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JONES' INDEX FOR FIXED POINT ALGEBRAS

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ABSTRACT. We show that if M is a II_1-factor and a countable discrete group G acts outerly on M then Jones' index $[M : M^G]$ of a pair of II_1-factors is equal to the order $|G|$ of G. It is also shown that for a subgroup H of G Jones' index $[M^H : M^G]$ is equal to the group index $[G : H]$ under certain conditions.

1. Introduction

Jones' index theory is one of the most important and interesting topics in recent operator algebra theory and much effort has been made to develop the theory in many connections with other areas of mathematics. The notion of bimodules for von Neumann algebras, Ocneanu's paragroup and sector theory were introduced and played an important role in Jones' index theory [4, 6, 11, 13].

In this paper, we present a relation between Jones' index for fixed point algebras and group index for countable discrete groups. We also give an example of Jones' index for fixed point algebra of group von Neumann algebra.

We first recall the definition of Jones' index. If M is a finite factor with faithful normal normalized trace τ, acting on a Hilbert space H with finite commutant M', then the coupling constant $\dim_M(H)$ of M is defined as $\tau([M'\xi])/\tau'([M\xi])$, where $\xi \in H$, $\xi \neq 0$ and τ' is a trace in M'. For a pair of finite factors $N \subset M$, V. Jones defined in [8] the index of N in M by $[M : N] = \dim_N(H)/\dim_M(H)$, which measures the relative size of M with respect to N. In the case of a pair of crossed product II_1-factors by an outer action of a pair of discrete groups, their

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Jones' index is just the group index. We prove in this paper the result which also holds for a pair of fixed point algebras.

For an action α of a group G on M we denote the fixed point algebra of M under the action α by $M^{\alpha}(G)$ (or M^G). We generalize the well known result of Jones' index that $[M : M^G] = |G|$, where M is a II_1-factor and G is a finite group of outer automorphisms of M ([8], Example 2.3.3). But for infinite groups, the situation is not that clear.

We establish equalities, $[M : M^G] = |G|$ and $[M^H : M^G] = [G : H]$ under certain conditions, where G is a countable discrete group and H is a subgroup of G. As an example, we investigate Jones' index for $\mathcal{L}(G)^{\Gamma_i}$'s for a non inner amenable discrete group G and Γ_i's ($i = 1, 2$), from which we see that $[G^{\Gamma_1} : G^{\Gamma_2}] = [\Gamma_2 : \Gamma_1]$. Jones' index for fixed point algebras seems to be a natural analogue of Jones' index for crossed product algebras.

2. Index for Fixed Point Algebras

From now on, M denotes a II_1-factor with faithful normal normalized trace τ. If $H \leq G$ is a pair of countable discrete groups of automorphisms for which the crossed products $M \rtimes_\alpha G$ and $M \rtimes_\alpha H$ are finite factors then $[M \rtimes_\alpha G : M \rtimes_\alpha H] = [G : H]$ holds ([8], Example 2.3.2). Similarly we investigate Jones' index for a pair of fixed point algebras $M^G \subset M^H$. Our main result is a generalization of the following fact which seems to be well known.

PROPOSITION 2.1. *If G is a finite group of outer automorphisms of M then for a subgroup H of G we have $[M^H : M^G] = [G : H]$.*

PROOF. Since H is a subgroup of a finite group G of outer automorphisms of M, $M^G \subset M^H$ is a pair of II_1-factors. Thus from the facts that $[M : M^G] = |G|$, $[M : M^H] = |H|$, $[M : M^G] = [M : M^H][M^H : M^G]$ and $[G : H] = |G|/|H|$ we have $[M^H : M^G] = [G : H]$. □

Many results for finite group case have been known in great detail in the literature [2, 3, 5, 12, 13] among others. Now let G be a countable discrete group and α a G-action on M. Note that if G is infinite then M^G may not be a II_1-factor, but it is interesting that we have the same results as finite case. In order to do this we study the induced action. Let H

be a normal subgroup of G with a quotient group $G/H = \{\overline{g_i} \mid g_i \in G\}$, where $\overline{g_i} = \{g_i h \mid h \in H\}, i = 1, 2, 3, \cdots$. Then for the fixed point algebra $M^H = \{x \in M \mid \alpha_h(x) = x, \forall h \in H\}$ we get the following lemma.

