Comm. Korean Math. Soc. **13** (1998), No. 1, pp. 73–76

UNSOLVABILITY OF SOME PARTIAL DIFFERENTIAL OPERATOR

RAKJOONG KIM

ABSTRACT. It is proved that the partial differential operator $D_x + ix^q D_y^2$ is not locally solvable in any open set which intersects the line $x = 0$, when $q = -\frac{2l-1}{2k-1}$ is not an integer.

§1. Introduction

It is well known that a partial differential operator $D_x + ix^k D_y$ is not locally solvable at the origin if k is an odd integer, whereas if k is an even integer, it is both solvable and analytic-hypoelliptic(cf. Treves [6]) For a Gel'fand-Shilov function x^λ Hoel [1] treated solvability and unsolvability of the partial differential operator $D_x + ix^\lambda D_y$. Kannai [3] proved that a partial differential operator $D_x + ixD_y^2$ is hypoelliptic but not locally solvable on the line $x = 0$. In this case even though x is replaced by $x^{2k-1} + o(x^{2k-1})$, where k is any positive integer, it is still not locally solvable (cf. Kim-Kim [5]). In the following the condition that k is an integer is relaxed.

We consider a partial differential operator of the type:

$$(1.1) \qquad Au \equiv D_x u + ix^q \ D_y^2 u, \qquad q = \frac{2l-1}{2k-1} > -1$$

where k, l are integers and $D_x = -i\,\frac{\partial}{\partial x}$, $D_y = -i\,\frac{\partial}{\partial y}$. We observe that x^q is locally integrable and symmetric with respect to the origin but not smooth anymore if q is not an integer. The purpose of this paper is to

Received August 16, 1997. Revised December 27, 1997

1991 Mathematics Subject Classification: 35A07, 35K65.

Key words and phrases: Solvability, unsolvability, partial differential operator of degenerate type.

This work was supported by Hallym Univ.

establish that the partial differential operator (1.1) is not locally solvable at the origin.

We now explain some notations. Let $U \times V$ be an open rectangular neighborhood of the origin and $\alpha > -1$. For an open interval U of the real line and for any compact interval $K_x \subset U$, $C_0^\alpha(U)$ is the inductive limit of the Banach spaces $C_0^\alpha(K_x)$ which are the completions of $C_0^\infty(K_x)$ with the norm

$$\|u\|_\alpha = \sup_{x,x'} \frac{|D^{[\alpha]}u(x) - D^{[\alpha]}u(x')|}{|x - x'|^{\alpha-[\alpha]}},$$

where $x, x' \in K_x$, $[\alpha]$ is the integer part of α.

For each j, $C_0^\alpha(K_x)\,\hat{\otimes}\,C_0^\infty(K_y)$ has seminorms $\sup_y \|D_t^j u(x,y)\|_\alpha$, and $C_0^\alpha(U)\,\hat{\otimes}\,C_0^\infty(V)$ is the inductive limit of the above spaces, so is metrizable. We interpret $D^{-1}u$ as any primitive of u.

§2. The Result

Now we state the main theorem in this paper:

THEOREM. *The partial differential equation $\mathcal{A}u = f$ is not locally solvable at the origin in the space $(C_0^q(U)\,\hat{\otimes}\,C_0^\infty(V)')$.*

PROOF. Assume that $\mathcal{A}u = f$ is solvable. Then according to Hörmander's theorem [2] for an open rectangular neighborhood ω of the origin and for $f \in C_0^\infty(\bar{\omega}), v \in C_0^q(\omega_x)\,\hat{\otimes}\,C_0^\infty(\omega_y)$ there is an integer N such that

$$(2.1) \qquad |\int\int fv\; dxdy| N \sum_{|\beta| \leq N} \sup|D^\beta f| \sup \|D_y^N \mathcal{A}^t v\|_q.$$

(see [1] and compare with [2]). We may assume that q is not an integer (see [3]). In order to prove that partial differential equation (1.1) is not locally solvable, it is sufficient to find functions f_λ, v_λ depending on λ such that

$$\lim_{\lambda\to\infty} |\int\int f_\lambda v_\lambda\; dxdy| \neq 0,$$

$$\lim_{\lambda\to\infty} N \sup|D^\beta f| \sup_y \|D_y^N \mathcal{A}^t v\|_q = 0, \qquad |\beta| \leq N.$$

Put $a(x) = x^q$, $b(x) = \int_0^x t^q\,dt$. As in [3] we take a density function u_λ depending on λ of the type

$$u_\lambda(x,y) = \frac{1}{\sqrt{2b\lambda+1}} \exp\left[\frac{-2b\lambda^2 - y^2\lambda - 2iy\lambda}{2(2b\lambda+1)}\right].$$

