Comm. Korean Math. Soc. **13** (1998), No. 1, pp. 77–84

SEMI-QUASITRIANGULARITY OF TOEPLITZ OPERATORS WITH QUASICONTINUOUS SYMBOLS

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Abstract. In this note we show that if T_φ is a Toeplitz operator with quasicontinuous symbol φ, if Ω is an open set containing the spectrum $\sigma(T_\varphi)$, and if $H(\Omega)$ denotes the set of analytic functions defined on Ω, then the following statements are equivalent:

(a) T_φ is semi-quasitriangular.
(b) Browder's theorem holds for $f(T_\varphi)$ for every $f \in H(\Omega)$.
(c) Weyl's theorem holds for $f(T_\varphi)$ for every $f \in H(\Omega)$.
(d) $\sigma(T_{f\circ\varphi}) = f(\sigma(T_\varphi))$ for every $f \in H(\Omega)$.

1. Introduction

Farenick and Lee ([5, Theorem 3.7]) showed that if T_φ is a Toeplitz operator with continuous symbol φ such that the winding number of φ with respect to each hole of $\varphi(\mathbb{T})$ is nonnegative (or is nonpositive), then $\sigma(T_{f\circ\varphi}) = f(\sigma(T_\varphi))$ for every analytic function f defined on an open set containing $\sigma(T_\varphi)$. In this note we extend this result to obtain the following theorem: if T_φ is a Toeplitz operator with quasicontinuous symbol φ, then T_φ is semi-quasitriangular if and only if $\sigma(T_{f\circ\varphi}) = f(\sigma(T_\varphi))$ for every analytic function f defined on an open set containing the spectrum of T_φ.

Let $\mathcal{L}(\mathcal{H})$ and $\mathcal{K}(\mathcal{H})$ denote the algebra of bounded linear operators and the ideal of compact operators on a complex Hilbert space $\mathcal{H}$, and let π denote the canonical map $\mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$. If $T \in \mathcal{L}(\mathcal{H})$ is

Received September 9, 1997. Revised November 22, 1997.
1991 Mathematics Subject Classification: 47B35, 47A10.
Key words and phrases: Toeplitz operators, quasicontinuous functions, semi-quasitriangular, Weyl's theorem, Browder's theorem.
This work was supported in part by the 63 Research Fund at Sung Kyun Kwan University and the KOSEF 971-0102-010-2.

a Fredholm operator, that is, if $\pi(T)$ is invertible in $\mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$, then $\ker T$ and $\ker T^*$ are finite-dimensional and the index of T is the integer

$$\text{ind}\,(T) = \dim\,\ker T - \dim\,\ker T^*.$$

Those Fredholm operators that have index zero are called Weyl operators. The essential spectrum $\sigma_e(T)$ and the Weyl spectrum $\omega(T)$ are defined as follows [6]:

$$\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Fredholm }\};$$
$$\omega(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Weyl}\}\ .$$

An operator $T \in \mathcal{L}(\mathcal{H})$ is called *Browder* if it is Fredholm "of finite ascent and descent." Recall that an operator T has finite ascent if there is a positive integer k for which $\ker T^k = \ker T^{k+n}$ for all positive integers n, and T has finite descent if there is a positive integer m such that the range of T^m equals the range of T^{m+n} for every positive integer n. An operator T is a Browder operator if T is Fredholm and $T - \lambda I$ is invertible for sufficiently small $\lambda \neq 0$ in $\mathbb{C}$ ([6, Theorem 7.9.3]). The Browder spectrum $\sigma_b(T)$ of T is defined by

$$\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Browder}\}\ :$$

evidently

$$\sigma_e(T) \subseteq \omega(T) \subseteq \sigma_b(T) = \sigma_e(T) \cup \text{acc}\ \sigma(T),$$

where we write acc(K) for the *accumulation points* of $K \subseteq \mathbb{C}$. If we write iso$(K) = K \setminus \text{acc}(K)$ and

(1) $$\pi_{00}(T) = \{\lambda \in \text{iso}\ \sigma(T) : 0 < \dim\ (T - \lambda I)^{-1}(0) < \infty\}$$

for the isolated eigenvalues of finite multiplicity, and

(2) $$p_{00}(T) = \sigma(T) \setminus \sigma_b(T)$$

for the *Riesz points* of T, then we have

$$\text{iso}\,\sigma(T) \setminus \sigma_e(T) = \text{iso}\,\sigma(T) \setminus \omega(T) = p_{00}(T) \subseteq \pi_{00}(T).$$

The authors of [7] and [8] use the notation of (1) for the concept of (2).