LEMMA 2.2. *α induces a G/H-action α^H on M^H satisfying*

$$M^G = (M^H)^{\alpha^H(G/H)}.$$

PROOF. For any $\overline{g_i} \in G/H$, we define $\alpha^H{}_{\overline{g_i}} = \alpha_{g_i}|_{M^H}$. Since for any $h \in H$ and $g_i \in G$, $hg_i = g_i h_1$ for some $h_1 \in H$, we have

$$\begin{aligned}\alpha_h(\alpha^H{}_{\overline{g_i}}(x)) &= \alpha_h(\alpha_{g_i}(x)) = \alpha_{hg_i}(x) = \alpha_{g_i h_1}(x) \\ &= \alpha_{g_i}\alpha_{h_1}(x) = \alpha_{g_i}(x) = \alpha^H{}_{\overline{g_i}}(x), \quad \forall x \in M^H.\end{aligned}$$

For any $\overline{g_i}$, $\overline{g_j} \in G/H$ with $\overline{g_i} = \overline{g_j}$, note that $g_j = g_i h$ for some $h \in H$, so we obtain

$$\begin{aligned}\alpha^H{}_{\overline{g_j}}(x) &= \alpha_{g_j}(x) = \alpha_{g_i h}(x) = \alpha_{g_i}(x) \\ &= \alpha^H{}_{\overline{g_i}}(x), \quad \forall x \in M^H.\end{aligned}$$

Moreover, we have

$$\begin{aligned}\alpha^H{}_{\overline{g_i}}\alpha^H{}_{\overline{g_j}}(x) &= \alpha^H{}_{\overline{g_i}}(\alpha_{g_j}(x)) = \alpha_{g_i}(\alpha_{g_j}(x)) \\ &= \alpha_{g_i g_j}(x) = \alpha^H{}_{\overline{g_i g_j}}(x) = \alpha^H{}_{\overline{g_i}\,\overline{g_j}}(x), \quad \forall x \in M^H.\end{aligned}$$

Thus α^H is a well-defined G/H-action on M^H.

Now it remains to show that $M^G = (M^H)^{\alpha^H(G/H)}$. If $x \in M^G$ then for any $\overline{g_i} \in G/H$, $\alpha^H{}_{\overline{g_i}}(x) = \alpha_{g_i}(x) = x$. Hence we have $x \in (M^H)^{\alpha^H(G/H)}$. Conversely, for any $x \in M^H$ satisfying $\alpha^H{}_{\overline{g_i}}(x) = x, \forall \overline{g_i} \in G/H$ and any $g \in G$, since $g = g_i h$ for some i and $h \in H$ we have

$$\alpha_g(x) = \alpha_{g_i h}(x) = \alpha_{g_i}\alpha_h(x) = \alpha_{g_i}(x) = \alpha^H{}_{\overline{g_i}}(x) = x$$

which implies $x \in M^G$. □

If for any $g \in G$ the restriction $\alpha_g|_{M^H}$ of α_g on M^H is an outer automorphism of M^H, then α^H is an outer action with $|\alpha^H(G/H)| = [G : H]$. But, in general, outerness of α and outerness of α^H are independent. It is known that if the quotient group G/H is finite and M^H is a II_1-factor then the outerness of α^H implies that M^G is a II_1-factor. But note that this is not true anymore for infinite groups G and G/H, in general. The following theorem is a generalization of the known result of Example 2.3.3 in [8] and gives a complete relation between Jones' index and group index even when G is infinite.

THEOREM 2.3. *Let G be a countable discrete group and H a normal subgroup of G. Let α be a G-action on M for which fixed point algebras M^G and M^H are II_1-factors.*
(a) *If α is outer then we have $[M : M^G] = |G|$.*
(b) *If the induced G/H-action α^H on M^H is outer then we have $[M^H : M^G] = [G : H]$.*

PROOF. (a) From Example 2.3.3 in [8], it is enough to prove when $|G| = \infty$. By Lemma 2.4 in [10] and Corollary 4.1 in [15], we have

$$\log |G| = H(M|M^G) \leq \log[M : M^G],$$

where $H(M|M^G)$ is the relative entropy for a pair of II_1-factors. Thus we have $[M : M^G] = \infty$ and $[M : M^G] = |G|$ as desired.

(b) Since α^H is an outer G/H-action on M^H by Lemma 2.2 and (a) we have $[M^H : M^G] = [M^H : (M^H)^{\alpha^H(G/H)}] = |\alpha^H(G/H)| = [G : H]$ as claimed. □

Now we consider an action α of an infinite countable discrete group G which is not outer with M^G, a II_1-factor. If the relative commutant $(M^G)' \cap M$ has a completely nonatomic part, then by Theorem 4.4 in [15] $H(M|M^G) = \infty$, where $H(M|M^G)$ is the relative entropy for a pair of II_1-factors. So $[M : M^G] = \infty$ since $\log[M : M^G] \geq H(M|M^G)$. But, if $(M^G)' \cap M$ is atomic with an infinite atomic set then $(M^G)' \cap M$ can not be finite dimensional, which implies $[M : M^G] = \infty$. The following proposition gives a sufficient condition for the same result as Theorem 2.3.(a) without outerness of an action.

