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REMARKS ON THE REIDEMEISTER NUMBERS FOR COINCIDENCES

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ABSTRACT. Let X, Y be connected, locally connected, semilocally simply connected and $f, g : X \to Y$ be a pair of maps. We find an upper bound of the Reidemeister number $R(f,g)$ by using the regular covering spaces.

1. Introduction

In [9] Degui Li gave an upper bound for the Reidemeister number $R(f)$. In this paper we present a similar upper bound for coincidences. The Reidemeister number of (f,g) is denoted by $R(f,g)$. A lower bound of $R(f,g)$ has been obtained in [3, 8] as follows:

$$|\mathrm{Coker}(f_{1*} - g_{1*})| \leq R(f,g)$$

where $f_{1*}, g_{1*} : H_1(X) \to H_1(Y)$ are the homomorphisms induced by f and g respectively between the 1-dimensional homology groups of X and Y. We obtain an upper bound of $R(f,g)$ as follows:

$$R(f,g) \leq |\mathrm{Coker}(f_{1*} - g_{1*})||D(\pi_1(Y, f(x_0)))|$$

where $D(\pi_1(Y, f(x_0)))$ is the commutator subgroup of the fundamental group $\pi_1(Y,\ f(x_0))$. The method used here follows that of Li.

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2. Preliminaries

Let X, Y be connected, locally connected and semilocally simply connected. Let $f, g : X \to Y$ be a pair of maps. Fix universal coverings $P : \widetilde{X} \to X$, $Q : \widetilde{Y} \to Y$. Denoted by $\pi_X := \pi_1(X), \pi_Y := \pi_1(Y)$ the groups of natural transformations of $\widetilde{X}$ and $\widetilde{Y}$ respectively. Let lift(f, g) be the set of all pairs of lifts $(\tilde{f}, \tilde{g})$ for which the following diagram commutes

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{(\tilde{f},\tilde{g})} & \widetilde{Y} \\ {\scriptstyle P}\big\downarrow & & \big\downarrow{\scriptstyle Q} \\ X & \xrightarrow{(f,g)} & Y \end{array}$$

If $\tilde{f}, \tilde{f}'$ are liftings of f, then there exists a unique $\alpha \in \pi_Y$ such that $\tilde{f}' = \alpha \circ \tilde{f}$. Hence for some $(\tilde{f}, \tilde{g}) \in$ lift(f, g), we have

$$\text{Lift}(f, g) = \{(\alpha \circ \tilde{f}, \beta \circ \tilde{g}) : \alpha, \beta \in \pi_Y\}.$$

Two lifting pairs $(\tilde{f}, \tilde{g})$ and $(\tilde{f}', \tilde{g}')$ are said to be cojugate if there exist $\gamma \in \pi_X$ and $\gamma' \in \pi_Y$ such that $(\tilde{f}', \tilde{g}') = \gamma' \circ (\tilde{f}, \tilde{g}) \circ \gamma^{-1}$. The equivalence classes by cojugacy are called *lifting classes of* (f, g) *on* $\widetilde{Y}$ and the lifting class of $(\tilde{f}, \tilde{g})$ is denoted by

$$[\tilde{f}, \tilde{g}] = \{\gamma' \circ (\tilde{f}, \tilde{g}) \circ \gamma^{-1} : \gamma \in \pi_X, \gamma' \in \pi_Y\}.$$

The set of all lifting classes is denoted by Lift$'(f, g)$ and $R(f, g) := |\text{Lift}'(f, g)|$ is called *Reidemeister number of* (f, g).

For every $\alpha \in \pi_X$ and some lifting $\tilde{f}$ of f, the composition $\tilde{f} \circ \alpha$ also a lifting of f, so there is unique element $\alpha' \in \pi_X$ such that $\alpha' \circ \tilde{f} = \tilde{f} \circ \alpha$. This correspondence

$$\begin{aligned} \tilde{f}_\pi : \pi_X &\longrightarrow \pi_Y \\ \alpha &\longmapsto \alpha' \end{aligned}$$

is determined by $\tilde{f}$ and is a homomorphism [6;p.24 Definition 1.1]. If π_Y is a commutative group, then $\tilde{f}_\pi$ does not depend on the choice of $\tilde{f}$.