DEFINITION 1. We say that *Weyl's theorem holds for* $T \in \mathcal{L}(\mathcal{H})$ if

$$\sigma(T) \setminus \omega(T) = \pi_{00}(T),$$

and that *Browder's theorem holds for* T if

$$\sigma(T) \setminus \omega(T) = p_{00}(T).$$

It is familiar ([1], [2]) that Weyl's theorem holds for all seminormal operators and all Toeplitz operators. Evidently "Weyl's theorem" implies "Browder's theorem". Recently, Browder's theorem has been considered in [7], [8] and [10]. We begin with:

LEMMA 2. *If* $T \in \mathcal{L}(\mathcal{H})$ *then the following are equivalent:*

(a) $ind\,(T - \lambda I)\, ind\,(T - \mu I) \geq 0$ *for each pair* $\lambda, \mu \in \mathbb{C} \setminus \sigma_e(T)$.

(b) $f\,\omega(T) = \omega\, f(T)$ *for each analytic function* f *defined on an open set containing* $\sigma(T)$.

Further if Browder's theorem holds for $T \in \mathcal{L}(\mathcal{H})$ *then*

$$\text{Browder's theorem holds for } f(T) \iff f\,\omega(T) = \omega\, f(T).$$

PROOF. By Theorem 5 of [7], the condition (a) is equivalent to the condition that $p\,\omega(T) = \omega\, p(T)$ for every polynomial p. But by an argument of Oberai [12, Theorem 2], we have that $p\,\omega(T) = \omega\, p(T)$ if and only if $f\,\omega(T) = \omega\, f(T)$. The proof of the second assertion is taken straight from a slight modification of the proof of [7, Theorem 4] which works with a polynomial p. □

We can rewrite the condition (a) in Lemma 2 in terms of the "spectral picture" ([13, Definition 1.22]) of the operator T, denoted $\mathcal{SP}(T)$, which consists of the set $\sigma_e(T)$, the collection of holes and pseudoholes in $\sigma_e(T)$, and the indices associated with these holes and pseudoholes. Thus if $\mathcal{SP}(T)$ has no pseudohole, then the condition (a) in Lemma 2 is ([13, Definition 4.8]) the condition that $T \in \mathcal{L}(\mathcal{H})$ be "semi-quasitriangular" in the sense that either T or T^* is quasitriangular. Recall ([13, Theorem 1.31]) that by the work of Apostol, Foias, and Voiculescu, T is quasitriangular if and only if $\mathcal{SP}(T)$ has no hole or pseudohole associate with a negative number.

The following lemma shows that the semi-quasitriangularity of essentially normal operators has a close relation to Browder's theorem.

LEMMA 3. *If $T \in \mathcal{L}(\mathcal{H})$ is essentially normal (i.e., $T^*T - TT^* \in \mathcal{K}(\mathcal{H})$) and if Browder's theorem holds for T, then the following are equivalent:*

(a) *T is semi-quasitriangular.*
(b) *Browder's theorem holds for $f(T)$,*

where f is an analytic function on an open set containing $\sigma(T)$.

PROOF. If T is essentially normal then by Proposition 2.16 of [13], for each $\lambda \in \mathbb{C}$,

$$T - \lambda I \text{ is Fredholm} \iff T - \lambda I \text{ is } \textit{semi-Fredholm},$$

in the sense that $\operatorname{ran}(T-\lambda I)$ is closed and either $\dim (T-\lambda I)^{-1}(0) < \infty$ or $\dim (T - \lambda I)^{*-1}(0) < \infty$. This implies that $\mathcal{SP}(T)$ has no pseudo-hole. Thus it follows from the remark below Lemma 2 that the semi-quasitriangularity of T is equivalent to the condition (a) in Lemma 2. But since Browder's theorem holds for T, the desired equivalence follows from Lemma 2. □

REMARK 4. (a) The condition "Browder's theorem holds for T" is essential in Lemma 3: even though T is essentially normal and semi-quasitriangular (even quasitriangular), Browder's theorem may not hold for T. For an example, let $\mathcal{H} = \ell_2 \oplus \ell_2$ and let $T = U \oplus U^*$, where U is the unilateral shift on ℓ_2. Then T is essentially normal. Further since $\sigma(T) = \operatorname{cl}\mathbb{D}$ (the closed unit disk) and $\omega(T) = \mathbb{T}$ (the unit circle), it follows that T is quasitriangular, but Browder's theorem does not hold for T because $\sigma(T) \setminus \omega(T) = \mathbb{D}$.