It is obvious that $\mathcal{A}^t u_\lambda = 0$. We consider a function $\phi \in C_0^\infty(R^2)$ such that

$$\phi(x,y) = \left\{\begin{cases} 1 & \text{if} \quad \sqrt{x^2+y^2} \le 1 \\ 0 & \text{if} \quad \sqrt{x^2+y^2} \ge 2. \end{cases}\right.$$

Choose function $F(x,y) \in C_0^\infty(R^2)$ such that

$$\int\int F(x,y)\,dxdy = A \neq 0.$$

First we set

$$f_\lambda(x,y) = \lambda^{-2N} F(\lambda^2 x, \lambda^2 y).$$

Then $f_\lambda(x,y)$ belongs to $C_0^\infty(\omega)$ for all large λ and

$$|D^\beta f_\lambda| \le |D^\beta F| \qquad \text{for all} \quad |\beta| \le N. \tag{2.2}$$

It follows that there exists a constant $c_1 > 0$ such that

$$\frac{2b\lambda^2 + y^2\lambda}{2(2b\lambda+1)} \ge \frac{c\delta^2\lambda}{2}$$

for $\delta \le \sqrt{x^2+y^2} \le 2\delta$. We may assume that $2\delta < 1$. Since

$$\begin{aligned}\mathcal{A}^t\left[\phi_\delta(x,y)u_\lambda(x,y)\right] &= -\delta^{-1}(D_x\phi_\delta)u_\lambda \\ &\quad + 2i\delta^{-1}a(x)(D_y\phi_\delta)(D_y u_\lambda) + \delta^{-2} i a(x)(D_y^2\phi_\delta)u_\lambda,\end{aligned}$$

where $\phi_\delta(x,y) \equiv \phi(\frac{x}{\delta}, \frac{y}{\delta})$, we see that there exist positive constants c_2, c_3 such that

$$\|D_y^N \mathcal{A}^t\left[\phi_\delta(x,y)u_\lambda(x,y)\right]\|_q \le c_2\delta^{-N-2-[q]}\,\lambda^{c_3}\, exp\left[-\frac{c\delta^2\lambda}{2}\right]. \tag{2.3}$$

We note that

$$\lim_{\lambda\to\infty} b(\frac{x}{\lambda^2})\,\lambda^2 = 0,$$

Thus it follows that

$$\begin{aligned}(2.4)\qquad &\lim_{\lambda\to\infty} \lambda^4 \int\int F(\lambda^2 x, \lambda^2 y)\phi_\delta(x,y)u_\lambda(x,y)dxdy \\ &= \lim_{\lambda\to\infty}\int\int F(x,y)\phi_\delta(\frac{x}{\lambda^2},\frac{y}{\lambda^2})\frac{1}{\sqrt{2b(\frac{x}{\lambda^2})\lambda+1}} \\ &\qquad \exp\left[\frac{-2b(\frac{x}{\lambda^2})\lambda^2 - \frac{y}{\lambda^3} + 2i\frac{y}{\lambda}}{2(2b(\frac{x}{\lambda^2})\lambda+1))}\right] dxdy \\ &= \int\int F(x,y)\phi(0,0)\,dxdy = A \neq 0.\end{aligned}$$

For a fixed N we take

$$v_\lambda(t,x) = \lambda^{2N+4}\phi_\delta(x,y)u_\lambda(t,x).$$

From $(2,2)$, $(2,3)$ and (2.4) our theorem follows. □

References

[1] C. Hoel, *A note on local solvability*, Duke Math. **40** (1973), 839-843.

[2] L. Hörmander, *Linear Partial Differential Operators*, Springer-Verlag, Berlin, 1963.

[3] Y. Kannai, *An unsolvable hypoelliptic differential operator*, Israel J. Math **9** (1971), 306-315.

[4] C. Nirenberg and F. Treves, *On local solvability of linear partial defferential equations, Part 1:Necessary conditions, Part 2: Sufficient conditions*, Comm. Pure Appl. Math. **23** (1970), 1-38, 459-510.

[5] R. J. Kim and J. S. Kim, *Unsolvable partial differential operators of degenerate type*, Bull. of Korean Math. Soc. **31** (1994), 181-185.

[6] F. Treves, *Analytic-hypoelliptic partial differential equations of principle type*, Comm. Pure Appl. Math. **24** (1971), 537-570.

Department of Mathematics
Hallym University
Chunchon, Kangwon 200-702, Korea