From now on, we work on regular covering spaces. Let $H \lhd \pi_X, K \lhd \pi_Y, f_\pi(H) \subset K$ and $g_\pi(H) \subset K$, where $H \lhd \pi_X, K \lhd \pi_Y$ denote that H, K are normal subgroups of π_X, π_Y, and f_π, g_π are the homomorphisms induced by f, g respectively. Up to isomorphism, there exist unique regular covering spaces $({}_H\widetilde{X}, {}_HP)$ and $({}_K\widetilde{Y}, {}_KQ)$ of X and Y respectively.

Let $\mathrm{Lift}(f, g, H, K)$ be the set of all lifting pairs $({}_H\tilde{f}, {}_H\tilde{g})$ of (f, g) for which the following diagram commutes

$$\begin{array}{ccc} {}_H\widetilde{X} & \xrightarrow{({}_H\tilde{f}, {}_H\tilde{g})} & {}_K\widetilde{Y} \\ {\scriptstyle {}_HP}\downarrow & & \downarrow{\scriptstyle {}_KQ} \\ X & \xrightarrow{(f,g)} & Y \end{array}$$

Especially, the set $\mathcal{D}({}_H\widetilde{X}, {}_HP) := \mathrm{Lift}(i_X, H)$ of all liftings of the identity $i_X : X \to X$ on ${}_H\widetilde{X}$ forms a group under the composition of maps. The correspondence

$$\begin{aligned} \lambda_H : \mathcal{D}({}_H\widetilde{X}, {}_HP) &\to \pi_X/H \\ \ell_{[\alpha]} &\mapsto [\alpha] := H\alpha \end{aligned}$$

is an isomorphism, where $\ell_{[\alpha]}(H\langle e_X \rangle) = [\alpha]$. Hence we will identify $\ell_{[\alpha]}$ with $[\alpha]$ and $\mathcal{D}({}_H\widetilde{X}, {}_HP)$ with π_X/H. And similarly, we will identify $\mathcal{D}({}_K\widetilde{Y}, {}_KQ)$ with π_Y/K. Then the set of liftings of the pair (f, g) on ${}_K\widetilde{Y}$ is denoted by

$$\mathrm{Lift}(f, g, H, K) = \{([\alpha] \circ {}_H\tilde{f}, [\beta] \circ {}_H\tilde{g}) : [\alpha], [\beta] \in \pi_Y/K\}$$

and the equivalence classes by cojugacy are called *lifting classes of* (f, g) *on* ${}_K\widetilde{Y}$ and the lifting class of $({}_H\tilde{f}, {}_H\tilde{g})$ is denoted by

$$[{}_H\tilde{f}, {}_H\tilde{g}] = \{[\gamma'] \circ ({}_H\tilde{f}, {}_H\tilde{g}) \circ [\gamma]^{-1} : [\gamma] \in \pi_X/H, [\gamma'] \in \pi_Y/K\}.$$

The set of all lifting classes is denoted by $\mathrm{Lift}'(f, g, H, K)$ and $R(f, g, H, K) := |\mathrm{Lift}'(f, g, H, K)|$ is called H, K*-Reidemeister number of* (f, g).

For regular covering spaces, we also have the correspondence

$$\begin{aligned} {}_H\tilde{f}_\pi : \pi_X/H &\rightarrow \pi_Y/K \\ [\alpha] &\mapsto [\alpha']. \end{aligned}$$

If $H = K = \{e\}$ then the following notations $({}_H\widetilde{X}, {}_HP)$, $({}_K\widetilde{Y}, {}_KQ)$ and λ_H are simply denoted by $(\widetilde{X}, P)$, $(\widetilde{Y}, Q)$ and λ respectively. And $\mathrm{Lift}(f, g, H, K)$, $\mathcal{D}({}_H\widetilde{X}, {}_HP)$, $\mathcal{D}({}_K\widetilde{Y}, {}_KQ)$ ${}_H\tilde{f}$, ${}_H\tilde{g}$, $\mathrm{Lift}'(f, g, H, K)$, $R(f, g, H, K)$, ${}_H\tilde{f}_\pi$ and ${}_H\tilde{g}_\pi$ is the same as $\mathrm{Lift}(f, g)$, $\pi_X, \pi_Y, \tilde{f}, \tilde{g}$, $\mathrm{Lift}'(f, g)$, $R(f, g)$, $\tilde{f}_\pi$ and $\tilde{g}_\pi$ respectively.

