SELECTION THEOREMS WITH $n$-CONNECTEDNESS

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Abstract. We give a generalization of the selection theorem of Ben-El-Mechaiekh and Oudadess to complete $LD$-metric spaces with the aid of the notion of $n$-connectedness. Our new selection theorem is used to obtain new results on fixed points and coincidence points for compact lower semicontinuous set-valued maps with closed values consisting of $\mathcal{D}$-sets in a complete $LD$-metric space.

0. Introduction

In 1991 Horvath [3] extended Michael's selection theorem [4] for closed convex valued lower semicontinuous maps to nonconvex values. In 1995 Ben-El-Mechaiekh and Oudadess [1] gave a generalized selection theorem by combining the result in [3] with [5] related to sets of topological dimension $\leq 0$. Using the concept of $n$-connectedness, we introduce $LD$-metric spaces which are more general than $l.c.$ metric spaces given in [3]. The purpose in this paper is first to extend the selection theorem in [1] to closed valued lower semicontinuous maps with $\mathcal{D}$-set values in a complete $LD$-metric space except possibly on a set of topological dimension $\leq 0$ and then to give new results on fixed points and coincidence points for compact lower semicontinuous set-valued maps with closed values consisting of $\mathcal{D}$-sets in a complete $LD$-metric space.

1. Preliminaries

Let $X$ and $Y$ be topological spaces. A set-valued map (simply, a map) $T : X \rightarrow Y$ is a function from $X$ into the set $2^Y$ of all nonempty...
subsets of $Y$; the map $T^- : Y \to X$ is defined by $T^- y := \{ x \in X : y \in Tx \}$ whenever $T$ is surjective. A map $T : X \to Y$ is said to be compact if its range $\bigcup_{x \in X} Tx$ is relatively compact in $Y$; and lower semicontinuous if $\{ x \in X : Tx \cap V \neq \emptyset \}$ is open in $X$ for every open set $V$ in $Y$. A continuous function $f : X \to Y$ is called a selection of $T : X \to Y$ whenever $f(x) \in Tx$ for every $x$ in $X$.

If $Z$ is a subset of a topological space $X$, then $\dim_X Z \leq 0$ means that $\dim E \leq 0$ for every set $E \subset Z$ which is closed in $X$, where $\dim E$ denotes the covering dimension of $E$.

A topological space $X$ is said to be $n$-connected for $n \geq 0$ if every continuous map $f : S^k \to X$ for $k \leq n$ has a continuous extension over $B^{k+1}$, where $S^k$ is the unit sphere and $B^{k+1}$ the closed unit ball in $\mathbb{R}^{k+1}$. Note that a contractible space is $n$-connected for every $n \geq 0$.

Given a set $Y$, let $\langle Y \rangle$ denote the collection of all nonempty finite subsets of $Y$. Let $\Delta_n = \text{co}\{e_0, \cdots, e_n\}$ be the standard simplex of dimension $n$, where $\{e_0, \cdots, e_n\}$ is the canonical basis of $\mathbb{R}^{n+1}$.

We introduce the following geometric structure as a generalization of convex sets with the aid of the notion of $n$-connectedness.

Let $Y$ be a topological space. A $D$-structure on $Y$ is a map $\mathcal{D} : \langle Y \rangle \to 2^Y$ such that it satisfies the following conditions:

1. for each $A \in \langle Y \rangle$, $\mathcal{D}(A)$ is nonempty and $n$-connected for all $n \geq 0$;
2. for each $A, B \in \langle Y \rangle$, $A \subset B$ implies $\mathcal{D}(A) \subset \mathcal{D}(B)$.

