)^{n+\varepsilon} = \alpha_\varepsilon \beta \gamma(b/a)^{n+\varepsilon}$$

whenever $j > i + 1$ and $\text{card} A \leq \beta(b/l_{i+1})^{n+\varepsilon} \gamma(l_{i+1}/a)^n = \beta \gamma(b/a)^n$ whenever $j = i + 1$.

We need the fact that $\mathbb{R}^n$ with the maximum norm is $(c_n, n)$-homogeneous for some $c_n > 0$. For $k \in \{i, j\}$ let $Q$ be a cube of rank $k$, let $F$ be an $n$-face of $Q$, let $\mathcal{L}$ be the family of all cubes of rank $k + 1$ contained in $Q$ and meeting $F$, let $S = \bigcup \mathcal{L}$, and let $\pi : S \to F$ be the ‘orthogonal projection’. Here $\pi$ is defined by conjugating the ordinary orthogonal projection of $\mathbb{R}^n$ onto $\mathbb{R}^n = \mathbb{R}^n \times 0$ with an isometry of $\mathbb{R}^m$ mapping $F$ onto $l_k I^n$ and $S$ onto $l_k I^n \times l_{k-1} I^{m-n}$. First let $k = j$. Then $a \leq \|\pi(x) - \pi(y)\| \leq l_j$ if $x, y \in S \cap S, x \neq y$. Hence, $\text{card}(A \cap S) \leq c_n(l_j/a)^n$. Thus, $\text{card}(A \cap Q) \leq r c_n l_j/a^n$, and we can choose $\gamma = r c_n$. Now let $k = i$. Express $\mathcal{L}$ as a disjoint union $\mathcal{L} = \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_{2^n}$ such that each family $\mathcal{L}_q$ is disjoint. Consider a
family \( \mathcal{L}_q \). If \( Q_1, Q_2 \in \mathcal{L}_q \), \( Q_1 \neq Q_2 \), \( x \in A \cap Q_1 \), and \( y \in A \cap Q_2 \), then \( l_{i+1} \leq \| \pi(x) - \pi(y) \| \leq b \). Hence, \( A \) meets at most \( c_n(b/l_{i+1})^n \) cubes of \( \mathcal{L}_q \). Since clearly \( A \) meets at most \( 3^m \) cubes of rank \( i \), it follows that \( A \) meets at most \( \beta(b/l_{i+1})^n \) cubes of rank \( i + 1 \) with \( \beta = 3^mr2^n c_n \).

Consider finally the case \( j = i \). In the previous notation with \( k = i \) we have \( a \leq \| \pi(x) - \pi(y) \| \leq b \) if \( x, y \in A \cap S, x \neq y \). Hence, \( \text{card}(A \cap S) \leq c_n(b/a)^n \). It follows that \( \text{card} A \leq 3^m r c_n(b/a)^n \). This completes the proof for the sufficiency part.

To prove the necessity part, suppose that \( \dim_A M = n \). Let \( \varepsilon > 0 \). Then there is \( C_\varepsilon > 0 \) such that \( M \) is \( (C_\varepsilon, n + \varepsilon) \)-homogeneous. Consider integers \( 0 \leq i < j \). Choose a cube \( Q \) of rank \( i \). Since \( n < m \) implies that \( Q \) has at least one pair of disjoint \( n \)-faces, we conclude that \( Q \) contains at least \( 2p_{i+1}^n \) cubes of rank \( i + 1 \) and, hence, at least \( \psi(i, j) = 2^{j-i}(p_{i+1} \ldots p_j)^n \) cubes of rank \( j \). For each cube \( R \subset Q \) of rank \( j \) let \( x_R \) be the vertex of \( R \) with smallest coordinates. Then \( x_R \in M \). Moreover, if \( R_1, R_2 \subset Q \) are different cubes of rank \( j \), then \( l_{j} \leq \| x_{R_1} - x_{R_2} \| \leq l_i \). Hence, \( Q \) contains at most \( C_\varepsilon(l_i/l_j)^{n+\varepsilon} \) cubes of rank \( j \). This implies \( \psi(i, j) \leq C_\varepsilon(l_i/l_j)^{n+\varepsilon} \). It follows that \( p_{i+1} \ldots p_j \geq C_\varepsilon^{-1/\varepsilon} 2^{(j-i)/\varepsilon} \). Thus, \( \varepsilon \in A(P) \). Hence, \( a(P) = 0 \).

**Corollary 3.9.** Let \( 0 \leq n \leq m \geq 1 \) and \( P_0 = (3, 4, 5, \ldots) \). Then \( \dim_A M_n^m(P_0) = n \). If \( q \geq 1 \) and \( X \subset M_n^m(P_0, q) \) is a bounded set, then \( \dim_A X \leq n \). If \( n = 0 \) and \( M_n^m \) is the set \( M_0^m(P_0) \) endowed with the ultrametric constructed in 2.11, then \( \dim_A M_n^m = 0 \).

**Lemma 3.10.** If \( X \subset \mathbb{R}^m \) is a compact set, then \( \text{dem} X \leq \dim_H X \leq \dim_A X \).

**Proof.** Choose \( k \in \{0, 1, \ldots, m \} \) with \( k \leq \dim_H X < k + 1 \); then \( \mathcal{H}^{k+1}(X) = 0 \), which implies \( \text{dem} X \leq k \) by [43, 6.15]. Hence, \( \text{dem} X \leq \dim_H X \).

**4. Antifractal metrization**

**Convention and Notation 4.1.** In this section we assume that the separable metrizable spaces \( X \) (possibly subsets of \( \mathbb{R}^m \)) under consideration are non-empty; the difference between the values \( \dim \emptyset = \text{dem} \emptyset = -1 \) and \( \dim_H \emptyset = \dim_B \emptyset = \overline{\dim}_B \emptyset = \dim_A \emptyset = 0 \) will then
not embarrass. We let $M(X)$, $M_{th}(X)$, and $M_{c}(X)$ denote, respectively, the set of all compatible metrics, compatible totally bounded metrics, or compatible complete metrics on $X$.

**Theorem 4.2.** Let $m \geq 1$ be an integer and $\varepsilon > 0$.

(a) If $X$ is a separable metric space, $n = \dim X < \infty$, $m \geq 2n + 1$, and $f : X \to \mathbb{R}^m$ a bounded continuous map, then there is an embedding $g : X \to \mathbb{R}^m$ with $\varrho(g, f) < \varepsilon$ such that $\dim_A gX = n$.

(b) If $X \subset \mathbb{R}^m$ is a compact set, then there is an embedding $g : X \to \mathbb{R}^m$ with $\varrho(g, \text{id}) < \varepsilon$ such that $\dim_A gX = \dim X$.

(c) If $X \subset \mathbb{R}^m$ is a set, $n \in \{m - 1, m\}$, and $\dim X \leq n$, then there is a bounded embedding $g : X \to \mathbb{R}^m$ such that $\dim_A gX \leq n$.

(d) If $X \subset \mathbb{R}^m$ is a compact set, then there is an $\varepsilon$-push $f$ of $(\mathbb{R}^m, X)$ such that $\dim_A fX \leq \text{dem } X$.

**Proof.** This follows from 2.9 and 3.9. □

**Theorem 4.3.** Every separable metrizable space $X$ can be metrized by a totally bounded metric for which $\dim_A X = \dim X$, or in other words, which is flat. If $\dim X = 0$, this metric can be chosen to be an ultrametric.

**Proof.** Suppose first that $\dim X = \infty$. Since $X$ is separable, we can metrize $X$ by a totally bounded metric. Then $\dim_A X = \infty$. Suppose now that $n = \dim X < \infty$. By 4.2(a) there is a bounded embedding $g : X \to \mathbb{R}^{2n+1}$ with $\dim_A gX = n$; then the metric on $X$ for which $g$ is isometric is totally bounded and satisfies $\dim_A X = n$. If $n = 0$, then $X$ can be embedded by 2.10.1 in the ultrametric space $M^1$ of 3.9, which yields the second part. □

**Corollary 4.4.** If $X$ is a separable metrizable space, then

\[
\dim X = \min \{ \dim_A(X, d) \mid d \in M(X) \} = \min \{ \dim_A(X, d) \mid d \in M_{th}(X) \}.
\]

**Theorem 4.5.** Every locally compact separable metrizable space $X$ can be metrized by a complete metric for which $\dim_A X = \dim X$. 
Proof. Since $X$ is completely metrizable, this is clear if $\dim X = \infty$. If $X$ is compact, this follows from 4.3. Thus, we may assume that $X$ is noncompact and that $n = \dim X < \infty$. Let $Y$ be the one-point compactification of $X$ by a point $\omega$; then $Y$ is metrizable and $\dim Y = n$ by [31, Corollary 2, p. 32]. Let $m = 2n + 1$. Then by 4.2(a) there is an embedding $g: Y \to \mathbb{R}^m$ with $\dim_A gY = n$. Replacing $g$ by $g - g(\omega)$ we may assume that $g(\omega) = 0$. Let $u: \mathbb{R}^m \setminus \{0\} \to \mathbb{R}^m \setminus \{0\}$ be the inversion of $\mathbb{R}^m$ in the unit sphere. Then $ugX$ is closed in $\mathbb{R}^m$ and $\dim_A ugX = \dim_A gX = n$. Hence, the metric on $X$ for which $ugX$ is isometric is the desired one.

**Corollary 4.6.** If $X$ is a locally compact separable metrizable space, then

$$\dim X = \min\{ \dim_A (X, d) \mid d \in M_c(X) \}.$$ 

**Theorem 4.7.** If $X \subset \mathbb{R}^m$ is a closed set, then there is a closed embedding $f: X \to \mathbb{R}^m$ such that $\dim_A fX = \dim X$.

Proof. We may assume $0 \notin X$. If $u$ is as in the previous proof, then $Y = uX \cup \{0\}$ is compact, and by 4.2(b) there is an embedding $g: Y \to \mathbb{R}^m$ with $g(0) = 0$ such that $\dim_A guX = \dim X$. Now $f = ugu|X$ is the desired embedding.

**Theorem 4.8.** Let $X$ be a compact set in $\mathbb{R}^m$. Then $\text{dem} X \leq \dim_A fX$ for all homeomorphisms $f: \mathbb{R}^m \to \mathbb{R}^m$, and for each $\varepsilon > 0$ there is an $\varepsilon$-push $f$ of $(\mathbb{R}^m, X)$ with $\text{dem} X = \dim_A fX$.

Proof. The first part follows from 3.10 since $\text{dem} fX = \text{dem} X$. The second part follows now from 4.2(d).

**Corollary 4.9.** If $X$ is a compact set in $\mathbb{R}^m$ then

$$\text{dem} X = \min\{ \dim_A fX \mid f: \mathbb{R}^m \to \mathbb{R}^m \text{ a homeomorphism} \}.$$ 

**Corollary 4.10.** A compact set $X \subset \mathbb{R}^m$ is tame if and only if $fX$ is flat for some homeomorphism $f: \mathbb{R}^m \to \mathbb{R}^m$. In other words, $X$ is wild if and only if $fX$ is fractal for each homeomorphism $f: \mathbb{R}^m \to \mathbb{R}^m$. 


Remark 4.11. Our results have many predecessors. Let $X$ be a separable metrizable space. First Pontrjagin and Schnirelmann [53] proved for a compact $X$ that $\dim X = \inf \{ \overline{\dim_0}(X, d) \mid d \in M(X) \}$. But, as observed by the author [41, 4.7], in the case $n = \dim X < \infty$ in [53] the compact space $X$ is embedded as a subspace of $\mathbb{R}^{2n+1}$ in such a way that, in fact, $\overline{\dim_B} X = \overline{\dim_B} X = n$. In [41] from this result and the existence of a dimension-preserving metrizable compactification of $X$ it was deduced that also an arbitrary $X$ admits a totally bounded metric for which $\overline{\dim_B} X = \dim X$ (and, hence, $\overline{\dim_H} X = \overline{\dim_B} X = \dim X$). In the same manner Bruijning [13] derived from [53] the weaker result that $\dim X = \inf \{ \overline{\dim_B}(X, d) \mid d \in M_{tb}(X) \}$. Marczewski [46, 31, Theorem VI(15)] proved directly (the proof is due to S. Eilenberg by [46, p. 87]) that $X$ admits a totally bounded metric for which $\overline{\dim_H} X = \dim X$; if $X \subset \mathbb{R}^2$ and $\dim X = 1$, by [46, p. 88] this metric can be obtained through embedding $X$ in a modification of the Sierpiński universal curve $M^2_1(P_M)$. Being apparently unaware of [53] but by applying Baire’s theorem on function spaces as it was also applied in [46], Prosser [54] proved for a compact $X$ that $\overline{\dim_B}(X, d) = \dim X$ for some $d \in M(X)$ (his $md(X)$ equals our $\overline{\dim_B} X$, but his equality (6) does not hold because its right-hand side equals $\overline{\dim_B} X$). Väisälä [67] proved the analogue of 4.8 where $\dim_A$ is replaced by $\overline{\dim_H}$; he made use of $M_n^m(P)$ with $P = (2^{i+1})_{i \geq 1}$. The author [41, 4.7] gave an alternative construction of $d \in M_{tb}(X)$ with $\overline{\dim_B}(X, d) = \dim X$ by resorting in the case $n = \dim X < \infty$ to $M_n^{2n+1}(P)$ with $P$ of [67]. This same proof, with $P = (3^i)_{i \geq 1}$, was also independently given by Nguyen To Nhu [55, 2-8], who later [51, Theorem 3] proved a stronger version concerning majorizability of the function $\varepsilon \mapsto N_\varepsilon(X)\varepsilon^n$ as $\varepsilon \to 0$.

It should also be mentioned that a result of Kahnert [33, Satz] implies that every locally compact separable metrizable topological group $G$ can be metrized by a left-invariant metric for which $\overline{\dim_B} U = \dim G$ for each compact neighbourhood $U$ of the neutral element of $G$ and for which, consequently, $\overline{\dim_H} G = \dim_B G = \dim_G$. A remark [33, p. 22] deals with a similar strengthening as [51, Theorem 3] referred to just.

Remarks 4.12. 1. Neither in 4.3 nor in 4.5 the condition $\dim_A X = \dim X$ can be strengthened in the case $n = \dim X < \infty$ to the condition
that \( X \) is \( n \)-homogeneous. This is clear if \( n = 0 \) because 0-homogeneity is equivalent to finiteness. Generally, recalling an example in [46], let \( C \) be the usual Cantor set and \( X = C \times I^n \); then \( X \) is a compact metrizable space with \( \dim X = n \). Now let \( X \) be metrized in an arbitrary way. Then \( \dim A = n \) implies that \( \mathcal{H}^n(A) > 0 \) for each of the uncountably many components \( A \) of \( X \) by [31, Theorem VII 2], and thus \( \mathcal{H}^n(X) = \infty \). Hence, \( X \) is not \( n \)-homogeneous by 3.4.

2. It remains open whether for every wild compact set \( X \subset \mathbb{R}^m \) and for \( n = \text{dim} X \) there is a homeomorphism \( f : \mathbb{R}^m \to \mathbb{R}^m \) such that \( fX \) is \( n \)-homogeneous. It is thus interesting to note that Keesling [34, pp. 371–372] has constructed a wild Cantor set \( X \subset \mathbb{R}^3 \) (necessarily of dimension 1) with \( 0 < \mathcal{H}^1(X) < \infty \).