The covering space homomorphisms

$$\begin{aligned} \varphi_X : (\widetilde{X}, P) &\longrightarrow ({}_H\widetilde{X}, {}_HP) \\ \langle c \rangle &\longmapsto H\langle c \rangle \end{aligned}$$

and

$$\begin{aligned} \varphi_Y : (\widetilde{Y}, Q) &\longrightarrow ({}_K\widetilde{Y}, {}_KQ) \\ \langle d \rangle &\longmapsto K\langle d \rangle \end{aligned}$$

induce a surjection

$$\begin{aligned} \varphi : \mathrm{Lift}(f, g) &\longrightarrow \mathrm{Lift}(f, g, H, K) \\ (\tilde{f}, \tilde{g}) &\longmapsto (\varphi(\tilde{f}), \varphi(\tilde{g})) \end{aligned}$$

where $(\varphi(\tilde{f}), \varphi(\tilde{g})) \circ \varphi_X = \varphi_Y \circ (\tilde{f}, \tilde{g})$. Especially, φ_X and φ_Y induce surjections

$$\begin{aligned} \varphi'_X : \pi_X &\longrightarrow \mathcal{D}({}_H\widetilde{X}, {}_HP) \\ \alpha &\longmapsto \varphi'_X(\alpha) \end{aligned}$$

and

$$\begin{aligned} \varphi'_Y : \pi_Y &\longrightarrow \mathcal{D}({}_K\widetilde{Y}, {}_KQ) \\ \beta &\longmapsto \varphi'_Y(\beta) \end{aligned}$$

where $\varphi'_X(\alpha) \circ \varphi_X = \varphi_X \circ \alpha$, $\varphi'_Y(\beta) \circ \varphi_Y = \varphi_Y \circ \beta$. Note that

$$(\varphi_X \circ \alpha)\langle e_X \rangle = \varphi_X(\alpha\langle e_X \rangle) = \varphi_X(\ell_\alpha \langle e_X \rangle) = \varphi_X(\alpha) = [\alpha],$$
$$\varphi'_X(\alpha) \circ \varphi_X \langle e_X \rangle = \varphi'_X(\alpha)(\varphi_X \langle e_X \rangle) = \varphi'_X(\alpha)(H\langle e_X \rangle).$$

Hence $\varphi'_X(\alpha)(H\langle e_X \rangle) = [\alpha] = \ell_{[\alpha]}(H\langle e_X \rangle)$. It follows that $\varphi'_X(\alpha) = \ell_{[\alpha]} = [\alpha]$. Thus we have

$$[\alpha] \circ \varphi_X = \varphi_X \circ \alpha.$$

Similarly, we have

$$[\beta] \circ \varphi_Y = \varphi_Y \circ \beta.$$

3. An upper bound of the Reidemeister number $R(f, g)$.

We now recall the set of the Redemeister classes $\nabla(f, g : x_0, r)$ for $f, g : X \to Y$. Let $\gamma \in \pi_X$ and $\alpha \in \pi_Y$. The formula

$$(*) \qquad (\gamma, \alpha) \mapsto (f \circ \gamma)\alpha r(g \circ \gamma)^{-1} r^{-1}$$

defines a left action of π_X on π_Y. The orbit space of this action is denoted by $\nabla(f, g : x_0, r)$ with elements orb(α). $(*)$ is called the *Reidemeister operation of* f, g *on* π_Y [4; section 1].