The pair $(Y, \mathcal{D})$ is called a $D$-space; a subset $Z$ of $Y$ is said to be a $\mathcal{D}$-set if $\mathcal{D}(A) \subset Z$ for each $A \in \langle Z \rangle$. A $D$-space $(Y, \mathcal{D})$ is called an $LD$-metric space if $(Y, d)$ is a metric space such that for each $\epsilon > 0$,

$$B(E, \epsilon) = \{ y \in Y : d(y, z) < \epsilon \text{ for some } z \in E \}$$

is a $\mathcal{D}$-set whenever $E \subset Y$ is a $\mathcal{D}$-set and open balls are $\mathcal{D}$-sets.

A $D$-space is a generalization of $c$-spaces in the sense of Horvath [3]. A simple example of a $D$-space but not a $c$-space is the space $Y$, obtained by forming the disjoint union of the comb space $X$ and another copy $X'$ of $X$ and identifying a point $x_0 = (0, 1) \in X$ with the corresponding point $x'_0 \in X'$, by setting $\mathcal{D}(A) := Y$ for every $A \in \langle Y \rangle$. 
It can be shown that any \( D \)-space becomes a generalized convex space introduced by Park and Kim [8].

A generalized convex space \((Y, \Gamma)\) consists of a topological space \(Y\) and a map \(\Gamma : \langle Y \rangle \to 2^Y\) such that the following conditions are satisfied:

1. for each \(A, B \in \langle Y \rangle\), \(A \subseteq B\) implies \(\Gamma(A) \subseteq \Gamma(B)\);
2. for each \(A \in \langle Y \rangle\) with \(|A| = n + 1\), there exists a continuous function \(\Phi_A : \Delta_n \to \Gamma(A)\) such that \(\Phi_A(\Delta_J) \subseteq \Gamma(J)\) for every \(J \in \langle A \rangle\), where \(\Delta_J\) denotes the face of \(\Delta_n\) corresponding to \(J \in \langle A \rangle\).

**Lemma 0.** A \( D \)-space \((Y, \mathcal{D})\) is a generalized convex space.

**Proof.** Since \((Y, \mathcal{D})\) is a \( D \)-space, it suffices to show that for each \(A \in \langle Y \rangle\) with \(|A| = n + 1\), there exists a continuous function \(f : \Delta_n \to \mathcal{D}(A)\) such that \(f(\Delta_J) \subseteq \mathcal{D}(J)\) for every \(J \in \langle A \rangle\). Let \(A = \{a_0, a_1, \cdots, a_n\} \in \langle Y \rangle\) be given such that \(e_i \in \Delta_{\{a_i\}}\). For each \(i \in \{0, 1, \cdots, n\}\), there exists a \(y_i \in \mathcal{D}(\{a_i\})\). Define a function \(f^0 : \Delta_n^0 \to \mathcal{D}(A)\) on the 0-skeleton of \(\Delta_n\) by \(f^0(e_i) := y_i\). Then the function \(f^0\) is continuous and \(f^0(\Delta_{\{a_i\}}) \subseteq \mathcal{D}(\{a_i\})\) for \(i = 0, 1, \cdots, n\).

Assume that a continuous function \(f^k : \Delta_n^k \to \mathcal{D}(A)\) on the \(k\)-skeleton of \(\Delta_n\) has been constructed such that \(f^k(\Delta_J) \subseteq \mathcal{D}(J)\) for all \(J \in \langle A \rangle\) with \(|J| \leq k + 1\).

Now let \(\Delta_J\) be a face of dimension \(k + 1\) of \(\Delta_n\) and let \(J_i := J \setminus \{a_i\}\) for each \(a_i \in J\). Let \(\partial \Delta_J\) be the boundary of \(\Delta_J\). Then \(\partial \Delta_J = \bigcup_{a_i \in J} \Delta_{J_i}\) is contained in the \(k\)-skeleton of \(\Delta_n\) and we have

\[
\begin{align*}
f^k(\partial \Delta_J) &\subseteq \bigcup_{a_i \in J} f^k(\Delta_{J_i}) \\
&\subseteq \bigcup_{a_i \in J} \mathcal{D}(J_i) \subseteq \mathcal{D}(J).
\end{align*}
\]