5. Porous sets

In this section we first show that the important class of porous sets in a Euclidean space can be characterized in terms of the Assouad dimensions of these sets. From our characterization of dimension based on Assouad dimension it then follows a characterization for a set in \( \mathbb{R}^m \) to be porous up to a self-homeomorphism of \( \mathbb{R}^m \). We finally present an upper bound to the Assouad dimension of a set whose porosity is close to its extremity. We employ notation and terminology from A.1 in the appendix.

**Porous sets 5.1.** Let \( Y \) be a metric space with \( d(Y) > 0 \), and let \( X \) be a subset of \( Y \). If there is \( \alpha \in (0, 1] \) such that for all \( x \in Y \) and \( r \in (0, \infty) \) with \( r \leq d(Y) \) there is \( y \in \overline{B}(x, r) \) with \( B(y, \alpha r) \cap X = \emptyset \), then we say that \( X \) is porous or also \( \alpha \)-porous in \( Y \). Note that \( X \) is \( \alpha \)-porous if and only if \( \overline{X} \) is \( \alpha \)-porous. It is easy to see that if \( X \) is only assumed to satisfy this \( \alpha \)-porosity condition for all points \( x \in X \) in place of all points \( x \in Y \), then \( X \) is still \( \alpha' \)-porous in \( Y \) with \( \alpha' = \alpha/(1 + \alpha) \in (0, \frac{1}{2}] \).

This term is adopted from Väisälä [69]; he has \( Y = \mathbb{R}^m \), \( m > 0 \), and proves that each \( \eta \)-quasisymmetric image in \( \mathbb{R}^m \) (in the sense of A.1) of an \( \alpha \)-porous set in \( \mathbb{R}^m \) is \( \alpha' \)-porous with \( \alpha' \) depending only on \( (\alpha, \eta) \). If the \( \alpha \)-porosity condition above is assumed only with the limitation \( r \leq r_0 \) for some \( r_0 \in (0, d(Y)) \), then, if \( Y \) is bounded or if \( X \) is bounded and \( Y \) connected, the set \( X \) is still \( \alpha' \)-porous in \( Y \) with
\( \alpha \)' depending only on \((\alpha, r_0/d(Y))\) or \((\alpha, r_0/d(X))\), respectively. From this it follows that the concept of thinness introduced by Granlund, Lindqvist, and Martio [29, 4.15] coincides with porosity in their results on harmonic measure and uniform domains. Sjögren [62, Theorem 1] characterized certain "convolution sets" in \( \mathbb{R}^m \) as closed porous sets. Kotochigov [36] and Dyn'kin [18], [19] have considered in interpolation problems closed sets \( X \) in \( Y = \{ z \in \mathbb{C} \mid |z| = 1 \} \) satisfying the condition (K) that \( \sup_{z \in J} d(z, X) \geq c|J| \) for each arc \( J \subset Y \) with some constant \( c \in (0, \frac{1}{2}) \) where \( |J| \) denotes the length of \( J \). By Dyn'kin [19], [20], a compact set \( X \) satisfying (K) (or its analogue in \( \mathbb{R} \)) supports a homogeneous measure; see 6.8. Now, (K) is obviously equivalent to \( X \) being porous in \( Y \) (with \( \alpha = 2c \) and \( c = \frac{1}{2}\alpha \) for \( Y = \mathbb{R} \)). Various weak local versions of porosity have also been applied in analysis; see, e.g., Sarvas [59].

It is known [59, 3.2] that if \( X \) is an \( \alpha \)-porous set in \( \mathbb{R}^m \), \( m > 0 \), then \( \dim_H X \leq c(\alpha, m) \) \( < m \). If \( X \), \( Y \), and \( c \) are as in (K), then by Dyn'kin [18, (5.2)] we have that \( \int_J d(z, X)^{-\alpha}|dz| \leq c'|J|^{1-\alpha} \) for each arc \( J \subset Y \) with some constants \( \alpha \in (0, 1) \) and \( c' > 0 \) depending only on \( c \) and from this result it can easily be deduced that \( \dim_H X \leq 1 - \alpha < 1 \); see also 5.4. (More results on dimensions of porous sets are given in [58], [65], [47], and [35]; see 5.4 and 5.3.)

We generalize both of these dimensional restrictions and also prove a converse in the following characterization of porous sets in a Euclidean space.

**Theorem 5.2.** Let \( X \) be a subset of \( \mathbb{R}^m \), \( m > 0 \). Then \( X \) is porous in \( \mathbb{R}^m \) if and only if its Assouad dimension is smaller than \( m \). More precisely, if \( X \) is \( \alpha \)-porous, then \( X \) is \((C, s)\)-homogeneous for some \( C \) and \( s < m \) depending only on \((\alpha, m)\), and conversely, if \( X \) is \((C, s)\)-homogeneous with \( s < m \), then \( X \) is \( \alpha \)-porous with \( \alpha \) depending only on \((C, s, m)\).

**Proof.** Suppose first \( X \) to be \( \alpha \)-porous. Fix an integer \( p > 1 \) with \( p > 4\sqrt{m}/\alpha \). Let \( q = p^m - 1 \) and \( s = \log q/\log p \); then \( 0 \leq s < m \). We show modifying the proof of [59, 3.2] that \( X \) is \( s \)-homogeneous. Consider \((a, b, A) \in T(X)\) as in A.1. Choose a closed cube \( Q_0 \supset A \) in \( \mathbb{R}^m \) of side length \( b \). For each integer \( i \geq 0 \) let \( \mathcal{D}_i \) be the decomposition of \( Q_0 \) into \( p^{im} \) closed cubes of side length \( b/p^i \). Consider \( i \geq 0 \) and a cube \( Q \subset \mathcal{D}_i \). Let \( x \) denote the centre and \( l \) the side length of \( Q \). For
$r = l/4$, there is $y \in \overline{B}(x, r)$ with $B(y, \alpha r) \cap X = \emptyset$. Since $B(y, \alpha r) \subset B(x, l/2) \subset Q$ and $l/\sqrt{m}/p < \alpha l/4 = \alpha r$, there is $Q' \in D_{i+1}$ with $Q' \subset B(y, \alpha r)$. Then $Q' \subset Q$ and $Q' \cap A = \emptyset$. Hence, $A \cap Q$ can be covered by $q^i$ cubes of the totally $p^n$ cubes of $D_{i+1}$ that are contained in $Q$. By induction it follows that for each $i \geq 0$ we can cover $A$ by $q^i$ of the cubes of $D_i$.

Now let $k$ be the smallest integer with $p^k > b\sqrt{m}/a$; then $k \geq 1$ and $p^k \leq pb\sqrt{m}/a$. If $Q \in D_k$, then $d(Q) = b\sqrt{m}/p^k < a$, and consequently $\text{card}(A \cap Q) \leq 1$. Hence, $\text{card} A \leq q^k = p^{ks} \leq C(b/a)^s$ with $C = (p\sqrt{m})^s$. Thus, $X$ is $(C, s)$-homogeneous.

Conversely, suppose $X$ to be $(C, s)$-homogeneous with $s < m$. Clearly there is $c_0 > 0$ depending only on $m$ such that for all $x \in \mathbb{R}^m$, $r > 0$, and $t \geq 1$ there is an $(r/t)$-discrete set $Y$ in $\overline{B}(x, r)$ with $\text{card} Y \geq c_0 t^m$. Fix $\alpha \in (0, 1/3]$ so small that $C(3/\alpha)^s < c_0 (1/3\alpha)^m$. Claim that $X$ is $\alpha$-porous. If not, then there are $x \in \mathbb{R}^m$ and $r > 0$ such that $B(y, \alpha r) \cap X \neq \emptyset$ for each $y \in \overline{B}(x, r)$. Choose an $3\alpha r$-discrete set $Y \subset \overline{B}(x, r)$ with $\text{card} Y \geq c_0 (1/3\alpha)^m$. For each $y \in Y$ choose a point $f(y) \in B(y, \alpha r) \cap X$. Then $\alpha r \leq |f(y) - f(z)| \leq 3r$ for all $y, z \in Y$ with $y \neq z$. Hence, $\text{card} Y = \text{card} \{f(Y) \leq C(3r/\alpha r)^s = C(3/\alpha)^s$, which yields a contradiction.

\[ \square \]

**Corollary 5.3.** Let $X$ be a subset of $\mathbb{R}^m$, $m > 0$. Then $fX$ is porous in $\mathbb{R}^m$ for some homeomorphism $f: \mathbb{R}^m \rightarrow \mathbb{R}^m$ if and only if the closure of $X$ has no interior points.

**Proof.** We may assume $X$ to be closed. The necessity being clear, suppose $X$ to be without interior points. We may assume $0 \notin X$. Let $u: \mathbb{R}^m \setminus \{0\} \rightarrow \mathbb{R}^m \setminus \{0\}$ be the inversion of $\mathbb{R}^m$ in the unit sphere. Then $X^* = uX \cup \{0\}$ is a compact set in $\mathbb{R}^m$ without interior points. It follows that $\dim X^* \leq m - 1$. Hence, by 4.8 there is a homeomorphism $g: \mathbb{R}^m \rightarrow \mathbb{R}^m$ with $g(0) = 0$ such that $\dim gX^* \leq m - 1$. Define a homeomorphism $f: \mathbb{R}^m \rightarrow \mathbb{R}^m$ setting $f(0) = 0$ and $f(x) = ugu(x)$ if $x \neq 0$. Then $\dim fX \leq m - 1$ by A.10(1). Now $fX$ is a porous set by 5.2.

\[ \square \]

**Strongly porous sets 5.4.** Fix a positive integer $m$. Like Salli [58, 3.1], define a subset $X$ of $\mathbb{R}^m$ to be $(\alpha, \delta)$-porous in $\mathbb{R}^m$ with $\alpha \in (0, 1]$ and $\delta \in (0, \infty)$ if for all $x \in X$ and $r \in (0, \delta)$ there is $y \in \mathbb{R}^m$
such that $B(y, \alpha r/2) \subset B(x, r) \setminus X$. It is easy to see that $X$ satisfies the $\alpha$-porosity condition of 5.1 for all $x \in X$ if and only if $X$ is $(\beta, \infty)$-porous with $\beta = 2\alpha/(1 + \alpha)$; cf. 58, 3.1]. Recall that in this case $X$ is $\alpha_1$-porous with $\alpha_1 = \alpha/(1 + \alpha) = \beta/2$, and note that here $\alpha \mapsto \beta$ is a self-homeomorphism of $(0, 1]$. It follows that 5.2 holds if "$\alpha$-porous" is replaced by "$(\alpha, \infty)$-porous". The latter parameterization of porosity, however, is more suitable than the former in the next application of Salli's results to a study of the behaviour of $\dim_A X$ when $X$ is strongly porous in the sense that $X$ is $(\alpha, \infty)$-porous with $\alpha$ close to 1.

For $\alpha \in (0, 1]$, let $D_H(\alpha)$, $D_P(\alpha)$, $D_B(\alpha)$, and $D_A(\alpha)$ denote the supremum of the numbers $\dim_H X$, $\dim_P X$, $\dim_B X$, or $\dim_A X$, respectively, over all $(\alpha, \infty)$-porous sets $X$ in $\mathbb{R}^m$, with $X$ bounded in the case of $D_B(\alpha)$. Then $m - 1 \leq D_H(\alpha) \leq D_P(\alpha) \leq D_B(\alpha) \leq D_A(\alpha) < m$. Let first $D$ be one of $D_H$, $D_P$, and $D_B$. Let $A(m) > 0$ be the constant depending only on $m$ provided by [58, 3.8.1] (with $n = m$), and let $B = \frac{1}{2} \log 2$. Then

$$m - 1 + \frac{B}{\log (1/(1 - \alpha))} \leq D(\alpha) \leq m - 1 + \frac{A(m)}{\log (1/(1 - \alpha))}$$

for each $\alpha \in [\frac{1}{2}, 1)$

by [58, 3.8.2]. Hence, $D(\alpha) \to m - 1$ as $\alpha \to 1$, and consequently $D(1) = m - 1$. Moreover, if $m = 1$, then

$$D(\alpha) = \frac{\log 2}{\log ((2 - \alpha)/1 - \alpha)}$$

for each $\alpha \in (0, 1)$

by [58, 3.8.2]. We now show that (5.5) and (5.6) also hold if $D = D_A$. To establish (5.5) for $D = D_A$, it suffices to show that if $\alpha \in [\frac{1}{2}, 1)$, if $X \subset \mathbb{R}^m$ is $(\alpha, \infty)$-porous, if $s$ denotes the upper bound in (5.5), and if $C = (2 + (2 - 2\alpha)^{-1/2})^m$, then $X$ is $(C, s)$-homogeneous. Thus, consider $(a, b, A) \in T(X)$. Apply [58, 3.8.1] with the substitution $n \mapsto m$, $E \mapsto A$, $\beta \mapsto 1 - \alpha$ (then $d \mapsto \cdot$), $\delta \mapsto b/(2 - 2\alpha)$, and $h \mapsto a/2$; this is possible as $A$ is a finite $(\alpha, \delta)$-porous set and $h \leq \beta \delta$. Then we get that $|B(A, a/2)| \leq (b/2)^{s-m} |B(A, \varrho)| (a/2)^{m-s}$ where $\varrho = (\beta/2)^{1/2}\delta = (8 - 8\alpha)^{-1/2}b$ and $\cdot$ denotes the Lebesgue measure on $\mathbb{R}^m$. Since $(B(x, a/2))_{x \in A}$ is a disjoint family in $B(A, a/2)$
and since \( B(A, \rho) \subset B(x_0, b + \rho) \) where \( x_0 \in A \) is arbitrary, this implies that 
\[
\text{card } A \cdot (a/2)^m \leq b^{s-m}(b + \rho)^{m-s} a^{m-s},
\]
so card \( A \leq C(b/a)^s \).

To establish (5.6) for \( D = D_A \), it suffices to show that if \( \alpha \in (0, 1) \), 
if \( X \subset \mathbb{R}^1 \) is \((\alpha, \infty)\)-porous, if \( s \in (0, 1) \) denotes the right-hand side 
of (5.6), if \( C_0 = \alpha^{-1}(\alpha/(1 - \alpha))^{s} \), and if \( C = \max(C_0, \alpha^{-1}) \), then 
\( X \) is \((C, s)\)-homogeneous. Thus, consider \((a, b, A) \in T(X)\). Claim 
\[
\text{card } A \leq C(b/a)^s.
\]
Since \( C \geq 1 \), we may assume that card \( A > 1 \). Let \( J = [\min A, \max A] \); then we may assume that \( b = |J| \). Let \( \theta = (1 - \alpha)/(2 - \alpha) \in (0, 1/2) \). We wish to apply [58, 3.5] with the substitution 
\( E \mapsto A, \alpha \mapsto \alpha \) (then \( d \mapsto s \)), \( \delta \mapsto (1 - \theta)b \), \( I \mapsto J \), and \( h \mapsto a/2 \); 
since \( A \) is \((\alpha, \delta)\)-porous and \(|J|/\delta = (1 - \theta)^{-1} \), this is possible whenever 
\( h \leq (2\theta)^{-1}(1 - 2\theta)|J| \) or equivalently \( b/a \geq (1 - \alpha)/\alpha \). In this case 
(which is valid for all \( a > 0 \) and \( b \geq a \) if and only if \( \alpha \geq 1/2 \)) we get that 
\[
|B(A, a/2)| \leq M^* b^s(a/2)^{1-s} \quad \text{where} \quad M^* = 2^{2-s} \alpha^{s-1}(2 - \alpha)^{-s} = 2^{1-s} C_0.
\]
Since \( A \) is \( a \)-discrete, this implies that card \( A \cdot a \leq C_0 b^s a^{1-s} \), so 
\( \text{card } A \leq C_0 (b/a)^s \leq C(b/a)^s \). On the other hand, if \( b/a \leq (1 - \alpha)/\alpha \), 
then card \( A \leq 1 + b/a \leq 1/\alpha \leq C(h/a)^s \).