Fix a reference pair (x_0, r) and $\alpha \in \pi_Y$. Let $\tilde{\omega}$ be the lifting of αr from $\widetilde{f(x_0)}$. Then there are unique liftings $\tilde{f}$, $\tilde{g}$ of f, g respectively such that $\tilde{f}(\langle e_X \rangle) = \tilde{\omega}(0)$, $\tilde{g}(\langle e_X \rangle) = \tilde{\omega}(1)$. If the lifting pair $(\tilde{f}, \tilde{g})$ of (f, g) is chosen, then $\tilde{f}_\pi = f_\pi, \tilde{g}_\pi = (\alpha r)_* \circ g_\pi$ where $(\alpha r)_*$ is the isomorphism induced by the path αr [6; p.25 Lemma 1.3]. In this case, the lifting $(\tilde{f}, \tilde{g})$ is called a *reference lifting of* (f, g) *corresponding* $\langle \alpha r \rangle$. Thus we have

LEMMA 1[5]. *There exists one-to-one correspondence*

$$\rho : \nabla(f, g : x_0, r) \longrightarrow \mathit{Lift}'(f, g)$$
$$\mathit{orb}(\alpha) \longmapsto [\tilde{f}, \tilde{g}].$$

PROOF. Let $\rho(\mathrm{orb}(\alpha)) = [\tilde{f}, \tilde{g}]$ and $\rho(\mathrm{orb}(\alpha')) = [\tilde{f}', \tilde{g}']$. If $\mathrm{orb}(\alpha) = \mathrm{orb}(\alpha')$, then $\alpha' = (f \circ \gamma)\alpha r(g \circ \gamma)^{-1} r^{-1}$ for some $\gamma \in \pi_X$. Since

$$f_\pi(\gamma) \circ \tilde{f} \circ \gamma^{-1}(\langle e_X \rangle) = \langle e_Y \rangle = \tilde{f}'(\langle e_X \rangle),$$

by the unique lifting property of covering space, we have

$$\tilde{f}' = f_\pi(\gamma) \circ \tilde{f} \circ \gamma^{-1}. \tag{3-1}$$

And

$$f_\pi(\gamma) \circ \tilde{g} \circ \gamma^{-1}(\langle e_X \rangle) = \langle (f \circ \gamma)\alpha r(g \circ \gamma)^{-1} \rangle = \tilde{g}'(\langle e_X \rangle).$$

By the unique lifting property of covering space, we get

$$\tilde{g}' = f_\pi(\gamma) \circ \tilde{g} \circ \gamma^{-1}. \tag{3-2}$$

By (3-1) and (3-2), we have

$$[\tilde{f}, \tilde{g}] = [\tilde{f}', \tilde{g}'].$$

Thus ρ is well defined.

On the other hand, define the function $\phi : \mathrm{lift}'(f, g) \to \nabla(f, g : x_0, r)$ by $\phi[\tilde{f}, \tilde{g}] = \mathrm{orb}((Q \circ \tilde{c})r^{-1})$, where $\tilde{c}$ is a path from $\tilde{f}(\tilde{x}_0)$ to $\tilde{g}(\tilde{x}_0)$. It is easy to see that ϕ is the inverse function of ρ. □

For convenience, the orbit of α will be denoted by

$$T(\alpha, \tilde{f}, \tilde{g}) := \{f_\pi(\gamma)\alpha\tilde{g}_\pi(\gamma^{-1}) : \gamma \in \pi_X\}$$

where $(\tilde{f}, \tilde{g})$ is a reference lifting of (f, g) corresponding $\langle r \rangle$. Then by Lemma 1,

$$\rho(T(\alpha, \tilde{f}, \tilde{g})) = [\tilde{f}, \alpha \circ \tilde{g}].$$

If π_Y is a commutative group, then $T(\alpha, \tilde{f}, \tilde{g})$ does not depend on the choice of $(\tilde{f}, \tilde{g})$ $\in \mathrm{Lift}(f, g)$. For regular covering spaces, we have ono-to-one correspondence

$$\bar{\rho} : \nabla_K^H(f, g : x_0, r) \longrightarrow \mathrm{Lift}'(f, g, H, K)$$
$$T([\alpha], {}_H\tilde{f}, {}_H\tilde{g}) \longmapsto [{}_H\tilde{f}, [\alpha] \circ {}_H\tilde{g}].$$

If π_Y / K is a commutative group, then $T([\alpha], {}_H\tilde{f}, {}_H\tilde{g})$ does not depend on the choice of $({}_H\tilde{f}, {}_H\tilde{g})$ $\in \mathrm{Lift}(f, g, H, K)$. In this case $T([\alpha], {}_H\tilde{f}, {}_H\tilde{g})$ is simply denoted by $T([\alpha], H, K)$.