Note that there is a homeomorphism \(h : E^{k+1} \to \Delta_J\) such that \(h(S^k) = \partial \Delta_J\). Since \(f^k \circ h_{|S^k} : S^k \to \mathcal{D}(J)\) is continuous and \(\mathcal{D}(J)\) is \(k\)-connected, the function \(f^k \circ h_{|S^k}\) has a continuous extension \(g^{k+1} : E^{k+1} \to \mathcal{D}(J)\). Thus, \(f_{J_i}^{k+1} := g^{k+1} \circ h_{J_i}^{-1} : \Delta_{J_i} \to \mathcal{D}(J)\) is continuous and \(f_{J_i}^{k+1}|_{\partial \Delta_J} = f^k|_{\partial \Delta_J}\).
If \( \Delta_J \) and \( \Delta_{J'} \) are \((k + 1)\)-dimensional faces of \( \Delta_n \), \( \Delta_J \neq \Delta_{J'} \) and \( \Delta_J \cap \Delta_{J'} \neq \emptyset \), then it is clear that
\[
f^{k+1}_{J\cap J'}|_{\Delta_J \cap \Delta_{J'}} = f^k|\Delta_J \cap \Delta_{J'} = f^{k+1}_{J\cap J'}|_{\Delta_J \cap \Delta_{J'}}.
\]
Therefore, on the \((k+1)\)-skeleton of \( \Delta_n \) we obtain a continuous function
\[
f^{k+1}_{\Delta_n} : \Delta^{k+1}_n \rightarrow \mathcal{D}(A)
\]
which has the property \( f^{k+1}_{\Delta_J} \subset \mathcal{D}(J) \) for all \( J \in (A) \) with \( |J| \leq k + 2 \). It follows by the induction on \( k \leq n \) that a continuous function \( f : \Delta_n \rightarrow \mathcal{D}(A) \) has been constructed such that
\[
f(\Delta_J) \subset \mathcal{D}(J) \quad \text{for every } J \in (A).
\]
This completes the proof.

\[ \square \]

2. Selection theorems

In this paper, paracompact spaces are assumed to be Hausdorff. The following proposition is a basic statement for the new selection theorem presented in this section.

**Proposition 1.** Let \( X \) be a paracompact space, \( \mathcal{R} \) a locally finite open covering of \( X \), \( (Y, \mathcal{D}) \) a D-space, and \( \eta : \mathcal{R} \rightarrow Y \) a function. Then there exists a continuous function \( g : X \rightarrow Y \) such that
\[
g(x) \in \mathcal{D}(\{\eta(U) : x \in U \text{ and } U \in \mathcal{R}\}) \quad \text{for each } x \in X.
\]

**Proof.** For any \( k \geq 1 \), \( (B^{k+1}, S^k) \) is homeomorphic to \((s, \partial s)\), where \( s \) is a \((k+1)\)-simplex and \( \partial s \) is its boundary (cf. [10], 3.1.22). Therefore, under the weak condition of \( n \)-connectedness instead of contractibility, we can verify our result along the lines of proof of Theorem 3.1 in [3]. \( \square \)

Having established Proposition 1, we now turn to the selection theorem. It begins with the following lemma on \( \epsilon \)-approximate selections.

**Lemma 2.** Let \( X \) be a paracompact space, \( (Y, \mathcal{D}) \) an LD-metric space, \( Z \) a subset of \( X \) with \( \dim_X Z \leq 0 \), and \( T : X \rightarrow Y \) a lower semicontinuous map such that \( T(x) \) is a \( \mathcal{D} \)-set for all \( x \notin Z \). Then for every \( \epsilon > 0 \), \( T \) admits an \( \epsilon \)-approximate selection, that is, a continuous
single-valued function $g_\epsilon : X \to Y$ such that $g_\epsilon(x) \in B(Tx, \epsilon)$ for every $x \in X$.