PROPOSITION 2.4. *Let M be a II_1-factor and G an infinite countable discrete group of automorphisms of M with fixed point algebra M^G, a II_1-factor. Assume that $(M^G)' \cap M$ has a projection p with mutually orthogonal infinite set $\{\theta(p) \mid \theta \in G\}$ then we have $[M : M^G] = \infty$.*

PROOF. Since $p \in (M^G)' \cap M$, for any $\theta \in G$, we have

$$x\theta(p) = \theta(xp) = \theta(px) = \theta(p)x, \ x \in M^G$$

which means that $\theta(p) \in (M^G)' \cap M$. If $\{\theta(p) | \theta \in G\}$ is an infinite set of mutually orthogonal projections, then $(M^G)' \cap M$ has infinitely many mutually orthogonal minimal projections, which means that $(M^G)' \cap M$ can not be finite dimensional. Thus $[M : M^G] = \infty$ holds by [8]. □

3. Index for Fixed Point Algebras of a Group von Neumann Algebra

For an example of fixed point algebras, we now will turn our attention to a certain group von Neumann algebra. We constuct an example of II_1-factors which gives an example for the main theorem in the previous section.

Let G be a countable discrete ICC group with identity e and Γ the character group of G with identity 1. Let $\mathcal{L}(G)$ be the group von Neumann algebra generated by the left regular representation λ of G on $\ell^2(G)$, in this case, $\mathcal{L}(G)$ is a II_1-factor. For $\gamma \in \Gamma$, let α_γ denote the *-automorphism of $\mathcal{L}(G)$ induced by γ, and α be the associated action of Γ into Aut $\mathcal{L}(G)$, which is definded by $\alpha_\gamma(\lambda(g)) = \gamma(g)\lambda(g), \ g \in G$. Then α is obviously outer by Proposition 22.13 in [16]. Here, we define $\mathcal{L}(G)^{\alpha_\gamma} = \{A \in \mathcal{L}(G) | \alpha_\gamma(A) = A\}$, and $G^\gamma = \{g \in G | \gamma(g) = 1\}$ then it follows that

$$\mathcal{L}(G)^\Gamma = \bigcap_{\gamma\in\Gamma} \mathcal{L}(G)^{\alpha_\gamma}, G^\Gamma = \bigcap_{\gamma\in\Gamma} G^\gamma.$$

Now we investigate Jones' index for a pair of fixed point algebras of a group von Neumann algebra of a non inner amenable group G with character group Γ. Let Γ' be a subgroup of Γ and $\alpha' = \alpha|_{\Gamma'}$. If G is not inner amenable, then $G^{\Gamma'}$ and G^Γ are not inner amenable ([1], Proposition 3.1). Since non inner amenable groups are automatically

ICC groups, $\mathcal{L}(G^{\Gamma'})$ and $\mathcal{L}(G^{\Gamma})$ are II_1-factors. So we obtain a pair of II_1-factors $\mathcal{L}(G^{\Gamma}) \subseteq \mathcal{L}(G^{\Gamma'})$ and a pair of groups $G^{\Gamma} \leq G^{\Gamma'}$.

THEOREM 3.1. *If G is a non inner amenable discrete group and Γ' a subgroup of Γ then $[\mathcal{L}(G)^{\Gamma'} : \mathcal{L}(G)^{\Gamma}] = [G^{\Gamma'} : G^{\Gamma}]$.*

PROOF. By Proposition 3.1 in [1], for any $\gamma \in \Gamma$, $\mathcal{L}(G)^{\alpha_\gamma} \cong \mathcal{L}(G^\gamma)$. This gives an isomorphism between $\mathcal{L}(G)^{\Gamma'}$ and $\mathcal{L}(G^{\Gamma'})$, which also gives an isomorphism between $\mathcal{L}(G)^{\Gamma}$ and $\mathcal{L}(G^{\Gamma})$, so we have

$$[\mathcal{L}(G)^{\Gamma'} : \mathcal{L}(G)^{\Gamma}] = [\mathcal{L}(G^{\Gamma'}) : \mathcal{L}(G^{\Gamma})] = [G^{\Gamma'} : G^{\Gamma}]. \qquad \square$$

Now we deduce the following from Theorem 2.3.