LEMMA 2. *If π_Y/K is a commutative group, then*

$$R(f,g,H,K) = |(\pi_Y/K)/T([\langle e_Y\rangle],H,K)|.$$

PROOF. Since π_Y/K is a commutative group, we have

$$\begin{aligned} T([\alpha],H,K) &= \{{}_H f_\pi([\gamma])[\alpha]_H\tilde{g}_\pi([\gamma]^{-1}) : [\gamma] \in \pi_X/H\} \\ &= [\alpha]\{{}_H f_\pi([\gamma])_H\tilde{g}_\pi([\gamma]^{-1}) : [\gamma] \in \pi_X/H\} \\ &= [\alpha]T([\langle e_Y\rangle],H,K) \end{aligned}$$

and $T([\langle e_Y\rangle],H,K)$ is a normal subgroup of π_Y/K. Hence we have

$$\begin{aligned} R(f,g,H,K) &= |\{[\alpha]T([\langle e_Y\rangle],H,K) : [\alpha] \in \pi_Y/K\}| \\ &= |(\pi_Y/K)/T([\langle e_Y\rangle],H,K)|. \end{aligned}$$

□

LEMMA 3. *If $\varphi(\tilde{f},\tilde{g}) = ({}_H\tilde{f},{}_H\tilde{g})$, then*

$$({}_H\tilde{f}_\pi,{}_H\tilde{g}_\pi)([\gamma]) = ([\tilde{f}_\pi(\gamma)],[\tilde{g}_\pi(\gamma)]).$$

PROOF. By the definition of ${}_H\tilde{f}_\pi$, we have

$$\begin{aligned} {}_H\tilde{f}_\pi([\gamma]) \circ {}_H\tilde{f} \circ \varphi_X &= {}_H\tilde{f} \circ [\gamma] \circ \varphi_X = {}_H\tilde{f} \circ \varphi_X \circ \gamma \\ &= \varphi(\tilde{f}) \circ \varphi_X \circ \gamma = \varphi_Y \circ \tilde{f} \circ \gamma \\ &= \varphi_Y \circ \tilde{f}_\pi(\gamma) \circ \tilde{f} = [\tilde{f}_\pi(\gamma)] \circ \varphi_Y \circ \tilde{f} \\ &= \tilde{f}_\pi(\gamma)] \circ {}_H\tilde{f} \circ \varphi_X. \end{aligned}$$

Thus we have ${}_H\tilde{f}_\pi([\gamma]) = [\tilde{f}_\pi(\gamma)]$. Similarly ${}_H\tilde{g}_\pi([\gamma]) = [\tilde{g}_\pi(\gamma)]$. □

LEMMA 4. $\varphi(\alpha \circ \tilde{f}, \beta \circ \tilde{g}) = ([\alpha] \circ \varphi(\tilde{f}), [\beta] \circ \varphi(\tilde{g}))$.

PROOF. By the definition of φ, we get

$$\begin{aligned}\varphi(\alpha \circ \tilde{f}) \circ \varphi_X &= \varphi_Y \circ \alpha \circ \tilde{f} \\ &= [\alpha] \circ \varphi_Y \circ \tilde{f} \\ &= [\alpha] \circ \varphi(\tilde{f}) \circ \varphi_X.\end{aligned}$$

Thus we get $\varphi(\alpha \circ \tilde{f}) = [\alpha] \circ \varphi(\tilde{f})$. Similarly $\varphi(\beta \circ \tilde{g}) = [\beta] \circ \varphi(\tilde{g})$. □

Note that φ :Lift$(f, g) \longrightarrow$Lift(f, g, H, K) induces a surjection

$$\begin{aligned}\bar{\varphi} : \text{Lift}'(f, g) &\longrightarrow \text{Lift}'(f, g, H, K) \\ [\tilde{f}, \tilde{g}] &\longmapsto [\varphi(\tilde{f}), \varphi(\tilde{g})].\end{aligned}$$