The proof of Lemma 2 proceeds in precisely the same fashion as Lemma 2 in [1], except that all $c$-sets in an l.c. metric space is replaced by $D$-sets in an $LD$-metric space.

The following main theorem is a generalization of Ben-El-Mechaiekh and Oudadess [1, Theorem 3] which generalizes Michael and Pixley [5, Theorem 1.1].

**Theorem 3.** Let $X$ be a paracompact space, $(Y, D)$ a complete $LD$-metric space, $Z$ a subset of $X$ with $\dim X Z \leq 0$, and $T : X \to Y$ a lower semicontinuous map with closed values such that $Tx$ is a $D$-set for all $x \notin Z$. Then $T$ admits a selection $g : X \to Y$.

**Proof.** Set $T_1 := T$. By Lemma 2, there is $\delta$ continuous function $g_1 : X \to Y$ such that

$$g_1(x) \in B(T_1x, \frac{1}{2})$$

for every $x \in X$.

Hence, a map $T_2 : X \to Y, x \mapsto T_1x \cap B(g_1(x), \frac{1}{2})$, is lower semicontinuous (cf. [4, Proposition 2.4]) and $T_2x$ is a $D$-set for all $x \notin Z$.

Assume that for $k = 1, \cdots, n$, a lower semicontinuous map $T_k : X \to Y$ has been defined and a continuous function $g_k : X \to Y$ has been chosen such that

$$T_1x = Tx$$

$$T_kx = T_{k-1}x \cap B(g_{k-1}(x), \frac{1}{2^{k-1}})$$

for $k = 2, \cdots, n$

are nonempty $D$-sets for all $x \notin Z$ and

$$g_k(x) \in B(T_kx, \frac{1}{2^k})$$

for every $x \in X$.

Hence, a map $T_{n+1} : X \to Y, T_{n+1}x := T_nx \cap B(g_n(x), \frac{1}{2^n})$, is lower semicontinuous and $T_{n+1}x$ is a $D$-set for all $x \notin Z$. By Lemma 2, there exists a continuous function $g_{n+1} : X \to Y$ such that

$$g_{n+1}(x) \in B(T_{n+1}x, \frac{1}{2^{n+1}})$$

for every $x \in X$. 

It follows by induction that there is a sequence of functions $g_n : X \rightarrow Y$ which has the above properties for all $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$ be arbitrary. Then there is a $y \in Y$ such that $y \in T_{n+1}x \cap B(g_{n+1}(x), \frac{1}{2^{n+1}})$ for all $x \in X$, hence we have

$$d(g_{n+1}(x), g_n(x)) \leq d(g_{n+1}(x), y) + d(y, g_n(x)) < \frac{1}{2^{n+1}} + \frac{1}{2^n}.$$ 

It is also clear that the sequence $(g_n)$ is a uniformly Cauchy sequence. Since $Y$ is complete, $(g_n)$ converges uniformly on $X$.

Define a map $g : X \rightarrow Y$ by

$$g(x) := \lim_{n \rightarrow \infty} g_n(x) \quad \text{for } x \in X.$$ 

Then $g$ is continuous and $g(x) \in Tx$ for every $x \in X$ since $Tx$ is closed. This completes the proof. $\square$

Using Theorem 3, we give a sufficient condition for a lower semicontinuous set-valued map with closed values to have the selection extension property.

**Corollary 4.** Let $(Y, \mathcal{D})$ be a complete LD-metric space such that $\mathcal{D}(\{y\}) = \{y\}$ for all $y \in Y$. Let $X$ be a paracompact space, $Z$ a subset of $X$ with $\dim_X Z \leq 0$, and $T : X \twoheadrightarrow Y$ a lower semicontinuous map with closed values such that $Tx$ is a $\mathcal{D}$-set for all $x \notin Z$. If $A$ is closed in $X$, then every selection $g$ for $T|_A$ extends to a selection for $T$. Here $T|_A$ denotes the restriction of $T$ to $A$.