Dyn'kin's method [18] discussed in 5.1 shows that \( \dim_A X \leq \log 2/ \log(2/(1 - \alpha)) \) for each \( \alpha \)-porous set \( X \) in \( \mathbb{R}^1 \) with \( \alpha \in (0, 1) \) (note that 
in the proof of [18, Lemma 2] we can replace \( c \) by \( \alpha = 2c \)). This is the 
same bound that follows from (5.6) with \( D = D_A \) whenever \( 2\alpha/(1 + \alpha) \) is substituted for \( \alpha \).

In the connection of (5.6) we have that \( C \to 2 \) as \( \alpha \to 1 \), and, 
indeed, a set \( X \subset \mathbb{R}^1 \) is \((1, \infty)\)-porous if and only if card \( X \leq 2 \) (but 
1-porous only if card \( X \leq 1 \)). For an arbitrary \( m \) we show that a 
\((1, \infty)\)-porous set in \( \mathbb{R}^m \) is \((C, m - 1)\)-homogeneous with \( C \) depending 
only on \( m \). The proof is based on the characterization of these sets 
given in [58, 3.3] as sets \( X \) with \( X \subset \partial_r(\partial_r(X)) \) for each \( r > 0 \) where 
\( \partial_r(X) = \{ x \in \mathbb{R}^m \mid d(x, X) = r \} \). We may assume that our \((1, \infty)\)-
porous set \( X \) is finite and has more than one point. Take \( r = d(X) \), 
and let \( Y = \partial_r(X) \). Since \( d(Y) \leq 3r \), there is an integer \( k(m) \geq 1 \) 
such that we can write \( Y = \bigcup_{i=1}^{k(m)} Y_i \) with \( d(Y_i) \leq r/2 \) for each \( i \). 
It follows that \( X \subset \partial_r(Y) \subset \bigcup_{i=1}^{k(m)} \partial_r(Y_i) \). Since the unit sphere 
\( S^{m-1} = \partial_1(\{0\}) \) in \( \mathbb{R}^m \) is \((C_m, m - 1)\)-homogeneous for some \( C_m \geq 1 \), 
the next lemma with \( r = 2 \) implies, as the proof of A.5 shows, that \( X \) is 
\((C, m - 1)\)-homogeneous with \( C = 3^{(m-1)/2} k(m) C_m \).
Lemma 5.7. Let $X \subset \mathbb{R}^m$ be a compact set with $0 \in X \subset \overline{B}(0,1)$, and let $r > 1$. Then the radial projection $p: \partial_r(X) \to S^{m-1}$, $x \mapsto x/|x|$, is a bi-Lipschitz homeomorphism with $((r-1)/(r+1))^{1/2}|x-y|/r \leq |p(x) - p(y)| \leq |x-y|/r$ for all $x, y \in \partial_r(X)$.

Proof. The function $x \mapsto d(x, X)$ maps each ray from 0 onto $[0, \infty)$; hence $p$ is a surjection. Clearly $r \leq |x| \leq r+1$ for each $x \in \partial_r(X)$. Consider $x, y \in \partial_r(X)$ with $x \neq y$. First, $|p(x) - p(y)| \leq |x-y|/(|x||y|)^{1/2} \leq |x-y|/r$ by [43, 2.12]. To obtain the other inequality, letting $H$ be the inner product of $\mathbb{R}^m$, define a closed half-space $H_a = \{ z \in \mathbb{R}^m \mid (z-x) \cdot (x-a) \geq 0 \}$ for each $a \in \overline{B}(0,1)$ and a closed cone $K = \bigcap_{|a| \leq 1} H_a = \bigcap_{|a|=1} H_a$ with vertex $x$. Then $y \notin K$, for otherwise $y \in H_a$ and thus $r \leq |y-x| < |a-y|$ for each $a \in X$ in contradiction with $d(y, X) = r$. This implies that if $t_x y = \{ x+t(y-x) \mid t \leq 0 \}$ is the ray from $x$ in the direction opposite to the direction of $y$ as seen from $x$ and if $\delta_{xy} = d(0, l_{xy})$ is its distance from 0, then $\delta_{xy} > (|x|^2-1)^{1/2}$. Now assuming $|x| \leq |y|$ this gives $p(x) \neq p(y)$.

Let $\alpha \in (0, \pi)$ be the angle between $x$ and $y$ and $\beta \in [0, \pi)$ the angle between $y-x$ and $x$. Then $|x-y| \sin \beta = |y| \sin \alpha$ and $\sin \alpha = |p(x) - p(y)| \cos(\alpha/2) \leq |p(x) - p(y)|$. In the case $\delta_{xy} < |x|$ we have $|x| < |y|$, $\alpha < \pi/2 < \beta$, and $\sin \beta = \delta_{xy}/|x| \geq (|x|^2-1)^{1/2}/|x| \geq (r^2-1)^{1/2}/r$; thus $|x-y| \leq r((r+1)/(r-1))^{1/2}|p(x) - p(y)|$. In the case that $\delta_{xy} = |x|$ and $\alpha \geq \pi$ we have $\delta_{xy} = |y|$ as $|y| \geq |x|$, $0 \leq (\pi/2) - \beta \leq \alpha/2 < \pi/2$, and, thus, $\sin \beta \geq \cos(\alpha/2)$; hence, $|x-y| \leq (r+1)|p(x) - p(y)|$. In the remaining case $\alpha = \pi$, too, we have $|x-y| \leq 2(r+1) = (r+1)|p(x) - p(y)|$. Finally, note that $r+1 \leq r((r+1)/(r-1))^{1/2}$.

Remark 5.8. Lemma 5.7 is due to Jussi Väisälä who, however, gave the bounds to $|p(x) - p(y)|/|x-y|$ only in the infinitesimal case $y \to x$. Brown [12, Lemma 1] had earlier proved that the map $p$ and then also its extension $p^*: \mathbb{R}^m \to \mathbb{R}^m$ which maps each ray from 0 linearly onto itself are homeomorphisms. It follows from 5.7 by [43, 2.13 and 2.16] that $p^*$ is bi-Lipschitz.

Remark 5.9. For $m > 0$, there is a constant $C_m > 0$ such that $D(\alpha) \leq m - C_m \alpha^m$ if $\alpha \in (0, 1)$ whenever $D = D_H$ or $D = D_B$ by
Trotsenko [65] and Martio and Vuorinen [47], respectively (by [65] this estimate holds, in fact, also if $D(\alpha)$ is the supremum of the numbers $\dim B X$ over all bounded $(\alpha, \infty)$-porous sets $X$ in $\mathbb{R}^m$). This polynomial formulation of the dimensional results of [65] and [47] is due to Koskela and Rohde [35], who also proved that the exponent $m$ in the estimate is the smallest possible one even if $D = D_H$ and $\alpha$ is close to 0. It seems that the methods of [65] and [47] are inadequate for $D_A$.

6. Homogeneous measures

In this section we discuss a conjecture that the Assouad dimension of a complete metric space can be characterized as the infimum of certain exponents associated with homogeneous measures on this space. By Theorem 6.9 due to Volfberg and Konyagin this conjecture holds true for compact spaces. In 6.13 we give a corrected proof for their more precise result. Theorem 6.10, for a compact set in a Euclidean space. In Theorem 6.18 we show that if we weaken definitions suitably, then the conjecture is true also for noncompact spaces. By 6.9 and our flat metrization result 4.3, we obtain in Theorem 6.15, a characterization of the topological dimension of a locally compact separable metrizable space in terms of homogeneous measures. An analogous characterization of the dimension of a compact set in a Euclidean space is achieved in Theorem 6.16. For notation and terminology consult also A.1 in the appendix.

6.1. In this section we change slightly our earlier definition of $(C, s)$-homogeneity in 3.2 as follows. If $X$ is a metric space and $N \geq 0$ and $s \geq 0$ are constants, we now say that $X$ is $(N, s)$-homogeneous whenever $\text{card } A \leq N \lambda^s$ for all $x \in X$, $r > 0$, and $\lambda \geq 1$ and for every $r$-discrete set $A \subset \overline{B}(x, \lambda r)$. The two different definitions imply each other with the same $s$ and with $C = N$ and $\overline{N} = 2^s C$.

6.2. A homogeneous measure on a metric space $X$ is a Borel measure $\mu$ on $X$ for which there are constants $c \geq 0$ and $s \geq 0$ such that

$$0 < \mu(\overline{B}(x, \lambda r)) \leq c \lambda^s \mu(\overline{B}(x, r)) < \infty$$

for all $r \in X$, $r > 0$, and $\lambda \geq 1$;

then we also say that $\mu$ is $(c, s)$-homogeneous or $s$-homogeneous. Condition (6.3) implies that $\mu$ has for the constant $D = 2^s c$ the doubling
property that
\begin{equation}
0 < \mu(\overline{B}(x, 2r)) \leq D \mu(\overline{B}(x, r)) < \infty \quad \text{for all } x \in X \text{ and } r > 0,
\end{equation}
and conversely (6.4) with a constant \( D \geq 1 \) implies (6.3) with \( (c, s) = (D, \log_2 D) \).

There is a wide literature on analysis on metric (or more generally quasimetric) spaces endowed with a measure having the doubling property, a pioneering work being [15]. We encounter in A.15 a 4-homogeneous Haar measure on the Heisenberg group.

Let \( \mu \) be a homogeneous measure on a metric space \( X \). Then \( \mu(\{x\}) = 0 \) for each nonisolated point \( x \) in \( X \) by [44, Theorem 1], and, as it can be shown, \( \mu(X) = \infty \) in case \( X \) is unbounded.

6.5. Working slightly later than Assouad but independently of him, Vol'berg and Konyagin [70], [71] introduced the same concept as Assouad dimension under the name of uniform metric dimension. As it is easily seen in [71, p. 630], a metric space carrying a \((c, s)\)-homogeneous measure is \((N, s)\)-homogeneous with \( N = 9^s c \) (in fact, even with \( N = 5^s c \)). Thus, for each metric space \( X \),
\begin{equation}
\dim_A X \leq \inf \{ s \geq 0 \mid \text{there is an } s\text{-homogeneous measure on } X \}.
\end{equation}
(The homogeneous dimension of a (quasi)metric space \( X \), a term recently introduced by Biroli and Mosco [8], could properly be defined as the right-hand side of (6.6).) In particular, \( \dim_A X < \infty \) if \( X \) carries a homogeneous measure. This latter fact was already proved by Coifman and Weiss [15, p. 67].

Assouad [4] asked whether the converse of this is true but observed that not always as on the countable perfect space \( \mathbb{Q} \subset \mathbb{R} \) of Assouad dimension 1 there can be no homogeneous measure (in fact, for any compatible metric). He then suggested the first part of the following conjecture.

**Conjecture 6.7**. Let \( X \) be a complete metric space of finite Assouad dimension. Then there is a homogeneous measure on \( X \). More precisely, (6.6) holds as an equality.

\footnote{Verified as generalizations of 6.9 for complete spaces and 6.10 for closed sets by the author jointly with E. Saksman in Every complete doubling metric space carries a doubling measure, Proc. Amer. Math. Soc. 126 (1998), 531-534.}
6.8. As Assouad [4] remarked, 6.7 can be reduced to the special case where $X$ is a closed subset of a Euclidean space. To see this, fix $p \in (0, 1)$. Then note that a Borel measure on a metric space $(X, d)$ which is $s$-homogeneous on the metric space $(X, d^p)$ is $ps$-homogeneous on $(X, d)$. Thus, as $\dim_A(X, d^p) = (1/p) \dim_A(X, d)$, we may replace the metric $d$ of $X$ by $d^p$. Now note that $X$ can consequently be bi-Lipschitz embedded in a Euclidean space by A.13.

Dyn'kin [19], [20] proved that the first part of 6.7 is true for compact sets $X \subset \mathbb{R}^1$ satisfying a metric condition equivalent to $\dim_A X < 1$ (see 5.1 and 5.2). He then also conjectured the first part of 6.7 for closed sets in $\mathbb{R}^m$.

Motivated by [19], Vol'berg and Konyagin [71] (results announced in [70]) established 6.7 for compact spaces $X$ and in a sharpened form for compact sets $X$ in $\mathbb{R}^m$ of Assouad dimension $m$ as follows.

**Theorem 6.9 ([71, Theorem 1]).** Let $X$ be a non-empty $(N, s)$-homogeneous compact metric space, and let $t > s$. Then there exists a $(c, t)$-homogeneous probability measure on $X$ with $c$ depending only on $(N, s, t)$.

**Theorem 6.10 ([71, Theorem 2]).** Let $X$ be a non-empty compact set in $\mathbb{R}^m$. Then there exists a $(c, m)$-homogeneous probability measure on $X$ with $c$ depending only on $m$.

6.11. In [71, Theorem 4] it is also constructed for all $s > 0$ and $m > s$ an $s$-homogeneous compact set $X \subset \mathbb{R}^m$ such that $\inf \{ t \mid \mu \text{ is } t \text{-homogeneous} \} > s$ for each homogeneous measure $\mu$ on $X$.

6.12. In the proof of 6.9 in [71] it is first constructed for a large enough number $A \geq 8$ a sequence $(\mu_j)_{j \geq 0}$ of probability measures on $X$ each $\mu_j$ being supported by an arbitrary but fixed $A^{-j}$-net $S_j \subset X$ and satisfying the $t$-homogeneity condition in a rough form. A weakly convergent subsequence is chosen, and the limiting Borel probability measure $\mu$ on $X$ is then shown to be $t$-homogeneous. However, in the analogous proof of 6.10 the definition of the sets $S_j$ in [71] is incorrect. For this reason we next give a proof for 6.10 indicating also the flaw.

6.13. **Proof of 6.10.** To conform to the notation of [71], we replace $\mathbb{R}^m$ by $\mathbb{R}^n$. On $\mathbb{R}^n$ we first use the maximum metric $\varrho(x, y) = \max_{1 \leq i \leq n} |x_i - y_i|$. Now in [71, p. 634] it is defined $S_j = 9^{-j} \mathbb{Z}^n \cap X$ for $j \geq 0$ and claimed that then $\text{card } S_{c, m+1} \leq \Omega^n$ for each $e \in S_m$,
where \( S_{e, m+1} = \mathcal{E}^{-1}(e) \) for a fixed choice function \( \mathcal{E}: S_{m+1} \to S_m \) minimizing \( q(g, \mathcal{E}(g)) \) for each \( g \in S_{m+1} \). But if, for example, \( n = 1 \) and \( X = \{ i/81 \mid i \in \mathbb{Z} \setminus 9\mathbb{Z}, |i| \leq 40 \} \cup \{0\} \), then \( S_0 = S_1 = \{0\}, S_2 = X, S_{0.2} = X \), and, thus, \( \text{card } S_{0.2} = 73 > 9 = 9^n \).