(b) Note that Weyl's theorem may not hold for $T \in \mathcal{L}(\mathcal{H})$ even though T is essentially normal and semi-quasitriangular, and Browder's theorem holds for T. For example, if $T : \ell_2 \to \ell_2$ is defined by

$$T(x_1, x_2, x_3, \cdots) = (\frac{1}{2}x_2, \frac{1}{3}x_3, \frac{1}{4}x_4, \cdots),$$

then, evidently, it is essentially normal and quasitriangular and furthermore

$$\sigma(T) = \omega(T) = \pi_{00}(T) = \{0\} \quad \text{and} \quad p_{00}(T) = \emptyset.$$

We review here a few essential facts concerning Toeplitz operators, using [3], [11] as a general reference. The Hilbert space $L^2(\mathbb{T})$ has a canonical orthonormal basis given by the trigonometric functions $e_n(z) = z^n$, for all $n \in \mathbb{Z}$, and the Hardy space $H^2(\mathbb{T})$ is the closed linear span of $\{e_n : n = 0, 1, \dots\}$. If P denotes the projection operator $L^2(\mathbb{T}) \to H^2(\mathbb{T})$, then for every $\varphi \in L^\infty(\mathbb{T})$, the operator T_φ on $H^2(\mathbb{T})$ defined by $T_\varphi g = P(\varphi g)$ for all $g \in H^2(\mathbb{T})$ is called the Toeplitz operator with symbol φ. Every Toeplitz operator has connected spectrum and essential spectrum, and $\sigma(T_\varphi) = \omega(T_\varphi)$ (cf. [2]). The sets $C(\mathbb{T})$ of all continuous complex-valued functions on $\mathbb{T}$ and $H^\infty(\mathbb{T}) = L^\infty(\mathbb{T}) \cap H^2(\mathbb{T})$ are Banach algebras. Recall that the subspace $H^\infty + C(\mathbb{T})$ is a closed subalgebra of $L^\infty(\mathbb{T})$ and that

(3) $T_\psi T_\varphi - T_{\psi\varphi} \in \mathcal{K}(H^2)$ for every $\varphi \in H^\infty + C(\mathbb{T})$ and $\psi \in L^\infty(\mathbb{T})$;

(4)
T_φ is Fredholm if and only if $\varphi^{-1} \in H^\infty + C(\mathbb{T})$ for every $\varphi \in H^\infty + C(\mathbb{T})$.

Recall also that the elements of the closed selfadjoint subalgebra QC, which is defined to be

$$QC = \left(H^\infty + C(\mathbb{T})\right) \cap \overline{\left(H^\infty + C(\mathbb{T})\right)},$$

are called *quasicontinuous functions.*

We now have:

THEOREM 5. *If $\varphi \in H^\infty + C(\mathbb{T})$, if Ω is an open set containing the spectrum $\sigma(T_\varphi)$, and if $H(\Omega)$ denotes the set of analytic functions defined on Ω, then the following statements are equivalent:*

(a) *Browder's theorem holds for $f(T_\varphi)$ for every $f \in H(\Omega)$.*
(b) *Weyl's theorem holds for $f(T_\varphi)$ for every $f \in H(\Omega)$.*
(c) $\sigma(T_{f\circ\varphi}) = f(\sigma(T_\varphi))$ *for every $f \in H(\Omega)$.*

In particular, if $\varphi \in QC$, then each condition of the above is equivalent to the following:

(d) *T_φ is semi-quasitriangular.*

PROOF. We first claim that

$$\varphi \in L^\infty(\mathbb{T}) \implies \pi_{00}(f(T_\varphi)) = \emptyset. \tag{5}$$