**Proof.** Let $g : A \rightarrow Y$ be a selection for $T|_A$. We define a map $T_g : X \twoheadrightarrow Y$ by

$$T_gx := \begin{cases} \{g(x)\} & \text{for } x \in A \\ Tx & \text{for } x \notin A. \end{cases}$$

Then $T_g$ is a lower semicontinuous map with closed values and $T_gx$ is a $\mathcal{D}$-set for all $x \notin Z$. By Theorem 3, $T_g$ has a selection $f : X \rightarrow Y$, which is a selection for $T$ that extends $g$ because $g : A \rightarrow Y$ is a selection for $T|_A$. $\square$
Corollary 5. Let \((Y, \mathcal{D})\) be a complete LD-metric space such that \(\mathcal{D}(\{y\}) = \{y\}\) for all \(y \in Y\). Let \(X\) be a paracompact space, \(A\) a closed subset of \(X\) and \(g : A \to Y\) a continuous function. Then there is a continuous function \(f : X \to Y\) which extends \(g\).

Proof. A map \(T : X \to Y\), defined by

\[
Tx := \begin{cases} 
\{g(x)\} & \text{for } x \in A \\
Y & \text{for } x \not\in A
\end{cases}
\]

is lower semicontinuous and its values are closed \(\mathcal{D}\)-sets. By Theorem 3, \(T\) has a continuous selection \(f : X \to Y\). Since \(f(x) \in Tx\) for all \(x \in X\), we obtain \(f|_A = g\).

\[
\square
\]

3. Applications to fixed points and coincidence points

We need the following theorem due to Park [7, Theorem 2].

Theorem 6. Let \(X\) be a compact Hausdorff space, \((Y, \Gamma)\) a generalized convex space and \(T : X \to Y\) a map with the property that there is a map \(S : X \to Y\) such that the following conditions are satisfied:

1. For each \(x \in X\), \(A \in \langle Sx \rangle\) implies \(\Gamma(A) \subset Tx\); and
2. \(X = \bigcup \{\text{int} S^{-} y : y \in Y\}\), where \(\text{int}\) denotes the interior.

Then \(T\) has a continuous selection \(f : X \to Y\). More precisely, there exist a simplex \(\Delta_n\) and two continuous functions \(p : X \to \Delta_n\) and \(q : \Delta_n \to Y\) such that \(f = q \circ p\) and \(f(X) \subset \Gamma(A)\) for some \(A \in \langle Y \rangle\) with \(|A| = n + 1\).

An immediate consequence of Theorem 6 and Brouwer’s fixed point theorem is in connection with fixed points and coincidence points for set-valued maps. Since \(D\)-spaces are generalized convex spaces by Lemma 0, Theorem 6 works for \(D\)-spaces.

Theorem 7. Let \(X\) be a compact Hausdorff space, \((Y, \mathcal{D})\) a \(D\)-space, \(S, T : X \to Y\) two maps such that the following conditions are satisfied:

1. \(A \in \langle Sx \rangle\) implies \(\mathcal{D}(A) \subset Tx\) for every \(x \in X\); 
2. \(X = \bigcup \{\text{int} S^{-} y : y \in Y\}\).
Then
(a) For any continuous function \( g : Y \to X \) there is a \( y_0 \in Y \) such that \( y_0 \in Tg(y_0) \).

(b) If \( R : X \to Y \) is a set-valued map such that \( R^- : Y \to X \) has a continuous selection, then there is an \( x_0 \in X \) such that \( Rx_0 \cap Tx_0 \neq \emptyset \).