To get a correct definition (this comes close to that in [70]), let \( A \geq 8 \) be an odd integer (we could fix \( A = 9 \)). Consider a non-empty compact set \( X \subset \mathbb{R}^n \). Define

\[
S_m = \{ x \in A^{-m}\mathbb{Z}^n \mid q(x, X) \leq \frac{1}{2}A^{-m} \} \quad \text{for } m = 0, 1, 2, \ldots.
\]

Then for each \( m \geq 0 \) the following conditions are satisfied:

1. If \( x, y \in S_m \) and \( x \neq y \), then \( q(x, y) \geq A^{-m} \).
2. If \( x \in X \), then \( q(x, S_m) \leq \frac{1}{2}A^{-m} \).
3. If \( g \in S_{m+1} \), then \( q(g, e) \leq \left( \frac{1}{2}(A-1)/A \right)A^{-m} \) for a unique \( e \in S_m \).
4. There is a unique map \( \mathcal{E}: S_{m+1} \to S_m \) with \( q(g, \mathcal{E}(g)) = q(g, S_m) \) for each \( g \in S_{m+1} \).
5. The sets \( S_{e, m+1} = \mathcal{E}^{-1}(e) \) with \( e \in S_m \) form a partition of \( S_{m+1} \) into non-empty disjoint sets.
6. We have \( \text{card } S_{e, m+1} \leq A^n \) for each \( e \in S_m \).

In (3') the uniqueness is clear. To prove the existence in (3'), let \( g \in S_{m+1} \), let \( B = \frac{1}{2}(A-1)/A \), and choose \( e \in A^{-m}\mathbb{Z}^n \) with \( q(g, e) \leq BA^{-m} \); then \( q(e, X) \leq q(g, e) + q(g, X) \leq BA^{-m} + \frac{1}{2}A^{-m-1} = \frac{1}{2}A^{-m} \), so \( e \in S_m \). For (5'), note that if \( e \in S_m \) and we choose \( x \in X \) with \( q(e, x) \leq \frac{1}{2}A^{-m} \), then clearly there is \( g \in A^{-m-1}\mathbb{Z}^n \) such that \( q(x, g) \leq \frac{1}{2}A^{-m-1} \) and \( q(g, x) \leq BA^{-m} \); hence, \( g \in S_{e, m+1} \), and so \( S_{e, m+1} \neq \emptyset \). We have (6') as \( \text{card } S_{e, m+1} \leq \text{card}\{ g \in A^{-m-1}\mathbb{Z}^n \mid q(g, e) \leq BA^{-m} \} = A^n \) for each \( e \in S_m \).

Theorem 2 in [71] can now be proved as follows. First note that \( \mathbb{R}^n \) is \((N, n)\)-homogeneous with \( N = N(\mathbb{R}^n, g, n) = 3^n \). We can follow the proof of [71, Theorem 1] with \( \gamma = \gamma' = n \), with \( A \) as above, and with the number \( N(X, g, \gamma) \) in [71, (11)] being replaced by \( N(S_m, q, n) \leq 3^n \); in fact, (6') above gives the desired substitute for [71, (2)]. Let \( Y_m = X \cup (\bigcup_{j=m}^{\infty} S_j) \) for \( m \geq 0 \). We construct as follows for each \( m \geq 0 \) a Borel probability measure \( \mu_m \) on \( Y = Y_0 \) with support \( S_m \). We let \( \mu_0 \) be the uniform probability distribution on \( S_0 \). Then, inductively, we substitute \( \mu_m \) for the measure \( f_0 \) on \( S_m \) in [71, Lemma] and let
\( \mu_{m+1} \) be the resulting measure \( f_1 \) on \( S_{m+1} \). Since \( Y \) is compact, by [7, Theorem 6.1] there is a subsequence \( (\mu_{m_\nu})_{\nu \geq 0} \) of the sequence \( (\mu_m)_{m \geq 0} \) that converges weakly to a Borel probability measure \( \mu \) on \( Y \). We show that \( \mu(X) = 1 \). First, if \( m \geq 0 \), then \( Y \setminus Y_m \) is open in \( Y \) and \( \mu_j(Y \setminus Y_m) = 0 \) for each \( j \geq m \), which imply that \( \mu(Y \setminus Y_m) \leq \liminf_{\nu \to \infty} \mu_{m_\nu}(Y \setminus Y_m) = 0 \) by [7, Theorem 2.1]. Thus, \( \mu(Y \setminus X) = \lim_{m \to \infty} \mu(Y \setminus Y_m) = 0 \).

To proceed, assume first that \( g(X) \leq 1 \). Then we can show as in the proof of [71, Theorem 1] that if \( x_0 \in X \), \( R > 0 \), and \( k \geq 1 \) (\( k \) a real number), then

\[(7') \quad \mu\left( \overline{B}(x_0, kR) \cap X \right) \leq c_n k^n \mu\left( \overline{B}(x_0, R) \cap X \right)\]

with \( c_n = 3^n 4^n A^{5n} \). In proving this we may assume that \( kR \leq 1 \) (not \( R \leq 1 \) as in [71]), because the case \( R \geq 1 \) is trivial and because the case \( R < 1 < kR \) reduces to the case \( kR = 1 \) if we consider \( k' = 1/R \in (1, k) \) in place of \( k \).

The reduction of the case \( g(X) > 1 \) to the case \( g(X) = 1 \) is obvious; the constant \( c_n \) in \( (7') \) remains the same. In fact, let \( \alpha = 1/g(X) \) and \( X^* = \alpha X \); since \( g(X^*) = 1 \), we then get a Borel probability measure \( \mu^* \) on \( X^* \) such that \( (7') \) is satisfied whenever \( (X^*, \mu^*) \) is substituted for \( (X, \mu) \); now by setting \( \mu(B) = \mu^*(\alpha B) \) for each Borel set \( B \subset X \) we obtain a Borel probability measure \( \mu \) on \( X \), which satisfies \( (7') \) as

\[
\mu\left( \overline{B}(x_0, kR) \cap X \right) = \mu^*\left( \overline{B}(\alpha x_0, k\alpha R) \cap X^* \right) \\
\leq c_n k^n \mu^*\left( \overline{B}(\alpha x_0, \alpha R) \cap X^* \right) \\
= c_n k^n \mu\left( \overline{B}(x_0, R) \cap X \right).
\]

Finally, when the metric \( g \) on \( \mathbb{R}^n \) is replaced by the Euclidean metric, the measure \( \mu \) constructed above satisfies \( (7') \) with the coefficient \( c_n \) multiplied by \( n^{n/2} \).

6.14. Theorems 6.9 and 6.10 can be generalized for certain totally bounded spaces. Thus, let \( X \) be a non-empty \((N, s)\)-homogeneous bounded metric space. Then the completion \( \tilde{X} \) of \( X \) is compact and \((N, s)\)-homogeneous. By 6.9, for each \( t > s \) there is a \((c, t)\)-homogeneous probability measure \( \mu \) on \( \tilde{X} \) with \( c \) depending only on \((N, s, t)\). Now
each point of $\Delta = \tilde{X} \setminus X$ is nonisolated in $\tilde{X}$. Hence, if $\Delta$ is countable, then $\mu(\Delta) = 0$, and in this case $\mu$ induces a $(c,t)$-homogeneous Borel probability measure on $X$. There is a similar strengthening of 6.10 for non-empty bounded sets $X \subset \mathbb{R}^m$ with $\overline{X} \setminus X$ countable. Note also that if $X \subset \mathbb{R}^1$ is the complement of a set of Lebesgue measure zero, then $\dim_A X = 1$ and the Lebesgue measure on $X$ is 1-homogeneous.

**Theorem 6.15.** Let $X$ be a non-empty locally compact separable metrizable space. If $s \geq 0$ is such that there is an $s$-homogeneous measure on $(X,d)$ for some compatible metric $d$ on $X$, then $\dim X \leq s$. Conversely, if $n = \dim X < \infty$, then $X$ can be metrized by a totally bounded metric having the property that for each $s > n$ there is an $s$-homogeneous probability measure on $X$.

**Proof.** The first part follows from (6.6) and the inequality $\dim X \leq \dim_A(X,d)$. In the second part we assume $X$ to be noncompact. Let $Y$ be the one-point compactification of $X$ by a point $\omega$; then $Y$ is metrizable and $\dim Y = n$. Endow $Y$ with a flat metric by 4.3. Let $s > n$. Then by 6.9 there is an $s$-homogeneous probability measure $\mu$ on $Y$. Now it suffices to observe that $\mu(\{\omega\}) = 0$ as the point $\omega$ is nonisolated. \qed

**Theorem 6.16.** Let $X$ be a non-empty compact set in $\mathbb{R}^m$. If $s \geq 0$ is such that there is an $s$-homogeneous measure on $fX$ for some homeomorphism $f : \mathbb{R}^m \to \mathbb{R}^m$, then $\dim X \leq s$. Conversely, if $n = \dim X$, then for each $\varepsilon > 0$ there is an $\varepsilon$-push $f$ of $(\mathbb{R}^m,X)$, with $f = \text{id}$ if $n = m$, such that for each $s > n$, or for $s = n$ if $n = m$, there is an $s$-homogeneous probability measure on $fX$.

**Proof.** The first part follows from (6.6) and 4.8, and the second part from 4.8 and 6.9 if $n < m$ or from 6.10 if $n = m$. \qed

6.17. In the following two theorems we apply 6.9 and 6.10 to verify a weak version of Conjecture 6.7. Let $X$ be a metric space. We say that $X$ is weakly $(N,s)$-homogeneous if $X$ satisfies the $(N,s)$-homogeneity condition in 6.1 with the limitation $r \leq 1$. Similarly, a weakly $(c,s)$-homogeneous measure on $X$ is a Borel measure $\mu$ on $X$ satisfying (6.3) with the limitation $r \leq 1$. If $X$ is bounded, these two weak concepts coincide with the ordinary ones, provided that $N$ and $c$ are multiplied
by \(d(X)^s\) in case \(d(X) > 1\). Likewise they coincide if \(s = 0\). In general they may differ. For example, let \(s > 0\) and endow \(X = \{1, 2, \ldots\}\) with the complete ultrametric \(d\) defined by \(d(x, y) = x^{1/s}\) if \(x > y\). Then \(X\) is weakly \((1, s)\)-homogeneous. To see this, let \(x \in X\), \(r \in (0, 1]\), and \(\lambda \geq 1\), and consider an \(r\)-discrete set \(A \subset \overline{B}(x, \lambda r)\) with \(\text{card} A > 1\). Then, since \(d(A) \leq \lambda\), choosing \(k \in \mathbb{N}\) with \(k \leq \lambda^s < k + 1\) we have \(A \subset \{1, \ldots, k\}\), so \(\text{card} A \leq \lambda^s\). On the other hand, if \(\lambda = 2^{1/s}\), \(k \geq 1\), and \(r_k = k^{1/s}\), then \(\{k, \ldots, 2k\}\) is an \(r_k\)-discrete set of \(k + 1\) points in \(\overline{B}(k, \lambda r_k)\), and consequently \(\dim A X = \infty\). Now it follows from 6.18 below that there are weakly homogeneous measures on \(X\), but none of them can be homogeneous.

In the same manner as (6.6) is established, it can be established that if a metric space \(X\) carries a weakly \((c, s)\)-homogeneous measure, then \(X\) is weakly \((N, s)\)-homogeneous with \(N\) depending only on \((c, s)\). We next prove a converse. Then we can also add a normalization condition for the measure. We say that a Borel measure \(\mu\) on \(X\) is \((c', c'')\)-normalized with constants \(0 < c' \leq c''\) if \(c' \leq \mu(\overline{B}(x, 1)) \leq c''\) for each \(x \in X\).

**Theorem 6.18.** Let \(X\) be a weakly \((N, s)\)-homogeneous complete metric space, and let \(t > s\). Then there exists a weakly \((c, t)\)-homogeneous \((c', c'')\)-normalized measure \(\mu\) on \(X\) with \((c, c', c'')\) depending only on \((N, s, t)\).

**Proof.** Note that every closed ball in \(X\) is totally bounded and thus compact. Choose a 1-net \(Z\) in \(X\). Consider \(z \in Z\). Denote \(B_z = \overline{B}(z, 6)\) and \(B'_z = \overline{B}(z, 3)\). We show \(B_z\) to be \((6^s N, s)\)-homogeneous. Thus, consider \(x \in B_z\), \(r > 0\), \(\lambda \geq 1\), and an \(r\)-discrete set \(A \subset \overline{B}(x, \lambda r) \cap B_z\). Then \(\text{card} A \leq N \lambda^s\) if \(r \leq 1\), and \(\text{card} A \leq N^s 6^s \leq 6^s N \lambda^s\) if \(r \geq 1\). Now by 6.9 there is a \((c_0, t)\)-homogeneous Borel probability measure \(\nu_z\) on \(B_z\) with \(c_0 \geq 1\) depending only on \((N, s, t)\). We extend \(\nu_z\) uniquely to a Borel probability measure on \(X\) still preserving its name. Define a continuous function \(\varphi_z : X \to I\) by \(\varphi_z(y) = 1\) if \(d(y, z) \leq 3\), by \(\varphi_z(y) = 0\) if \(d(y, z) \geq 6\), and by \(\varphi_z(y) = ((6 - d(y, z))/3)^t\) otherwise. Define a Borel measure \(\mu_z\) on \(X\) putting \(\mu_z(A) = \int_A \varphi_z d\nu_z\) for each Borel set \(A \subset X\). Now we show that the Borel measure \(\mu = \sum_{z \in Z} \mu_z\) on \(X\) is the desired one.

First, for \(x \in X\) denote \(B^*_x = \overline{B}(x, 1)\) and \(Z_x = \{ z \in Z \mid B_z \cap B^*_x \neq \emptyset \}\).
\(\emptyset\}. Then \(Z_x \subset \bar{B}(x, 7)\), and consequently \(\text{card } Z_x \leq 7^sN\).

To prove that \(\mu\) is normalized, consider \(x \in X\). Choose \(z \in Z\) with \(d(x, z) < 1\). Then \(B_z \subset \bar{B}(x, 7)\) and \(B_z^* \subset \bar{B}_z^*\). Hence, \(1 = v_z(\bar{B}(x, 7)) \leq 7^t c_0 v_z(B_z^*)\) and, thus, \(\mu(B_z^*) \geq \mu_z(B_x) \geq 7^{-t} c_0^{-1}\). On the other hand, \(\mu(B_z^*) = \sum_{z \in Z_x} \mu_z(B_x^*) \leq 7^s N\) as \(\mu_z(X) \leq 1\) for each \(z \in Z\). Thus, \(\mu\) is \((c', c'')\)-normalized with \(c' = 7^{-t} c_0^{-1}\) and \(c'' = 7^s N\) (assuming \(N \geq 1\) also if \(X = \emptyset\)).

For the weak homogeneity, consider \(x \in X, r \in (0, 1]\), and \(\lambda \geq 1\). We study first the case \(\lambda r \leq 1\). We show that for each \(z \in Z\) there is \(u \in Z\) such that \(\mu_z(\bar{B}(x, \lambda r)) \leq 7^t c_0 \lambda^t \mu_u(\bar{B}(x, r))\). From this it then follows that \(\mu(\bar{B}(x, \lambda r)) = \sum_{z \in Z_x} \mu_z(\bar{B}(x, \lambda r)) \leq c_1 \lambda^t \mu(\bar{B}(x, r))\) with \(c_1 = 7^s N 7^t c_0 \geq 1\).

Thus, let \(z \in Z\). Denote \(\delta = d(x, z)\) and \(\Delta = 3 - \delta - \lambda r\).