If f is a constant function or $\varphi = \lambda$ for some $\lambda \in \mathbb{C}$, then (5) is evident. Thus we suppose f is a nonconstant analytic function and φ is a nonconstant function. Observe that $\sigma(f(T_\varphi)) = f(\sigma(T_\varphi))$ is connected because $\sigma(T_\varphi)$ is connected. Assume $\sigma(f(T_\varphi))$ is a singleton set. Then $\sigma(T_\varphi)$ must also be a singleton set: if it were not so then by the Identity Theorem in the elemetary complex analysis, f would be a constant function. But since the only quasinilpotent Toeplitz operator is 0, we must have that $\varphi = \lambda$ for some $\lambda \in \mathbb{C}$, a contradiction. Therefore $\sigma(f(T_\varphi))$ is a connected set which is not a singleton set. Thus $\sigma(f(T_\varphi))$ has no isolated points. This proves (5). Now the equivalence of (a) and (b) immediately follows from (5). For the equivalence of (b) and (c), observe by (4) that, for every $\lambda \notin \sigma(T_\varphi)$, $\varphi - \lambda$ is invertible in $H^\infty + C(\mathbb{T})$; hence, $(\varphi - \lambda)^{-1} \in H^\infty + C(\mathbb{T})$. Then, by (3), we have that if $\lambda \notin \sigma(T_\varphi)$ then

$$T_{\varphi-\lambda} T_{(\varphi-\lambda)^{-1}} - I \in K(H^2), \quad \text{so that} \quad T_{\varphi-\lambda}^{-1} - T_{(\varphi-\lambda)^{-1}} \in K(H^2).$$

Therefore we have that, for $\lambda, \mu \in \mathbb{C}$,

$$T_{\varphi-\mu} T_{\varphi-\lambda}^{-1} - T_{(\varphi-\mu)(\varphi-\lambda)^{-1}} \in K(H^2) \quad \text{whenever } \lambda \notin \sigma(T_\varphi)\,.$$

The arguments above extend to rational functions to yield: if r is any rational function with all of its poles outside of $\sigma(T_\varphi)$, then $r(T_\varphi) - T_{r\circ\varphi} \in K(H^2)$. Suppose that f is an analytic function on an open set containing $\sigma(T_\varphi)$. Now applying Runge's theorem with f gives

$$T_{f\circ\varphi} - f(T_\varphi) \in K(H^2). \tag{6}$$

Because the Weyl spectrum is stable under the compact perturbations, it follows from (6) that

$$\omega(f(T_\varphi)) = \omega(T_{f\circ\varphi}) = \sigma(T_{f\circ\varphi}). \tag{7}$$

But since, by (5), $\pi_{00}(f(T_\varphi)) = \emptyset$, it follows from (7) that Weyl's theorem holds for $f(T_\varphi)$ if and only if $\omega(f(T_\varphi)) = \sigma(f(T_\varphi))$ if and only if

$\sigma(T_{f\circ\varphi}) = f(\sigma(T_\varphi))$. This gives the equivalence of (b) and (c). For the second assertion, it suffices to prove the equivalence of (d) and (a). If $\varphi \in QC$ then we can see that the self-commutator $[T_\varphi, T_\varphi^*]$ is compact, so that T_φ is essentially normal. But since Browder's theorem holds for every Toeplitz operator, the equivalence of (d) and (a) follows from Lemma 3. □

If $\varphi \in L^\infty(\mathbb{T})$ has the property that $\bigcap_{\epsilon>0} \mathrm{cl}\big[\varphi(\lambda_0 - \epsilon, \lambda_0 + \epsilon)\big]$ is contained in some line segment L_{λ_0} for each $\lambda_0 \in \mathbb{T}$, then we call φ *Douglas function* ([4], [5]). Then the conditions (a), (b), and (d) in Theorem 5 are also equivalent for Toeplitz operators T_φ with Douglas symbol φ because $[T_\varphi, T_\varphi^*] \in \mathcal{K}(H^2)$ (cf. the footnote on page 23 of [4]). We believe this argument can be extended for Toeplitz operators with "generalized Douglas symbols" introduced in [9].

ACKNOWLEDGEMENTS. We thank D. Farenick for comments concerning the subject of this paper.

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