Proof. (a) Let \( g : Y \to X \) be a continuous function. By Theorem 6, \( T \) has a continuous selection \( f : X \to Y \) and there exist continuous functions \( p : X \to \Delta_n \) and \( q : \Delta_n \to Y \) such that \( f = q \circ p \). The continuous function \( \varphi : \Delta_n \to \Delta_n, z \mapsto p \circ g \circ q(z) \), has a fixed point \( z_0 \), by Brouwer’s fixed point theorem. Setting \( y_0 = q(z_0) \), we have

\[
y_0 = (q \circ p \circ g \circ q)(z_0) = (f \circ g)(y_0) \in Tg(y_0).
\]

(b) Let \( h : Y \to X \) be a continuous selection for \( R^- \). By (a), there is a \( y_0 \in Y \) such that \( y_0 \in Th(y_0) \) and also \( h(y_0) \in R^- y_0 \). If \( x_0 := h(y_0) \), then \( Rx_0 \cap Tx_0 \neq \emptyset \). This completes the proof. \( \square \)

Using the selection theorems above, we establish the existence of fixed points and coincidence points for compact lower semicontinuous set-valued maps with closed values in a complete \( l.D \)-metric space.

**Theorem 8.** Let \( (Y, D) \) be an \( l.D \)-metric space and suppose that for every \( \epsilon > 0 \) there are two maps \( S, T : Y \to Y \) such that the following conditions are satisfied:

1. \( A \in \{Sy \} \) implies \( D(A) \subset Ty \) for every \( y \in Y \);
2. \( Y = \bigcup \{ \text{int } S^{-} y : y \in Y \} \); and
3. \( y \in B(Ty, \epsilon) \) for all \( y \in Y \).

Then any compact continuous function \( g : Y \to Y \) has a fixed point.

Proof. Let \( \epsilon > 0 \). Applying Theorem 7 to \( T|_{g(Y)} \), there is a point \( y_\epsilon \) in \( Y \) such that \( y_\epsilon \in Tg(y_\epsilon) \), hence by (3), \( d(g(y_\epsilon), y_\epsilon) < \epsilon \). Since \( g(Y) \) is relatively compact in \( Y \) and \( g \) is continuous, it is easy to verify that there exists a \( y_0 \in Y \) such that \( g(y_0) = y_0 \). \( \square \)

**Remark.** Theorem 8 remains true if \( Y \) is a Hausdorff uniform space with a \( D \)-structure \( D \) on \( Y \).
Corollary 9. Let \((Y, \mathcal{D})\) be an LD-metric space such that \(\mathcal{D}(\{y\}) = \{y\}\) for all \(y \in Y\). Then any compact continuous function \(g : Y \to Y\) has a fixed point.

Proof. Apply Theorem 8 with \(S = T\) and \(T.x := Y\) for every \(x \in X\). \(\square\)

Theorem 10. Let \((Y, \mathcal{D})\) be a complete LD-metric space such that \(\mathcal{D}(\{y\}) = \{y\}\) for all \(y \in Y\) and let \(Z\) be a subset of \(Y\) with \(\dim_y Z \leq 0\). Then any compact lower semicontinuous map \(T : Y \to Y\) with closed values such that \(Ty\) is a \(\mathcal{D}\)-set for all \(y \notin Z\) has a fixed point.

Proof. By Theorem 3, \(T\) has a continuous selection \(g : Y \to Y\). Since \(g\) is compact, by Corollary 9, \(g : Y \to Y\) has a fixed point. Thus, \(y_0 = g(y_0) \in Ty_0\) for some \(y_0 \in Y\). \(\square\)

Corollary 11. Let \((Y, \mathcal{D})\) be a complete LD-metric space, and \(Z\) a subset of \(Y\) with \(\dim_y Z \leq 0\) such that \(\mathcal{D}(\{y\}) = \{y\}\) for all \(y \in Y\). Let \(T : Y \to Y\) be a compact map with closed values such that \(Tx\) is a \(\mathcal{D}\)-set for all \(x \notin Z\) and \(T^{-1}y\) is open for all \(y \in Y\). Then \(T\) has a fixed point.