**Case 1:** \(\Delta \geq r\). Then \(x \in B_z\).

**Subcase 1.1:** \(\delta - \lambda r > 3\). Then \(\bar{B}(x, \lambda r) \subset B_z \setminus B_z^*\). Since \(\varphi_z(y) \leq ((\Delta + 2\lambda r)/3)^t\) for \(y \in \bar{B}(x, \lambda r)\) and \(\varphi_z(y) \geq ((\Delta + \lambda r - r)/3)^t > 0\) for \(y \in \bar{B}(x, r)\), we get that \(\mu_z(\bar{B}(x, r)) > 0\) and

\[
\frac{\mu_z(\bar{B}(x, \lambda r))}{\mu_z(\bar{B}(x, r))} \leq \frac{((\Delta + 2\lambda r)/3)^t}{((\Delta + \lambda r - r)/3)^t} \cdot \frac{\nu_z(\bar{B}(x, \lambda r))}{\nu_z(\bar{B}(x, r))} \leq \left(\frac{1 + 2\lambda}{\lambda - r}\right)^t c_0 \lambda^t \leq 3^t c_1 \lambda^t.
\]

**Subcase 1.2:** \(\delta - \lambda r \leq 3\). Now, if \(y \in \bar{B}(x, r)\), then \(d(y, z) \leq \delta + r \leq 3 + \lambda r + r \leq 5\), so \(\varphi_z(y) \geq 3^{-t}\). Thus,

\[
\mu_z(\bar{B}(x, \lambda r)) \leq \nu_z(\bar{B}(x, \lambda r)) \leq c_0 \lambda^t \nu_z(\bar{B}(x, r)) \leq 3^t c_0 \lambda^t \mu_z(\bar{B}(x, r)).
\]

**Case 2:** \(\Delta < r\). Now, if \(y \in \bar{B}(x, \lambda r)\), then \(6 - d(y, z) \leq \Delta + 2\lambda r \leq 3\lambda r\), so \(\varphi_z(y) \leq (\lambda r)^t\). This implies \(\mu_z(\bar{B}(x, \lambda r)) \leq (\lambda r)^t\). Choose \(u \in Z\) with \(d(x, u) < 1\). Then \(B_u \subset \bar{B}(x, 7)\) and \(\bar{B}(x, r) \subset B_u^*\).

Hence,

\[
1 = \mu_u(\bar{B}(x, 7)) \leq c_0 (7/r)^t \nu_u(\bar{B}(x, r)) = c_0 (\lambda r)^t \mu_u(\bar{B}(x, r)).
\]

It follows that \(\mu_z(\bar{B}(x, \lambda r)) \leq 7^t c_0 \lambda^t \mu_u(\bar{B}(x, r))\).
The case $\lambda r \leq 1$ being settled, we now study the case $\lambda r \geq 1$. Choose a 1-net $Y$ in $\overline{B}(x, \lambda r)$; then $\text{card } Y \leq N(\lambda r)^s$ and $\overline{B}(x, \lambda r) \subset \bigcup_{y \in Y} B_y^*$. Thus,

$$
\mu\left(\overline{B}(x, \lambda r)\right) \leq N(\lambda r)^sc'' \leq N(\lambda r)^s(c''/c')\mu(B_x^*) \\
\leq N(\lambda r)^t(c''/c')c_1(1/r)^t\mu\left(\overline{B}(x, r)\right) = c\lambda^t \mu\left(\overline{B}(x, r)\right)
$$

with $c = Nc_1c''/c' \geq c_1$.

Finally, note that if $x \in X$ and $r > 0$, then $c' \leq \mu\left(\overline{B}(x, r)\right) \leq c''cr^t$ if $r \geq 1$ and $(c'/c)r^t \leq \mu\left(\overline{B}(x, r)\right) \leq c''$ if $r \leq 1$. \hfill \Box

**Theorem 6.19.** Let $X$ be a closed set in $\mathbb{R}^m$. Then there exists a weakly $(c, m)$-homogeneous $(c', c'')$-normalized measure $\mu$ on $X$ with $(c, c', c'')$ depending only on $m$.

**Proof.** It suffices to observe that $\mathbb{R}^m$ is $(N, m)$-homogeneous for some $N$ and that in the previous proof we can now choose by 6.10 the measures $\nu_z$ to be $(c_0, m)$-homogeneous with $c_0$ depending only on $m$. \hfill \Box

6.20. Theorem 6.19 is due to Jonsson [32, Proposition 1]; he assumes the limitation $\lambda r \leq 1$ in the weak $(c, m)$-homogeneity condition for $\mu$, but, as we saw, this can readily be improved to $r \leq 1$. The proofs of 6.18 and 6.19 are adaptations of that in [32].

6.21. Bylund [14] has recently dealt with two-sided homogeneity conditions. He modifies the proof of 6.9 to get in [14, Theorem 1] a corresponding result for a non-empty compact set $X \subset \mathbb{R}^m$. His method can easily be generalized for an arbitrary non-empty compact metric space $X$. In what follows we state this more general result without giving a detailed proof.

Consider triples $(x, r, \lambda)$ where $x \in X$, $r > 0$, and $\lambda \geq 1$. For each such triple $(x, r, \lambda)$, denote $R(x, r, \lambda) = \max\{\text{card } A \mid A \text{ is an } r\text{-discrete subset of } \overline{B}(x, \lambda r)\}$. We assume there to be constants $0 < q \leq s$, $M > 0$, and $N > 0$ such that $M\lambda^q \leq R(x, r, \lambda) \leq N\lambda^s$ for each $(x, r, \lambda)$ with $\lambda r \leq 1$. Let $P \geq 1$ be a constant with $P \geq \max\{\text{card } A \mid A \text{ is a } 1\text{-discrete subset of } X\}$. Then $X$ is easily seen to be $(NP, s)$-homogeneous. Similarly, it is easy to see that if $r_0 > 1$ and $M' = \min(M, 1)r_0^{-q} > 0$, then $R(x, r, \lambda) \geq M'\lambda^q$ for each $(x, r, \lambda)$ with $\lambda r \leq r_0$. 
Now for all numbers \( q' \in (0, q) \) and \( s' > s \) there exist constants \( c_1 \in (0, 1] \), \( c_2 \geq 1 \), and \( c_3 \in (0, 1] \) depending only on \((q, s, q', s', M, N, P)\) and a Borel probability measure \( \mu \) on \( X \) such that (a) \( \mu(\overline{B}(x, \lambda r)) \geq c_1 \lambda^{q'} \mu(\overline{B}(x, r)) \) for each \((x, r, \lambda)\) with \( \lambda r \leq 1 \), (b) \( \mu \) is \((c_2, s')\)-homogeneous, and (c) \( \mu \) is \((c_3, 1)\)-normalized. It also easily follows that if \( r_0 > 1 \), then substituting \( c_1' = c_1 r_0^{-q'} \in (0, 1] \) for \( c_1 \) permits the limitation in (a) to be \( \lambda r \leq r_0 \).

Assouad [4, 4.2] has shown that in the special case \( q = s \) we can take \( q' = s' = s \).

The analogue of this result where \( X \subset \mathbb{R}^m \) and \( s = s' = m \) remains open.

In [14, Theorem 2] Bylund proves a similar result for a closed set \( X \subset \mathbb{R}^m \) which has a certain cover by sets satisfying the downward homogeneity condition with the same constants. It implies 6.18 for \( X \) a closed subset of \( \mathbb{R}^m \) with \( s < m \). Also this result can be generalized for \( X \) an arbitrary complete metric space.

6.22. It is tempting to try to reduce Conjecture 6.7 for a closed set in a Euclidean space to the compact case using the inversion invariance of Assouad dimension and a suitable weight function, but my attempts have failed. To conclude this section, we prove a result which could possibly be applied to attack 6.7, namely the quasisymmetry invariance of the doubling property for a measure. This result is due to Assouad [4, 2.10, 2.13, 2.15, and 4.9] in the case of metric spaces that are connected or more generally homogeneously dense in the sense of [66, 3.8]. See A.1 for the definition of quasisymmetric embeddings.

**Theorem 6.23.** Let \((X, d)\) and \((X, \varrho)\) be metric spaces such that id: \((X, \varrho) \to (X, d)\) is an \( \eta \)-quasisymmetric homeomorphism, and let \( \mu \) be a Borel measure on the space \( X \) having the doubling property (6.4) on \((X, d)\) with a constant \( D \). Then \( \mu \) has the doubling property on \((X, \varrho)\) with a constant \( D' \) that depends only on \( \cdot, D, \eta \).

**Proof.** We have assumed \( \eta(1) \geq 1 \) and may assume \( D \geq 1 \). Refer to balls in \((X, d)\) or \((X, \varrho)\) by subscripts \( d \) or \( \varrho \), respectively. Let \( x \in X \) and \( r > 0 \). Since \( \overline{B}_d(x, r) \) is a bounded neighbourhood of \( x \) in \((X, d)\) by [66, 2.6], we conclude that \( 0 < \mu(\overline{B}_d(x, r)) < \infty \). Assuming \( \overline{B}_\varrho(x, 2r) \neq \overline{B}_\varrho(x, r) \) as we may, denote \( \beta = \sup \{ d(x, y) : y \in \overline{B}_\varrho(x, 2r) \} \) and \( \alpha = \inf \{ d(x, y) : y \in X \setminus \overline{B}_\varrho(x, r) \} \); then
$0 < \alpha \leq \beta < \infty$. Choose $k \in \mathbb{N}$ with $2^{k-1} \leq \beta / \alpha < 2^k$. Since 
$\beta \leq 2^{k+1} \frac{1}{2} \alpha$, we get

$$
\mu(\overline{B}_\varrho(x, 2r)) \leq \mu(\overline{B}_d(x, \beta)) \leq D^{k+1} \mu(\overline{B}_d(x, \frac{1}{2} \alpha)) \leq D^{k+1} \mu(\overline{B}_\varrho(x, r)).
$$

To find a bound to $k$, choose sequences $(y_j)$ in $\overline{B}_\varrho(x, 2r)$ and $(z_j)$ in $X \setminus \overline{B}_\varrho(x, r)$ with $d(x, y_j) \to \beta$ and $d(x, z_j) \to \alpha$. Then

$$
\frac{d(x, y_j)}{d(x, z_j)} \leq \eta \left( \frac{\varrho(x, y_j)}{\varrho(x, z_j)} \right) \leq \eta \left( \frac{2r}{r} \right) = \eta(2),
$$

whence $\beta / \alpha \leq \eta(2)$. Thus, $\mu(\overline{B}_\varrho(x, 2r)) \leq D' \mu(\overline{B}_\varrho(x, r))$ with $D' = D^{2+\log_2 \eta(2)} \geq 1$.

\[\square\]

**Appendix A. Assouad dimension: Elementary properties and the bi-Lipschitz embeddability problem**

In this appendix we first prove Theorem A.3 on characterizations of metric spaces of finite Assouad dimension and Theorem A.5 on basic properties of Assouad dimension, as promised in 3.3. These results are all due to Assouad ([2, Partie I], [3, Remarque (2) and Proposition 2]) except the conditions (4) and (4') in A.3 and (12) in A.5 due to the present author (cf. [66, 3.20]) and (9) in A.5 due to Tukia and Väisälä [66]. In [2] Assouad introduced the condition $\dim_{\lambda} X < \infty$ for a metric space $X$ using the equivalent doubling condition A.3(3), applied by many authors already earlier, whereas he did not define the number $\dim_{\lambda} X$ until in [3]. However, [3] provides nc proofs. Moreover, [2] and [3] deal with the more complicated case of quasimetric spaces; a quasimetric is defined otherwise in the same way as a metric but the right-hand side of the triangle inequality is multiplied by a constant. It is for these reasons that I wanted to give here proofs for A.3 and A.5. We do not include in A.5 Assouad’s result [3, Proposition 2(c)] about so-called factors of quasimetrics.

In Theorem A.10 referred to in 3.3 we prove that Assouad dimension is invariant under inversions of normed spaces and study its behaviour under the more general quasi-Möbius embeddings of Väisälä [68].

In Theorem A.12 we prove a result which gives an external characterization, based on Lebesgue measure, for the Assouad dimension of
a subset of a Euclidean space. This characterization was mentioned in 3.6.

A metric space bi-Lipschitz embeddable in a Euclidean space has necessarily a finite Assouad dimension. We recall in Theorem A.13 the partial converse of this fact due to Assouad that if a metric space has a finite Assouad dimension, then the metric space modified from it by raising its metric to any positive power smaller than 1 admits a bi-Lipschitz embedding in a Euclidean space. The modified space has larger Assouad dimension (if positive) than the original space has and is thus in this sense more fractal. The consequence A.13(4) of this result for quasisymmetric embeddings is due to the author [66, 3.20]. Finally in A.14 and A.15 we review more recent research on this bi-Lipschitz embeddability problem; A.14 is devoted to ultrametric spaces, in which case no modification of the space is needed, and A.15 to an example showing that in general, however, bi-Lipschitz embeddability does not hold for the original space.

In presenting Theorems A.3, A.5, and A.13, I was benefited by an unpublished M.Sc. thesis (in Finnish) of Pekka Alestalo at the University of Helsinki in 1989 dealing with Assouad dimension.

**Notation and Terminology A.1.** All metrics are denoted by $d$ if not otherwise stated. Let $X$ be a metric space. If $A \subset X$, then $d(A)$ denotes the diameter of $A$ and $d(x, A)$ the distance of a point $x \in X$ from $A$. For $x \in X$ and $r > 0$ we let $B(x, r)$ denote the open and $\overline{B}(x, r)$ the closed ball with centre $x$ and radius $r$. Consider $Y \subset X$ and $\varepsilon > 0$. We say that $Y$ is $\varepsilon$-discrete if $d(x, y) \geq \varepsilon$ whenever $x, y \in Y$ and $x \neq y$, that $Y$ is $\varepsilon$-dense (in $X$) if for each $x \in X$ there is $y \in Y$ with $d(x, y) < \varepsilon$, and that $Y$ is an $\varepsilon$-net if $Y$ is both $\varepsilon$-discrete and $\varepsilon$-dense. It is well-known that $X$ contains $\varepsilon$-nets for each $\varepsilon > 0$. We let $T(X)$ denote the set of all triples $(a, b, A)$ where $0 < a \leq b$ are reals and $A \subset X$ is a non-empty $a$-discrete set with $d(A) \leq b$. Recall from 3.2 that if $C \geq 0$, $s \geq 0$, and card $A \leq C(b/a)^s$ for each $(a, b, A) \in T(X)$, then $X$ is said to be $(C, s)$-homogeneous; here we also then call the pair $(C, s)$ a homogeneity parameter of $X$.

The following two concepts are introduced by Tukia and Väisälä in [66]. A metric space $X$ is said to be homogeneously totally bounded if there is an increasing function $k : [\frac{1}{2}, \infty) \to [1, \infty)$ such that for all $t \geq \frac{1}{2}$ and $r > 0$, each closed ball of radius $r$ can be covered by $q$ sets
of diameter smaller than \( r/t \) with \( q \leq k(t) \); then we also say that \( X \) is \( k \)-HTB. We say that an embedding \( f: X \to Y \) between metric spaces is quasisymmetric if there is a self-homeomorphism \( \eta \) of \([0, \infty)\) for which \( d(a, x) \leq td(b, x) \) implies that \( d(f(a), f(x)) \leq \eta(t) d(f(b), f(x)) \) for all \( a, b, x \in X \) and \( t \geq 0 \); then we also say that \( f \) is \( \eta \)-quasisymmetric. We always assume \( \eta(1) \geq 1 \). For example, if \( f \) is \( L \)-bi-Lipschitz, i.e., if \( L \geq 1 \) and if \( d(x, y)/L \leq d(f(x), f(y)) \leq Ld(x, y) \) for all \( x, y \in X \), then \( f \) is \( \eta \)-quasisymmetric with \( \eta(t) = L^2 t \). A converse also holds; see A.2. If \( f \) is \( \eta \)-quasisymmetric, then \( f^{-1}: fX \to X \) is \( \eta' \)-quasisymmetric with \( \eta'(t) = \eta^{-1}(t^{-1})^{-1} \) for \( t > 0 \). An id-quasisymmetric embedding multiplies distances by a constant and is called a similarity.