LEMMA 5. *φ induces a surjective transformation*

$$\begin{aligned}\varphi_\nabla : \nabla(f, g : x_0, r) &\longrightarrow \nabla_K^H(f, g : x_0, r) \\ T(\alpha, \tilde{f}, \tilde{g}) &\longmapsto T([\alpha], \varphi(\tilde{f}), \varphi(\tilde{g}))\end{aligned}$$

and the diagram

$$\begin{array}{ccc} \nabla(f, g : x_0, r) & \xrightarrow{\rho} & \mathit{Lift}'(f, g) \\ {\scriptstyle \varphi_\nabla}\downarrow & & \downarrow{\scriptstyle \bar{\varphi}} \\ \nabla_K^H(f, g : x_0, r) & \xrightarrow{\bar{\rho}} & \mathit{Lift}'(f, g, H, K) \end{array}$$

commutes.

PROOF. Let $T(\alpha, \tilde{f}, \tilde{g}) \in \nabla(f, g : x_0, r)$. Then $(\tilde{f}, \tilde{g})$ is the reference lifting of (f, g) corresponding $\langle r \rangle$. Note that

$$\varphi(\tilde{f})(H\langle e_X \rangle) = \varphi(\tilde{f}) \circ \varphi_X(\langle e_X \rangle) = \varphi_Y \circ \tilde{f}(\langle e_X \rangle) = \varphi_Y(\langle e_Y \rangle) = K\langle e_Y \rangle$$

and

$$\varphi(\tilde{g})(H\langle e_X \rangle) = \varphi(\tilde{g}) \circ \varphi_X(\langle e_X \rangle) = \varphi_Y \circ \tilde{g}(\langle e_X \rangle) = \varphi_Y\langle r \rangle = K\langle r \rangle.$$

Thus $(\varphi(\tilde{f}), \varphi(\tilde{g})) := ({}_H\tilde{f}, {}_H\tilde{g})$ is the reference lifting of (f, g) corresponding $K\langle r\rangle$ for regular coverings.

Suppose that $T(\alpha, \tilde{f}, \tilde{g}) = T(\alpha', \tilde{f}', \tilde{g}')$, $\varphi_\nabla(T(\alpha, \tilde{f}, \tilde{g})) = T([\alpha], {}_H\tilde{f}, {}_H\tilde{g})$ and $\varphi_\nabla(T(\alpha', \tilde{f}', \tilde{g}')) = T([\alpha'], {}_H\tilde{f}', {}_H\tilde{g}')$. Note that

$$\begin{aligned} {}_Hf_\pi([\gamma])[\alpha]_H\tilde{g}_\pi([\gamma]^{-1}) &= [f_\pi(\gamma)\alpha\tilde{g}_\pi(\gamma^{-1})] \\ &= [f'_\pi(\delta)\alpha'\tilde{g}'_\pi(\delta^{-1})] \\ &= {}_Hf'_\pi([\delta])[\alpha']_H\tilde{g}'_\pi([\delta^{-1}]) \end{aligned}$$

for some $\gamma, \delta \in \pi_X$. Then we have

$$T([\alpha], {}_H\tilde{f}, {}_H\tilde{g}) = T([\alpha'], {}_H\tilde{f}', {}_H\tilde{g}').$$

Thus φ_∇ is well-defined.

Let $T(\alpha, \tilde{f}, \tilde{g}) \in \nabla(f, g : x_0, r)$. Then by Lemma 4,

$$\bar{\varphi} \circ \rho(T(\alpha, \tilde{f}, \tilde{g})) = \bar{\varphi}([\tilde{f}, \alpha \circ \tilde{g}]) = [\varphi(\tilde{f}), \varphi(\alpha \circ \tilde{g})] = [\varphi(\tilde{f}), [\alpha] \circ \varphi(\tilde{g})],$$

$$\bar{\rho} \circ \varphi_\nabla(T(\alpha, \tilde{f}, \tilde{g})) = \bar{\rho}(T([\alpha], \varphi(\tilde{f}), \varphi(\tilde{g}))) = [\varphi(\tilde{f}), [\alpha] \circ \varphi(\tilde{g})].$$

Thus

$$\bar{\varphi} \circ \rho = \bar{\rho} \circ \varphi_\nabla. \qquad \square$$

Let $\alpha \in \pi_Y$. Then the Reidemeister operation of f, g on π_Y induces an operation of H on $K\alpha$ by restriction. Let $\mathrm{Orb}(K\alpha)$ denote the orbits of this operation. Then the inclusion of K into π_Y induces a function $j_* : \mathrm{Orb}(K\alpha) \to \nabla(f, g : x_0, r)$ and

$$j_*(\mathrm{Orb}(K\alpha)) = \varphi_\nabla^{-1}(\varphi_\nabla(T(\alpha, \tilde{f}, \tilde{g})))$$

(see [2; p.271 (1.9)]).