Corollary 12. Let \(X\) be a paracompact space, \((Y, \mathcal{D})\) a complete LD-metric space such that \(\mathcal{D}(\{y\}) = \{y\}\) for all \(y \in Y\), \(Z\) a subset of \(X\) with \(\dim_X Z \leq 0\), and let \(S, T : Y \to Y\) be two maps such that the following conditions are satisfied:

1. \(T\) is a compact lower semicontinuous map with closed values such that \(Tx\) is a \(\mathcal{D}\)-set for all \(x \notin Z\);
2. \(S^{-1} : Y \to X\) has a continuous selection.

Then there is an \(x_0 \in X\) such that \(Sx_0 \cap Tx_0 \neq \emptyset\).

Proof. Let \(g : Y \to X\) be a continuous selection for \(S^{-1}\). The composition \(T \circ g : Y \to Y\) is compact, lower semicontinuous. By Theorem 10, there is a \(y_0 \in Y\) such that \(y_0 \notin Tg(y_0)\). Since \(g(y_0) \in S^{-1}y_0\), we have \(Sg(y_0) \cap Tg(y_0) \neq \emptyset\). \(\square\)

With the help of \(\mathcal{D}\)-functions, we give a fixed point theorem which is a generalization of a result in [2].
Let \((X, D)\) be a \(D\)-space. A continuous function \(f : X \times X \to \mathbb{R}\) is said to be a \(D\)-function if it has the following properties:

1. For every \(x \in X\) and every \(\lambda \in \mathbb{R}\), \(\{y \in X : f(x, y) > \lambda\}\) is a \(D\)-set.
2. \(f(x, x) \geq 0\) for all \(x \in X\).

**Theorem 13.** Let \((X, D)\) be a compact Hausdorff \(D\)-space. Suppose that for any \((x_1, x_2) \in X \times X\) with \(x_1 \neq x_2\) there is a \(D\)-function \(f : X \times X \to \mathbb{R}\) such that \(f(x_1, x_2) < 0\). Then any compact continuous function \(g : X \to X\) has a fixed point.

**Proof.** For \(\lambda < 0\) and \(D\)-function \(f\), let

\[
T_\lambda(f) = \{(x, y) \in X \times X : f(x, y) > \lambda\}.
\]

Then \(T_\lambda(f)\) is a graph of the multimap \(x \mapsto \{y \in X : f(x, y) > \lambda\}\) having open inverses and \(D\)-set values.

For \(\lambda_i < 0\) and \(D\)-functions \(f_i, i = 1, \ldots, n\), \(\bigcap_{i=1}^{n} T_{\lambda_i}(f_i)\) is a graph of the multimap \(x \mapsto \{y \in X : f_i(x, y) > \lambda_i\}\) for all \(i\) having open inverses and \(D\)-set values. Since \(Y\) is compact, there exists a unique uniform structure on \(Y\) (cf. [9], II 3.6 Satz 1).

Now let \(V\) be an open entourage and \((x_1, x_2) \in (X \times X) \setminus V\). By assumption, there is a \(D\)-function \(f\) and a number \(\lambda < 0\) such that \(f(x_1, x_2) < \lambda\). Therefore, we have \((x_1, x_2) \notin T_\lambda(f)\). The collection

\[
\{(X \times X) \setminus T_\lambda(f) : \lambda < 0 \text{ and } f \text{ is a } D\text{-function}\}
\]

covers the closed set \((X \times X) \setminus V\). By the compactness of \(X \times X\), there are finitely many \(D\)-functions \(f_1, \ldots, f_n\) and numbers \(\lambda_1, \ldots, \lambda_n < 0\) such that

\[
(X \times X) \setminus V \subset (X \times X) \setminus \bigcap_{i=1}^{n} T_{\lambda_i}(f_i)
\]

hence \(\bigcap_{i=1}^{n} T_{\lambda_i}(f_i) \subset V\). By Theorem 8, any compact continuous function \(g : X \to X\) has a fixed point. \(\square\)
References


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