**Remark A.2.** For A.10(4) we need the observation that if \( d \) and \( d' \) are metrics on a set \( X \) such that \( \text{id}: (X, d) \to (X, d') \) is an \( \eta \)-quasisymmetric homeomorphism with \( \eta \) linear, then \( d \) and \( d' \) are bi-Lipschitz equivalent. To see this, assume \( \text{card } X > 1 \), let \( \eta(t) = ct \) for \( t \geq 0 \) with \( c \geq 1 \), and let \( \alpha \) be the infimum and \( \beta \) the supremum of the ratios \( d'(x, y)/d(x, y) \) for distinct points \( x, y \) in \( X \). If \( x \neq y \) and \( a \neq b \) are points in \( X \), then

\[
\frac{d'(x, y)}{d'(a, b)} = \frac{d'(x, y)}{d'(x, b)} \cdot \frac{d'(x, b)}{d'(a, b)} \leq c \frac{d(x, y)}{d(x, b)} \cdot c \frac{d(x, b)}{d(a, b)} = c^2 \frac{d(x, y)}{d(a, b)}
\]

whenever \( x \neq b \), while \( d'(x, y)/d'(a, b) \leq cd(x, y)/c(a, b) \) whenever \( x = b \). Hence, \( 0 < \beta \leq c^2 \alpha < \infty \), and \( \alpha d \leq d' \leq \beta d \).

**Theorem A.3.** Let \( X \) be a metric space. Then the following conditions are equivalent.

1. There are \( C, s \geq 0 \) such that \( X \) is \((C, s)\)-homogeneous.

2. For each \( \lambda \in (0, 1) \) there is \( n \in \mathbb{N} \) such that if \( r > 0 \), then each open ball of radius \( r \) in \( X \) can be covered by \( q \) open balls of radius \( \lambda r \) in \( X \) with \( q \leq n \).

2' The condition (2) with open balls replaced by closed balls.

3. There are \( \lambda \in (0, 1) \) and \( n \in \mathbb{N} \) such that if \( r > 0 \), then each open ball of radius \( r \) in \( X \) can be covered by \( q \) open balls of radius \( \lambda r \) in \( X \) with \( q \leq n \).

3' The condition (3) with open balls replaced by closed balls.

4. There is an increasing function \( k: \left[ \frac{1}{2}, \infty \right) \to [1, \infty) \) such that \( X \) is \( k \)-HTB.
There are $c, \sigma \geq 0$ such that $X$ is $k$-HTB with $k(t) = ct^\sigma$.

Moreover, consider the following data in these conditions: the pair $(C, s)$ in (1), the function $\lambda \rightarrow v$ in (2) and (2'), the pair $(\lambda, n)$ in (3) and (3'), the function $k$ in (4), and the pair $(c, \sigma)$ in (4'). Then for each pair of these conditions, the datum of the second condition can be chosen to depend only on the datum of the first condition.

**Proof.** We prove the theorem in the quantitative form as described at the end of its statement.

(1) $\Rightarrow$ (2): Let $X$ be $(C, s)$-homogeneous and $\lambda \in (0, 1)$. We show that (2) holds with $n$ being the integer part of $C(2/\lambda)^s$. Let $x \in X$ and $r > 0$. Choose an $\lambda r$-net $Y$ in $B(x, r)$. Then $B(x, r) \subset \bigcup_{y \in Y} B(y, \lambda r)$ and $\text{card} \ Y \leq C(2r/\lambda r)^s = C(2/\lambda)^s$.

(1) $\Rightarrow$ (2'): As (1) $\Rightarrow$ (2) with the same $n$; only replace $B(x, r)$ by $\overline{B(x, r)}$ and $B(y, \lambda r)$ by $\overline{B(y, \lambda r)}$.

(2) $\Rightarrow$ (3) and (2') $\Rightarrow$ (3'): Choose $\lambda = \frac{1}{2}$.

(3) $\Rightarrow$ (1): Let (3) be satisfied with datum $(\lambda, n)$. We may assume that $n > 0$. Let $s = \log_{1/\lambda} n$ and $C = 4^s n$. We show $X$ to be $(C, s)$-homogeneous. Let $(a, b, A) \in T(\lambda)$. Choose $k \in \mathbb{N}$ with $\lambda^k < a/4b \leq \lambda^{k-1}$. Choose $x \in A$; then $A \subset \overline{B(x, 2b)}$. By (3) and since $2b\lambda^k < a/2$, there is $Y \subset X$ such that $\text{card} \ Y \leq n^k$ and $B(x, 2b) \subset \bigcup_{y \in Y} \overline{B(y, a/2)}$. Since $\text{card} (A \cap B(y, a/2)) \leq 1$ for each $y \in Y$, this yields $\text{card} \ A \leq n^k = n(\lambda^{k-1})^{-s} \leq C(b/a)^s$.

(3') $\Rightarrow$ (1): As (3) $\Rightarrow$ (1) but with $B(x, 2b)$ replaced by $\overline{B(x, b)}$; then we could even improve $C$ to $C = 2^s n$, which is also true in (3) $\Rightarrow$ (1).

(1) $\Rightarrow$ (4'): Let $X$ be $(C, s)$-homogeneous. We show $X$ to be $k$-HTB with $k(t) = 6^s C t^s$. Let $x \in X$, $r > 0$, and $t \geq \frac{1}{2}$. For $\lambda = 1/3t$ let $n \leq C(2/\lambda)^s = k(t)$ be the integer given by the proof of (1) $\Rightarrow$ (2'). Then by (2') there are $x_1, \ldots, x_n \in X$ for which $\overline{B(x, r)} \subset \bigcup_{i=1}^n \overline{B(x_i, \lambda r)}$, and here $d(\overline{B(x_i, \lambda r)}) < r/t$.

(4') $\Rightarrow$ (1): Let $X$ be $k$-HTB with $k(t) = ct^\sigma$. Then $X$ is $(c, \sigma)$-homogeneous. For, if $(a, b, A) \in T(X)$, then applying the $k$-HTB property of $X$ with $r = b$ and $t = b/a$ yields $\text{card} \ A \leq k(t) = c(b/a)^\sigma$.

(4') $\Rightarrow$ (4): Trivial.

(4) $\Rightarrow$ (3'): If $X$ is $k$-HTB, then (3') is satisfied with $\lambda = \frac{1}{2}$ and with $n$ the integer part of $k(2)$.
Remark A.4. In A.3(1) the datum \((C, s)\) cannot be replaced by the single number \(s\), that is, by \(s\)-homogeneity. To see this, note that every finite metric space \(X\) is 0-homogeneous but the smallest number \(C\) for which \(X\) is \((C, 0)\)-homogeneous is \(C = \text{card} \ X\).

Theorem A.5. Let \((X, d)\) be a metric space. The following conditions are basic properties of Assouad dimension. With the last three of them excluded, comparing Assouad dimension with other dimensions, all they have a more precise formulation in terms of homogeneity parameters as made explicit in the proof.

1. If \(d'\) is a metric on \(X\) with \(\alpha d \leq d' \leq \beta d\) for some \(0 < \alpha \leq \beta\), then \(\dim(A)(X, d') = \dim(A)(X, d)\).

2. If \(Y \subset X\), then \(\dim(A)(Y) \leq \dim(A)(X)\) and \(\dim(A)(\overline{X}) = \dim(A)(X)\).

3. If \(X = \bigcup_{i=1}^{n} X_i\) with \(n \geq 1\), then \(\dim(A)(X) = \max_{1 \leq i \leq n} \dim(A)(X_i)\).

4. If \(X_1, \ldots, X_n\) are non-empty metric spaces with \(n \geq 1\), if \(X = X_1 \times \cdots \times X_n\), and if \(d\) is any of the (mutually bi-Lipschitz equivalent) standard product metrics on \(X\), then \(\max_{1 \leq i \leq n} \dim(A)(X_i) \leq \dim(A)(X) \leq \sum_{i=1}^{n} \dim(A)(X_i)\).

5. If \(X_1 = \cdots = X_n = Y\) in (4), then \(\dim(A)(X) = n \dim(A)(Y)\).

6. If \(X \subset \mathbb{R}^m\) and \(X\) has interior points, then \(\dim(A)(X) = m\).

7. If \(p \in (0, 1)\), then \(\dim(A)(X, d^p) = (1/p) \dim(A)(X, d)\).

8. If \(d'\) is a metric on the set \(X\), then \(\dim(A)(X, c'd') \leq \dim(A)(X, d) + \dim(A)(X, d')\).

9. If \(d'\) is a metric on \(X\) such that \(\text{id} : (X, d) \to (X, d')\) is a quasisymmetric homeomorphism and if \(\dim(A)(X, d) < \infty\), then \(\dim(A)(X, d') < \infty\).

10. Always \(\dim(H)(X) \leq \dim(A)(X)\).

11. If \(X\) is bounded, then \(\overline{\dim(H)}(X) \leq \dim(A)(X)\).

12. Always \(\dim(P)(X) \leq \dim(A)(X)\).

Proof. We only prove the respective claims for the homogeneity parameters.

1. Let \((X, d')\) be \((C, s)\)-homogeneous. We show \((X, d)\) to be \((C', s)\)-homogeneous with \(C' = C(\beta/\alpha)^s\). Let \((a, b, A) \in T(X, d)\). Then \((\alpha a, \beta b, A) \in T(X, d')\). Hence, \(\text{card} \ A \leq C(\beta/\alpha)^{s} \leq C'(b/a)^{s}\). For the converse note that \(\beta^{-1}d' \leq d \leq \alpha^{-1}d'\) and \(\alpha^{-1}/\beta^{-1} = \beta/\alpha\).

2. Clearly, if \(X\) is \((C, s)\)-homogeneous, then \(Y\) is \((C, s)\)-homogeneous. Now suppose \(Y\) to be \((C, s)\)-homogeneous. We prove \(\overline{Y}\) to be \((C, s)\)-
homogeneous. Let \((a, b, A) \in T(Y)\). Consider \(\varepsilon \in (0, a/2)\). Choose a map \(f: A \to Y\) with \(d(x, f(x)) < \varepsilon\) for each \(x \in A\). Then \(f\) is injective and \((a-2\varepsilon, b+2\varepsilon, fA) \in T(Y)\). Hence, \(\text{card } A \leq C((b+2\varepsilon)/(a-2\varepsilon))^s\). Thus, \(\text{card } A \leq C(b/a)^s\).

(3) Let \(X_i\) be \((C_i, s_i)\)-homogeneous for \(1 \leq i \leq n\). We show \(X\) to be \((C, s)\)-homogeneous with \(C = \sum_{i=1}^n C_i\) and \(s = \max_{1 \leq i \leq n} s_i\). Let \((a, b, A) \in T(X)\). Then \((a, b, A \cap X_i) \in T(X_i)\) if \(A \cap X_i \neq \emptyset\), and hence \(\text{card } (A \cap X_i) \leq C_i(b/a)^{s_i}\) for each \(i\), which implies that \(\text{card } A \leq C(b/a)^s\).

(4) By (1) assume that \(d(x, y) = \max_{1 \leq i \leq n} d(x_i, y_i)\) for all \(x = (x_1, \ldots, x_n)\) and \(y = (y_1, \ldots, y_n)\) in \(X\). Since each \(X_i\) can be isometrically embedded in \(X\), the first inequality in (4) is clear. Now let \(X_i\) be \((C_i, s_i)\)-homogeneous for \(1 \leq i \leq n\). We show \(X\) to be \((C, s)\)-homogeneous with \(s = \sum_{i=1}^n s_i\) and \(C = 4^s \prod_{i=1}^n C_i\). Let \((a, b, A) \in T(X)\). Consider \(a' \in (0, a)\). If \(1 \leq i \leq n\) and if \(B_i \subset X_i\) is the projection of \(A\), then, since \(d(B_i) \leq b\), from A.3(2') with \(\lambda = a'/2b\) and \(r = b\) we get a natural number \(m_i \leq C_i(2, \lambda)^{s_i} = 4^{s_i} C_i(b/a')^{s_i}\), and sets \(B_{ij} \subset B_i\) with \(d(B_{ij}) \leq a'\) for \(1 \leq j \leq m_i\) such that \(B_i = \bigcup_{j=1}^{m_i} B_{ij}\). Let \(J = \prod_{i=1}^n \{1, \ldots, m_i\}\). If \(j = (j_1, \ldots, j_n) \in J\), let \(A_j = A \cap (B_{1j_1} \times \cdots \times B_{nj_n})\); then \(d(A_j) \leq a' < a\), whence \(\text{card } A_j \leq 1\). Since \(A = \bigcup_{j \in J} A_j\), we thus get that \(\text{card } A \leq m_1 \cdots m_n \leq C(b/a')^s\). Hence, \(\text{card } A \leq C(b/a)^s\).

(5) Let \(X = Y^n\) be \((C, s)\)-homogeneous, and let \(d\) be as in the proof of (4). We show \(Y\) to be \((C^{1/n}, s/n)\)-homogeneous. Let \((a, b, A) \in T(Y)\). Then \((a, b, A^n) \in T(X)\). Hence, \(\text{card } A^n \leq C(b/a)^s\), implying that \(\text{card } A \leq C^{1/n}(b/a)^{s/n}\).

(6) It is easy to see that \(\text{card } A \leq 1 + b/a \leq 2b/a\) for each \((a, b, A) \in T(R)\). Thus, \(R\) is \((2, 1)\)-homogeneous. If \(Z = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\} \subset I = [0, 1]\) and \(Z\) is \((C, s)\)-homogeneous, then for each \(n \geq 1\) considering the set \(A = \{1/k \mid n \leq k \leq 2n\}\) \(Z\) gives \(r + 1 \leq C(2n - 1)^s\), which yields \(s \geq 1\). Hence, \(\dim_A I = \dim_A Z = 1\). It follows that \(R^m\) is \(m\)-homogeneous and \(\dim_A I^m = m\) for each \(m \geq 0\). Thus, (6) holds. Moreover, it is easy to see that \(R^m\) with the maximum norm \(\| \cdot \|\) is \((2^m, m)\)-homogeneous with \(2^m\) the best possible constant; as \(\|x\| \leq |x| \leq m^{1/2}\|x\|\), this implies \(R^m\) with the usual norm to be \((2^m m^{m/2}, m)\)-homogeneous.

(7) If \((X, d)\) is \((C, s)\)-homogeneous, then \((X, d^p)\) is \((C, s/p)\)-homoge-
neous. In fact, if \((a, b, A) \in T(X, d^p)\), then \((a^{1/p}, b^{1/p}, A) \in T(X, d)\), so \(\text{card } A \leq C(b^{1/p}/a^{1/p})^s = C(b/a)^{s/p}\). Conversely, if \((X, d^p)\) is \((C, s)\)-homogeneous, then \((X, d)\) is \((C, ps)\)-homogeneous. In fact, if \((a, b, A) \in T(X, d)\), then \((a^p, b^p, A) \in T(X, d^p)\), so \(\text{card } A \leq C(b^p/a^p)^s = C(b/a)^{ps}\).