THEOREM 3.2. *Let G be a non inner amenable discrete group with character group Γ and $\Gamma_1 \leq \Gamma_2$ a pair of countable discrete subgroups of Γ. If the induced Γ_2/Γ_1-action on $\mathcal{L}(G)^{\Gamma_1}$ is outer then we have*

$$[\mathcal{L}(G)^{\Gamma_1} : \mathcal{L}(G)^{\Gamma_2}] = [\Gamma_2 : \Gamma_1].$$

PROOF. Since G^{Γ_i}'s are ICC groups and $\mathcal{L}(G)^{\Gamma_i} \cong \mathcal{L}(G^{\Gamma_i})$, $i = 1, 2$, $\mathcal{L}(G)^{\Gamma_i}$'s are II_1-factors. Since Γ is abelian, Γ_2/Γ_1-action on $\mathcal{L}(G)^{\Gamma_1}$ is well defined.
By Theorem 2.3, we have $[\mathcal{L}(G)^{\Gamma_1} : \mathcal{L}(G)^{\Gamma_2}] = [\Gamma_2 : \Gamma_1]$. $\square$

In addition to the above properties we have the following.

COROLLARY 3.3. *Let G be a non inner amenable discrete group with character group Γ and $\Gamma_1 \leq \Gamma_2$ a pair of countable discrete subgroups of Γ such that the quotient group Γ_2/Γ_1 is isomorphic to a subgroup of the character group of G^{Γ_1}. Then we have $[G^{\Gamma_1} : G^{\Gamma_2}] = [\Gamma_2 : \Gamma_1]$.*

PROOF. Since the restriction of outer action is also outer, there is an outer Γ_2/Γ_1-action on $\mathcal{L}(G^{\Gamma_1}) \cong \mathcal{L}(G)^{\Gamma_1}$. The assertion follows from Theorem 3.1 and Theorem 3.2. $\square$

We will end this paper with an example which satisfies all the conditions considered above. Let G be a free group on n generators $a_1, \cdots, a_n$. Then G is not inner amenable and we have $\Gamma \cong T^n$ under the isomorphism given by $\gamma \to (\gamma(a_1), \cdots, \gamma(a_n))$, where T denotes the circle group ([1], 4.1).

EXAMPLE 3.4. Consider a free group $G = F_2 =< a, b >$ and $\Gamma \cong T^2$ where $T = \{e^{2\pi ix} \mid x \in (0,1]\}$. For some $n \in N$, let Γ_1 and Γ_2 be

$$\Gamma_1 \cong \{ \ (e^{2\pi ix}, 1) \mid x \in (0,1] \cap Q \ \},$$

$$\Gamma_2 \cong \{(e^{2\pi ix}, e^{2\pi i \frac{k}{n}}) \mid x \in (0,1] \cap Q, k = 1, \cdots, n\}.$$

Then $\Gamma_1 \leq \Gamma_2$ is a pair of countable discrete subgroups of Γ with $[\Gamma_2 : \Gamma_1] = n < \infty$. Observe that Γ_2/Γ_1 is isomorphic to the following subgroup Γ_0 of the character group of G^{Γ_1}

$$\Gamma_0 \cong \{(1, e^{2\pi i \frac{k}{n}}) \mid k = 1, \cdots, n\}.$$

Thus Γ_2/Γ_1-action on $\mathcal{L}(G^{\Gamma_1})$ is outer and $\mathcal{L}(G^{\Gamma_1}) \cong \mathcal{L}(G)^{\Gamma_1}$ implies that the induced Γ_2/Γ_1-action on $\mathcal{L}(G)^{\Gamma_1}$ is outer.

We show that they satisfy the equality in Corollary 3.3. Recall that an element $g \in G$ is a reduced word of a, b and for any $\gamma \in \Gamma$, $\gamma(g) = \gamma(a)^p\gamma(b)^q$ for some integers p, q. Note that $g \in G^{\Gamma_1}$ means $\gamma(a)^p = 1$ for $\gamma \in \Gamma_1$ and $g \in G^{\Gamma_2}$ means $\gamma(a)^p = 1$ and $\gamma(b)^q = 1$ for $\gamma \in \Gamma_2$. Thus for any $g \in G^{\Gamma_1}$

$$g \in G^{\Gamma_2} \iff (e^{2\pi i \frac{k}{n}})^q = 1, \ k = 1, 2, \cdots, n \iff q \equiv 0 \pmod{n}.$$

So for any $q \not\equiv 0 \pmod{n}$, there is some $\gamma \in \Gamma_2$ such that $\gamma(b)^q \neq 1$. Thus we obtain

$$G^{\Gamma_1} = g_0 G^{\Gamma_2} \cup g_1 G^{\Gamma_2} \cup \cdots \cup g_{n-1} G^{\Gamma_2},$$

where $g_i \in \{g \in G^{\Gamma_1} \mid \gamma(g) = \gamma(a)^p\gamma(b)^q, \ q \equiv i \pmod{n} \ \ \forall \gamma \in \Gamma_2\}$, for each $i = 0, 1, \cdots, n-1$. Hence $[G^{\Gamma_1} : G^{\Gamma_2}] = [\Gamma_2 : \Gamma_1]$ follows.

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