Lemma 6. $|\bar{\varphi}^{-1}([\varphi(\tilde{f}), \varphi(\tilde{g})])| \leq |K|$.

Proof. By Lemma 5 and the above observation,

$$\begin{aligned} |\bar{\varphi}^{-1}([\varphi(\tilde{f}), \varphi(\tilde{g})])| &= |\varphi_\nabla^{-1}(T([\alpha], \varphi(\tilde{f}), \varphi(\tilde{g})])| \\ &= |j_*(\mathrm{Orb}(K\alpha))| \\ &\leq |\mathrm{Orb}(K\alpha)| \\ &\leq |K|. \end{aligned}$$

$\square$

LEMMA 7. *If $D(\pi_Y) \subset K$, then $KT(\langle e_Y\rangle, \tilde{f}, \tilde{g})$ is a normal subgroup of π_Y.*

PROOF. Let $t_1, t_2 \in KT(\langle e_Y\rangle, \tilde{f}, \tilde{g})$. Then there exist $k_1, k_2 \in K$ and $\gamma_1, \gamma_2 \in \pi_X$ such that $t_1 = k_1 f_\pi(\gamma_1)\tilde{g}_\pi(\gamma_1^{-1})$, $t_2 = k_2 f_\pi(\gamma_2)\tilde{g}_\pi(\gamma_2^{-1})$. Observe that

$$\begin{aligned}
t_1t_2^{-1} &= k_1 f_\pi(\gamma_1)\tilde{g}_\pi(\gamma_1^{-1})\tilde{g}_\pi(\gamma_2)f_\pi(\gamma_2^{-1})k_2^{-1}\\
&= k_1 f_\pi(\gamma_1)\tilde{g}_\pi(\gamma_1^{-1}\gamma_2)f_\pi(\gamma_2^{-1})k_2^{-1}\\
&= [k_1 f_\pi(\gamma_1)\tilde{g}_\pi(\gamma_1^{-1}\gamma_2)f_\pi(\gamma_2^{-1}), k_2^{-1}]k_2^{-1}k_1 f_\pi(\gamma_1)\tilde{g}_\pi(\gamma_1^{-1}\gamma_2)f_\pi(\gamma_2^{-1})\\
&= [k_1 f_\pi(\gamma_1)\tilde{g}_\pi(\gamma_1^{-1}\gamma_2)f_\pi(\gamma_2^{-1}), k_2^{-1}]k_2^{-1}k_1[f_\pi(\gamma_1), \tilde{g}_\pi(\gamma_1^{-1}\gamma_2)f_\pi(\gamma_2^{-1})]\\
&\quad \tilde{g}_\pi(\gamma_1^{-1}\gamma_2)f_\pi(\gamma_2^{-1})f_\pi(\gamma_1)\\
&= [k_1 f_\pi(\gamma_1)\tilde{g}_\pi(\gamma_1^{-1}\gamma_2)f_\pi(\gamma_2^{-1}), k_2^{-1}]k_2^{-1}k_1[f_\pi(\gamma_1), \tilde{g}_\pi(\gamma_1^{-1}\gamma_2)f_\pi(\gamma_2^{-1})]\\
&\quad [\tilde{g}_\pi(\gamma_1^{-1}\gamma_2), f_\pi(\gamma_2^{-1})f_\pi(\gamma_1)]f_\pi(\gamma_2^{-1}\gamma_1)\tilde{g}_\pi((\gamma_2^{-1}\gamma_1)^{-1}) \in KT(\langle e_Y\rangle, \tilde{f}, \tilde{g}).
\end{aligned}$$

Hence $KT(\langle e_Y\rangle, \tilde{f}, \tilde{g})$ is a subgroup of π_Y.