(8) Define a product metric \(\sigma(x, y) = d(x_1, y_1) + d'(x_2, y_2)\) on the set \(X \times X\). Then the diagonal map \((X, d + d') \to (X \times X, \sigma)\) is an isometric embedding. Hence, the claim follows from (4).

(9) Let id: \((X, d) \to (X, d')\) be an \(\eta\)-quasisymmetric homeomorphism. If \((X, d)\) is \((C, s)\)-homogeneous, then \((X, d')\) is \((C', s')\)-homogeneous with \((C', s')\) depending only on \((C, s, \eta)\). In fact, A.3 reduces this claim to the result [66, 2.10] that if \((X, d)\) is \(k\)-HTB, then \((X, d')\) is \(k'\)-HTB with \(k'\) depending only on \((k, \eta)\).

(10) Write \(X = \bigcup_{i=1}^{\infty} X_i\) with each \(X_i\) bounded. Since \(\dim_H X_i \leq \dim_A X_i \leq \dim_A X\) for each \(i\) by 3.4 and since \(\dim_H X = \sup_{i \geq 1} \dim_H X_i\), the claim follows.

(11) This property (observed in [4]) follows from 3.4.

(12) Write \(X = \bigcup_{i=1}^{\infty} X_i\) with bounded \(X_i\)’s. Then \(\dim_B X \leq \sup_{i \geq 1} \dim_B X_i \leq \dim_A X\) by (11).

\[\square\]

**EXAMPLES A.6.** 1. We give an example of two metric spaces \(X_1\) and \(X_2\) for which \(\dim_A(X_1 \times X_2) = \max(\dim_A X_1, \dim_A X_2) < \dim_A X_1 + \dim_A X_2\). Let \(X_1 = \mathbb{Z} \subset \mathbb{R}\) and \(X_2 = I \subset \mathbb{R}\). Then \(X_i\) is \((2,1)\)-homogeneous and \(\dim_A X_i = 1\) for each \(i\). Let \(X = \mathbb{Z} \times I\), and metrize \(X\) by \(d((m, x), (n, y)) = \max(|m - n|, |x - y|)\). It suffices to show that \(X\) is \((4,1)\)-homogeneous. Let \((a, b, A) \in T(X)\). Let \(A_n = \{x \in I \mid (n, x) \in A\}\) for \(n \in \mathbb{Z}\), and let \(B = \{n \in \mathbb{Z} \mid A_n \neq \emptyset\}\). If \(a > 1\), then \(\text{card } A_n = 1\) for each \(n \in B\) and \((a, b, B) \in T(\mathbb{Z})\); thus \(\text{card } A = \text{card } B \leq 2b/a\). If \(b < 1\), then \(B = \{n\}\) for some \(n \in \mathbb{Z}\); now \((a, b, A_n) \in T(I)\), and thus \(\text{card } A = \text{card } A_n \leq 2b/a\). Suppose finally that \(a \leq 1 \leq b\). Now \((1, b, B) \in T(\mathbb{Z})\) implying that \(\text{card } B \leq 2b\). If \(n \in B\), then \(A_n\) is \(a\)-discrete, so \((a, 1, A_n) \in T(I)\) implying that \(\text{card } A_n \leq 2/a\). Hence, \(\text{card } A \leq 4b/a\).

Here \(X_1\) is noncompact, but Larman [37, Theorem 9] has constructed for each \(\varepsilon > 0\) two countable compact sets \(X_1, X_2 \subset \mathbb{R}\) of Assouad dimension 1 such that \(\dim_A(X_1 \times X_2) \leq 1 + \varepsilon\).

2. Lipschitz maps, even if they are homeomorphisms, can raise Assouad dimension. This answers a question of H. Movahedi-Lankarani.
For example, if $d$ is the $\{0, 1\}$-valued ultrametric on $\mathbb{N}$, then $\dim_{A}(\mathbb{N}, d) = \infty$ while $d(x, y) \leq |x - y|$ for all $x, y \in \mathbb{N}$. To find an example among compact spaces, we rewrite with $\alpha = 1$ the example in [42,3.6]. Thus, let $d$ be the ultrametric on $X = \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ for which $d(x, y) = x^{2}$ if $x > y$; then $\dim_{A}(X, d) = \infty$ while $|x - y|^{2} \leq d(x, y) \leq 2|x - y|$ for all $x, y \in X$.

**Terminology A.7.** We first recall terminology of Väisälä [68]. Let $X$ be a metric space. The one-point extension of $X$ by a point $\omega \notin X$ "at infinity" is a topological space $\check{X} = X \cup \{\omega\}$ such that the open sets of $\check{X}$ are the open sets of $X$ and the complements in $\check{X}$ of the bounded closed sets of $X$. Then $\check{X}$ is metrizable; see below. If $Z \subset X$, then $\check{Z}$ is identified with the subspace $Z \cup \{\omega\}$ of $\check{X}$. The cross ratio of a quadruple $(x_1, x_2, x_3, x_4)$ of distinct points in $\check{X}$ is the positive number

$$|x_1, x_2, x_3, x_4| = \frac{d(x_1, x_3)d(x_2, x_4)}{d(x_1, x_4)d(x_2, x_3)}$$

with the factors containing $\omega$ omitted. If $Z \subset \check{X}$, $Y$ is a metric space, $f: Z \rightarrow \check{Y}$ is an embedding, $\theta$ is a self-homeomorphism of $[0, \infty)$ with $\theta(1) \geq 1$, and $|f(x_1), f(x_2), f(x_3), f(x_4)| \leq \theta(|x_1, x_2, x_3, x_4|)$ for each quadruple $(x_1, x_2, x_3, x_4)$ of distinct points in $Z$ then we say that $f$ is quasi-Möbius or also $\theta$-quasi-Möbius. In this case $f^{-1}: fZ \rightarrow Z$ is $\theta'$-quasi-Möbius with $\theta'(t) = \theta^{-1}(t^{-1})^{-1}$ for $t > 0$. If $f$ is id-quasi-Möbius, then $f$ preserves all cross ratios and is said to be Möbius.

Let $E$ be a normed vector space (always over $\mathbb{R}$) with norm $|\cdot|$. For $a \in E$ and $r > 0$, the inversion of $E$ in the sphere $S(a, r) = \{x \in E | |x - a| = r\}$ is the homeomorphism $v: E \rightarrow E$ defined by $v(x) = a + r^{2}(x - a)/|x - a|^{2}$ if $x \neq a, \omega$, by $v(a) = \omega$, and by $v(\omega) = a$; then $v^{-1} = v$. Let $\lambda = 1$ if $E$ is an inner product space and $\lambda = 3$ otherwise. If $u$ is the inversion of $E$ in the unit sphere $S(0, 1)$, that is, $u(x) = x/|x|^{2}$ for $x \neq 0, \omega$, then

$$\frac{|x - y|}{\lambda|x||y|} \leq |u(x) - u(y)| \leq \frac{\lambda|x - y|}{|x||y|} \quad \text{for all } x, y \in E \setminus \{0\}$$

by [68, (1.7) and (1.8)]. It follows that every inversion of $E$ is Möbius if $E$ is an inner product space and $\theta$-quasi-Möbius with $\theta(t) = 81t$.
for $t \geq 0$ otherwise. Here $81$ is the best possible constant if $E$ is, for example, $\mathbb{R}^2$ with the $l^1$-norm or $l^\infty$-norm; cf. [27, Exemple].

We introduce a new term. If an embedding $f : Z \to \hat{Y}$ for $Z \subset \hat{X}$ is $\theta$-quasi-Möbius with $\theta$ linear, then we say that $f$ is pseudo-Möbius; if here $\theta(t) = ct$ for $t \geq 0$ with $c \geq 1$, that is, if $|f(x_1), f(x_2), f(x_3), f(x_4)| \leq c|x_1, x_2, x_3, x_4|$ whenever $x_1, x_2, x_3, x_4 \in Z$ are distinct, then we also say that $f$ is $c$-pseudo-Möbius. In this case $f^{-1} : fZ \to Z$ is again $c$-pseudo-Möbius. Like Möbius embeddings and quasi-Möbius embeddings, pseudo-Möbius embeddings form a category. As we have seen, inversions of normed spaces are pseudo-Möbius. From this it follows that $\hat{X}$ admits a $81$-pseudo-Möbius homeomorphism onto a bounded metric space; see [68, 1.10].

We next extend the familiar concept of spherical (or chordal) metric of $\mathbb{R}^m$ [72, 1.12] to that of $\hat{E}$ with $E$ an arbitrary inner product space. Thus, identify $E$ with the subspace $E \times \{0\}$ of the inner product space $F = E \times \mathbb{R}$, and let $c = (0, 1) \in F$. Then the inversion $\nu$ of $F$ in the sphere $S(c, 1)$ maps $\hat{E}$ onto the sphere $S(2c, 1/2)$. The metric $q(x, y) = |\nu(x) - \nu(y)|$ on $\hat{E}$ induced by this stereographic projection is called the spherical metric of $\hat{E}$. The identity map of the extended space $\hat{E}$ onto the metric space $(\hat{E}, q)$ is Möbius. We have

$$q(x, y) = |x - y|(1 + |x|^2)^{-1/2}(1 + |y|^2)^{-1/2} \quad \text{if } x, y \in E,$$

$$q(x, \omega) = (1 + |x|^2)^{-1/2} \quad \text{if } x \in E.$$

**Remarks A.9. 1.** By [68, 3.2], an $\eta$-quasisymmetric embedding $f : X \to Y$ of metric spaces is $\theta$-quasi-Möbius with $\theta \geq \eta$ depending only on $\eta$. Since quasisymmetric embeddings preserve bounded sets [66, 2.6], the extension $\hat{f} : \hat{X} \to \hat{Y}$ of $f$ by $\hat{f}(\omega) = \omega$ is an embedding, and obviously $\hat{f}$, too, is $\theta$-quasi-Möbius. In particular, if $f$ is a similarity, then $\hat{f}$ is Möbius. Conversely, every quasi-Möbius embedding is a composition of quasisymmetric embeddings possibly extended as just and restrictions of inversions of normed spaces; see [68, (3.16)] for a (quantitatively unsatisfactory) factorization based on the fact that quasi-Möbius embeddings between bounded metric spaces are quasisymmetric [68, 3.12], or extract from the proof of A.10(3) a quantitatively more explicit result.
Möbius embeddings \( f: Z \to \hat{Y} \) for \( Z \subset \hat{X} \) with \( X \) and \( Y \) inner product spaces have an analogous characterization as compositions of similarities and inversions; see \( [68, \text{I}9] \). To obtain a similar characterization for pseudo-Möbius embeddings, note first that if \( f: X \to Y \) is \( L \)-bi-Lipschitz, then its extension \( \hat{f}: \hat{X} \to \hat{Y} \) is \( L^4 \)-pseudo-Möbius. Now, conversely, every pseudo-Möbius embedding is a composition of bi-Lipschitz embeddings possibly extended as just and restrictions of inversions of normed spaces; this can be deduced from the proof of A.10(4). Earlier Renggli \([55, \text{Satz} \ 3]\) has shown that a pseudo-Möbius homeomorphism \( f: C \to C \) with \( f(\omega) = \omega \) is bi-lipschitz on \( C \).

2. Inner product spaces can be characterized as normed spaces where some inversion and consequently all inversions are Möbius. To see this, let \( E \) be a normed space where the inversion \( u \) in the unit sphere is Möbius. Now it suffices to show that \( |x + ty| = |tx + y| \) for all \( x, y \in E \) and \( t \in \mathbb{R} \) with \( |x| = |y| = 1 \) as this is a classical characteristic property of inner product spaces \([25]\). We may assume that \( t \neq 0 \) and \( x \neq \pm y \), in which case \( tx \neq \pm y \). Then \( |u(tx), u(y), u(-y), u(\omega)| = |tx, y, -y, \omega| \), which when simplified yields the required equality.

Frunză and Frunză \([27]\) have characterized inner product spaces as normed spaces where inversions are conformal.

**Theorem A.10.** Assouad dimension has the following properties related to quasi-Möbius embeddings. All these conditions have a more precise formulation in terms of homogeneity parameters as made explicit in the proof.

1. If \( E \) is a normed space, if \( u: E \setminus \{0\} \to E \setminus \{0\} \) is the inversion \( x \mapsto x/|x|^2 \) of \( E \) in the unit sphere, and if \( X \subset E \setminus \{0\} \), then \( \dim_A uX = \dim_A X \).

2. If \( E \) is an inner product space, if \( d \) is the norm metric and \( q \) the spherical metric on \( E \), and if \( X \subset E \), then \( \dim_A(X, d) = \dim_A(X, q) \).

3. If \( X \) and \( Y \) are metric spaces, if \( Z \subset \hat{X} \), if \( f: Z \to \hat{Y} \) is a quasi-Möbius embedding, and if \( \dim_A(Z \cap X) < \infty \), then \( \dim_A(f(Z \cap Y)) < \infty \).

4. If \( f \) in (3) is pseudo-Möbius, then \( \dim_A(f(Z \cap Y)) = \dim_A(Z \cap X) \).

**Proof.** (1) Let \( \lambda \) be as in (A.8). Suppose \( uX \) to be \((C, s)\)-homogeneous. It suffices to show \( X \) to be \((\mu C, s)\)-homogeneous with \( \mu \geq 1 \) depending only on \((s, \lambda)\). If \( s = 0 \), this is true with \( \mu = 1 \); see 4.12.1
and A.4. Assuming \( s > 0 \) consider \((a, b, A) \in T(X)\) with \( A \) finite. Let \( \alpha = \min\{|x| \mid x \in A\} \) and \( \beta = \max\{|x| \mid x \in A\} \). Then \( a/\lambda \beta^2 \leq |u(x) - u(y)| \leq \lambda b/\alpha^2 \) for all \( x, y \in A \) with \( x \neq y \) by (A.8). It follows that \( \operatorname{card} A = \operatorname{card} uA \leq C((\lambda b/\alpha^2)/(a/\lambda \beta^2))^{2s} = \lambda^{2s}C(\beta/\alpha)^{2s}(b/a)^s. \)

In the case \( b \leq \alpha \) we have \( \beta \leq \alpha + b \leq 2\alpha \) yielding \( \operatorname{card} A \leq (2\lambda)^{2s}C(b/a)^s. \)

Now assuming \( b \geq \alpha \) choose \( k \in \mathbb{N} \) with \( 2^{k-1} \alpha \leq b < 2^k \alpha \). Then \( \beta \leq \alpha + b \leq 2b < 2^{k+1} \alpha \). Write \( A \) as the union of the sets \( A_i = \{x \in A \mid 2^{i-1} \alpha \leq |x| < 2^{i} \alpha \} \) for \( i \in \{1, \ldots, k+1\} \). Here \( d(A_i) < 2^{i+1} \alpha \), so \( \operatorname{card} A_i \leq \lambda^{2s}C(2^i \alpha/2^{i-1} \alpha)^{2s}(2^i \alpha/\alpha)^s = (4\lambda)^{2s}C2^{(i-1)s}(\alpha/a)^s. \)

Hence,

\[
\operatorname{card} A \leq (4\lambda)^{2s}C \sum_{i=1}^{k+1} 2^{(i-1)s}(\alpha/a)^s \leq \mu C(2^{k-1}\alpha/a)^s \leq \mu C(b/a)^s,
\]

where \( \mu = (8\lambda)^{2s}/(2^s - 1) \geq (2\lambda)^{2s}. \)

We conclude that \( X \) is \((\mu C, s)\)-homogeneous.

(2) Let \( F, e, \) and \( v \) be as in the construction of \( q \) in A.7. Then \( X \subset F \setminus \{c\} \), and \( v \) maps \((X, q)\) isometrically \( \text{onto} vX \). Hence, it follows from (1) that if one of the spaces \((X, d)\) and \((X, q)\) is \((C, s)\)-homogeneous, then the other is \((\mu C, s)\)-homogeneous where \( \mu \geq 1 \) is the constant of (1) corresponding to \( s \) and \( \lambda = 1 \).

(3) Let \( f \) be \( \theta \)-quasi-Möbius and \( Z \cap X \) be \((C, s)\)-homogeneous. We show \( fZ \cap Y \) to be \((C', s')\)-homogeneous with \((C', s')\) depending only on \((C, s, \theta)\). Since every metric space can be isometrically embedded in a Banach space, we may assume that \( X \) and \( Y \) are Banach spaces.

Suppose first that \( \omega \in Z \) and \( f(\omega) = \omega \). Then clearly the homeomorphism \( f_1: Z \cap X \to fZ \cap Y \) induced by \( f \) is \( \theta \)-quasisymmetric \([68, 3.10]\). Hence, A.5(9) implies that \( fZ \cap Y \) is \((C', s')\)-homogeneous with \((C', s')\) depending only on \((C, s, \theta)\).

Suppose then that \( f(a) = \omega \) for some \( a \in Z \cap X \). As the translation \( x \mapsto x - a \) of \( X \) is isometric, we may assume \( a = 0 \). Let \( u \) be the inversion of \( X \) in the unit sphere. Let \( \mu \) be the constant of (1) corresponding to \( s \) and \( \lambda = 3 \). Then \( uZ \cap (X \setminus \{0\}) = u[(Z \cap X) \setminus \{0\}] \) is \((\mu C, s)\)-homogeneous and \( uZ \cap X \) as the union of this set and \( uZ \cap \{0\} \) thus \((\mu C + 1, s)\)-homogeneous. Now \( g = fu: uZ \to \hat{Y} \) is a \( \theta_1 \)-quasi-Möbius embedding with \( \theta_1(t) = \theta(81t) \) for \( t \geq 0 \) such that \( g(\omega) = \omega \) and \( fZ = g[uZ] \). Hence, the claim follows from the previous case.
Suppose finally that \( fZ \subset Y \). Assuming \( Z \neq \emptyset \) choose \( a \in Z \); then replacing \( f \) by \( f - f(a) \) we may assume \( f(a) = 0 \). Let \( u' \) be the inversion of \( Y \) in the unit sphere. Then \( h = u'f : Z \to \bar{Y} \) is \( \theta_2 \)-quasi-Möbius with \( \theta_2(t) = 81\theta(t) \) for \( t \geq 0 \) and with \( h(a) = \omega \). Hence, by the previous cases \( h[Z \setminus \{a\}] = hZ \cap Y \) is \((C_1, s_1)\)-homogeneous with \((C_1, s_1)\) depending only on \((C, s, \theta)\). Let \( \mu_1 \) be the constant of \((1)\) corresponding to \( s_1 \) in place of \( s \) and \( \lambda = 3 \). Then \( fZ \setminus \{0\} = u'h[Z \setminus \{a\}] \) is \((\mu_1C_1, s_1)\)-homogeneous. Hence, \( fZ \) is \((\mu_1C_1 + 1, s_1)\)-homogeneous.

(4) Let \( f \) be \( c \)-pseudo-Möbius and \( Z \cap X \) be \((C, s)\)-homogeneous. It suffices to show that now \( fZ \cap Y \) is \((C', s)\)-homogeneous with \( C' \) depending only on \((C, s, c)\). For this it is enough to make the following remarks about the proof of \((3)\) in the present situation.

In the case that \( \omega \in Z \) and \( f(\omega) = \omega \), by \textit{A.2} there is \( \alpha > 0 \) such that \( \alpha|x - y| < |f(x) - f(y)| < c^2 \alpha|x - y| \) for all \( x, y \in Z \cap X \); then \( fZ \cap Y \) is \((c^2sC, s)\)-homogeneous

In the case that \( f(a) = \omega \) for some \( a \in Z \cap X \) we have \( \theta_1(t) = 81ct \) for \( t \geq 0 \). Now \( fZ \cap Y \) is \((C', s)\)-homogeneous with \( C' = (81c)^{2s}((\mu C + 1) \geq c^{2s}C \).

Finally, in the case \( fZ \subset Y \) we have \( \theta_2(t) = 81ct \) for \( t \geq 0 \) and can therefore choose \( s_1 = s \) and \( C_1 = (81^2c)^{2s}((\mu C + 1) \). Now \( \mu_1 = \mu \), and \( fZ \) is thus \((C', s)\)-homogeneous with \( C' = \mu C_1 \geq 1 = (81^2c)^{2s}((\mu^2C + \mu) + 1).

To summarize, with \( \mu^* = 4(81^2c)^{2s}\mu^2 \geq 1 \) depending only on \((s, c)\), the set \( fZ \cap Y \) is otherwise always \((\mu^*C, s)\)-homogeneous except when \( Z = \{\omega\} \) and \( fZ \subset Y \) as then \( C = 0 \) and \( C' = 1 \) are the best constants.

\textbf{Notation A.11.} For \( X \subset \mathbb{R}^m \), \( x \in X \), \( r > 0 \), and \( \lambda \geq 1 \) we define a set \( V(X, x, r, \lambda) = \{ y \in \mathbb{R}^m \mid d(y, X) \leq r, |y - x| \leq \lambda r \} \) and let \( |V(X, x, r, \lambda)| \) denote its Lebesgue measure.

\textbf{Theorem A.12.} Let \( X \subset \mathbb{R}^m \) be a set. If \( X \) is \((C, s)\)-homogeneous and \( s \leq m \), then there is \( L \geq 0 \) depending only on \((C, m)\) such that

\[ |V(X, x, r, \lambda)| \leq L r^m \lambda^s \quad \text{for all } x \in X, \ r > 0, \ \text{and } \lambda \geq 1. \]

Conversely, if \( X \) satisfies this condition for some \( L \) and \( s \leq m \), then \( X \) is \((C, s)\)-homogeneous with \( C \) depending only on \((L, m)\).
Proof. Let \( \omega \) denote the Lebesgue measure of the open unit ball in \( \mathbb{R}^m \). Assume first \( X \) to satisfy the condition for some \( L \) and \( s \leq m \). Let \( (a,b,A) \in T(X) \). Fix \( x_0 \in A \), and let \( r = a/2 \) and \( \lambda = (b + r)/r = 1 + 2b/a \leq 3b/a \). Then the open balls \( B(x,r) \) in \( \mathbb{R}^m \) with \( x \in A \) form a disjoint family in \( V(X,x_0,r,\lambda) \). Thus, \( \text{card } A \omega_r \mathbb{R}^m \leq |V(X,x_0,r,\lambda)| \leq Lr^m \lambda^s \) implying \( \text{card } A \leq (3^s L/\omega)(b/a)^s \). Hence, \( X \) is \((C,s)\)-homogeneous with \( C = 3^s L/\omega \).

Conversely, assume \( X \) to be \((C,s)\)-homogeneous with \( s \leq m \). Let \( x_0 \in X \), \( r > 0 \), and \( \lambda \geq 1 \). Choose an \( r \)-net \( Z \) in \( B(x_0,3\lambda r) \cap X \). Consider \( y \in V(X,x_0,r,\lambda) \). Choose \( x \in X \) with \( |y - x| < 2r \), and note that \( |y - x_0| \leq \lambda r \). Then \( |x - x_0| < 2r + \lambda r \leq 3\lambda r \), so there is \( z \in Z \) with \( |x - z| < r \). Now \( |y - z| < 3r \). Hence, \( V(X,x_0,r,\lambda) \subset \bigcup_{z \in Z} B(z,3r) \). Since \( \text{card } Z \leq C(6\lambda r/r)^s = 6^s C \lambda^s \), this implies that \( |V(X,x_0,r,\lambda)| \leq 6^s C \lambda^s \omega(3r)^m \leq 1 \cdot r^m \lambda^s \) with \( L = 18^s C \omega \). □

**Theorem A.13.** Let \((X,d)\) be a metric space. Then the following conditions are equivalent.

1. There are \( C, s \geq 0 \) such that \( X \) is \((C,s)\)-homogeneous.

2. For each \( p \in (0,1) \) there are \( m \in \mathbb{N} \), \( L \geq 1 \), and an \( L\)-bi-Lipschitz embedding \( f : (X,d^p) \to \mathbb{R}^m \).

3. There are \( p \in (0,1) \), \( m \in \mathbb{N} \), \( L \geq 1 \), and an \( L\)-bi-Lipschitz embedding \( f : (X,d^p) \to \mathbb{R}^m \).

4. There are \( m \in \mathbb{N} \), a self-homeomorphism \( \eta \) of \([0,\infty)\), and an \( \eta\)-quasisymmetric embedding \( f : X \to \mathbb{R}^m \).

Moreover, consider the following data in these conditions: the pair \((C,s)\) in (1), the function \( p \mapsto (m,L) \) in (2), the triple \((p,m,L)\) in (3), and the pair \((m,\eta)\) in (4). Then for each pair of these conditions, the datum of the second condition can be chosen to depend only on the datum of the first condition.

Theorem A.13 with (4) excluded is due to Assouad ([2, Proposition 1.30], [3, Remarque (2)], [5, Proposition 2.6]). Of course, (2) \(\Rightarrow\) (3) is trivial, and (3) \(\Rightarrow\) (1), also if \( p = 1 \), follows from A.5 with \( s = pm \). Since \( f : (X,d) \to \mathbb{R}^m \) in (3) is \( \eta\)-quasisymmetric with \( \eta(t) = L^2 t^p \), we conclude that (3) \(\Rightarrow\) (4) holds. The implication (4) \(\Rightarrow\) (1) obtains, for by A.5(9) we may assume that \( X \subset \mathbb{R}^m \) with \( f \) the inclusion map.

Assouad’s proof for A.13 is so natural that recently Stephen Semmes, being unaware of it, rediscovered it (unpublished).
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Remark A.14. For ultrametric spaces there is a definitive bi-Lipschitz embeddability result. Let $X$ in A.13 be an ultrametric space. Then, since $(X, d^p)$ is an ultrametric space for each $p > 0$, it follows from A.5(7) and A.13 that $(0, 1)$ can be replaced by $(0, \infty)$ in (2) and (3) ([3, Remarque (2) and Proposition 3(g)]). More precisely, by the author and Movahedi-Lankarani [42, 3.8] (an idea from [48]), if $X$ is $(C, s)$-homogeneous and $m > s$, then there is an $L$-bi-Lipschitz embedding $f : X \to \mathbb{R}^m$ with $L$ depending only on $(C, s)$, and conversely by Luosto [40], if there is an $L$-bi-Lipschitz embedding $f : X \to \mathbb{R}^m$ with $m > 0$, then $X$ is $(C, s)$-homogeneous with $s < m$, with $(C, s)$ depending only on $(m, L)$, and with $s \to 0$ as $L \to 1$. The first part implies that every ultrametric space of finite Assouad dimension admits a quasisymmetric embedding in $\mathbb{R}^1$ [42, 3.9]. Movahedi-Lankarani and Wells [49] proved independently the following weaker form of the second part: If a subset of $\mathbb{R}^m$, where $m > 0$, is bi-Lipschitz homeomorphic to an ultrametric space, then it is of Lebesgue measure zero. More strongly, the Hausdorff dimension of this set is smaller than $m$ by S. Semmes [49, 1.5].

Assouad’s question A.15. Assouad [2] asked whether every metric space of finite Assouad dimension can be bi-Lipschitz embedded in a Euclidean space, that is, whether $(1) \Rightarrow (2)$ in A.13 is also true for $p = 1$ (at least in a nonquantitative form). The answer is now known to be negative. Semmes [61, 7.1] observed that this follows from the later theorem due to Pausu [52, Théorème 2] on differentiability of Lipschitz maps $f : G \to G'$ of Carnot groups endowed with their Carnot-Caratheodory metrics. This fact had also become known to Assouad.

More explicitly, let $G = H$ be the Heisenberg group, which we take to be $\mathbb{C} \times \mathbb{R}$ endowed with the product $(z, s)(w, t) = (z + w, s + t + 2 \text{Im } z \overline{w})$ and with the norm $|(z, s)| = (|z|^4 + s^2)^{1/4}$. Then $e = (0, 0)$ is the neutral element, $(z, s)^{-1} = (-z, -s)$, and setting $d(x, y) = |x^{-1}y|$ for $x, y \in H$ gives a left-invariant metric $d$ on $H$, which is topologically equivalent to the Euclidean metric and bi-Lipschitz equivalent to the Carnot-Caratheodory metric. Moreover, if $t > 0$, setting $\delta_t(z, s) = (tz, t^2s)$ defines an automorphism $\delta_t$ of $H$, which is a dilation in the sense that $d(\delta_t(x), \delta_t(y)) = td(x, y)$. 
Pick an arbitrary \( m \in \mathbb{N} \), and let \( G' \) be the additive group \( \mathbb{R}^m \), which is a Carnot group with its Euclidean norm and dilations \( x \mapsto tx \) for \( t > 0 \). Suppose that there is an \( L \)-bi-Lipschitz embedding \( f : H \to \mathbb{R}^m \) for some \( L \). Define \( f_t(x; y) = (f(x\delta_t(y)) - f(x))/t \) for \( t > 0 \) and \( x, y \in H \). Since \( f \) is locally Lipschitz, Pansu’s theorem implies that for almost every point \( x \in H \) with respect to the Lebesgue measure, which is now the Haar measure, the limit \( f_x(y) = \lim_{t \to 0} f_t(x; y) \) exists for each \( y \in H \) uniformly on compact sets, \( f_x\delta_t = tf_x \) for each \( t > 0 \), and \( f_x : H \to \mathbb{R}^m \) is a group homomorphism. Since \( \mathbb{R}^m \) is commutative whereas \( H \) is not, the kernel of \( f_x \) is then nontrivial. But, along with \( f_t(x; \cdot) \) for all \( t > 0 \), the map \( f_x \) is \( L \)-bi-Lipschitz and thus injective, which yields a contradiction.

Now note that \( \dim_A H < \infty \). In fact, from the compactness of \( \overline{B}(e, 1) \), the left-invariance of the metric \( d \), and the existence of the dilations \( \delta_t \) it easily follows that \( H \) has the property A.3(3) with \( \lambda = \frac{1}{2} \). We compute further the exact value of \( \dim_A H \). Let \( |\cdot| \) denote the Lebesgue measure on \( H \), and let \( \omega = |B(e, 1)| \). Since the Jacobian determinant of \( \delta_t \) is \( r^d \) at every point, we have \( |B(x, r)| = \omega r^d \). Consequently, \( |\cdot| \) is a 4-homogeneous measure on \( H \) in the sense of 6.2. Thus, \( \dim_A H \leq 4 \) by (6.6). On the other hand, since \( |A| \leq \omega d(A)^4 \) for each bounded closed set \( A \subset H \), it is easily seen that \( \dim_H H \geq 4 \) [24, Principle 4.2]. Hence, \( \dim_A H = \dim_H H = 4 \).

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