For $\alpha \in KT(\langle e_Y\rangle, \tilde{f}, \tilde{g})$ and $t \in \pi_Y$, then $\alpha = kf_\pi(\gamma)\tilde{g}(\gamma^{-1})$ for some $k \in K, \gamma \in \pi_X$. Thus

$$\begin{aligned}
t\alpha t^{-1} &= tkf_\pi(\gamma)\tilde{g}_\pi(\gamma^{-1})t^{-1}\\
&= k't f_\pi(\gamma)\tilde{g}_\pi(\gamma^{-1})t^{-1}, \qquad \text{where } k't = tk, k' \in K\\
&= k'[tf_\pi(\gamma)\tilde{g}_\pi(\gamma^{-1}), t^{-1}]f_\pi(\gamma)\tilde{g}_\pi(\gamma^{-1}) \in KT(\langle e_Y\rangle, \tilde{f}, \tilde{g}).
\end{aligned}$$

Hence $KT(\langle e_Y\rangle, \tilde{f}, \tilde{g})$ is a normal subgroup of π_Y. □

THEOREM 1. *If $D(\pi_Y) \subset K$, then*

$$R(f,g) \leq |K||\pi_Y/KT(\langle e_Y\rangle, \tilde{f}, \tilde{g})|.$$

PROOF. Note that

$$\begin{aligned}
R(f,g) &= |\mathrm{Lift}'(f,g)| = |\bigcup\{\bar{\varphi}^{-1}([{}_H\tilde{f}, {}_H\tilde{g}]) : [{}_H\tilde{f}, {}_H\tilde{g}] \in \mathrm{Lift}'(f,g,H,K)\}|\\
&= \sum |\bar{\varphi}^{-1}([{}_H\tilde{f}, {}_H\tilde{g}])|
\end{aligned}$$

the summation being over all lifting classes $[{}_H\tilde{f}, {}_H\tilde{g}] \in \mathrm{Lift}'(f,g,H,K)$. By Lemma 2 and Lemma 6,

$$\begin{aligned}
R(f,g) &\leq |K||\mathrm{Lift}'(f,g,H,K)| = |K|R(f,g,H,K)\\
&= |K||(\pi_Y/K)/T([\langle e_Y\rangle], H, K)|.
\end{aligned}$$

Note that

$$\begin{aligned} T([\langle e_Y\rangle], H, K) &= \{_H f_\pi([\gamma])_H \tilde{g}_\pi([\gamma]^{-1}) : [\gamma] \in \pi_X/H\} \\ &= \{[f_\pi(\gamma)\tilde{g}_\pi(\gamma^{-1})] : \gamma \in \pi_X\} \\ &= \{K f_\pi(\gamma)\tilde{g}_\pi(\gamma^{-1}) : \gamma \in \pi_X\} \\ &= KT(\langle e_Y\rangle, \tilde{f}, \tilde{g})/K. \end{aligned}$$

Hence by Lemma 7, we have

$$R(f,g) \le |K||(\pi_Y/K)/(KT(\langle e_Y\rangle, \tilde{f}, \tilde{g})/K)| = |K||\pi_Y/KT(\langle e_Y\rangle, \tilde{f}, \tilde{g})|.$$

□

COROLLARY 1. $R(f,g) \le |D(\pi_Y)||\pi_Y/D(\pi_Y)T(\langle e_Y\rangle, \tilde{f}, \tilde{g})|$.

THEOREM 2. $|\pi_Y/D(\pi_Y)T(\langle e_Y\rangle, \tilde{f}, \tilde{g})| \le R(f,g)$.

PROOF. It is well known that

$$|\mathrm{Coker}(f_{1*} - g_{1*})| \le R(f,g)$$

where $f_{1*}, g_{1*} : H_1(X) \to H_1(Y)$ are the homomorphisms induced by f and g respectively between the 1-dimensional homology groups of X and Y. Hence it is sufficient to show that

$$|\mathrm{Coker}(f_{1*} - g_{1*})| = |\pi_Y/D(\pi_Y)T(\langle e_Y\rangle, \tilde{f}, \tilde{g})|.$$

Let us consider the commutative diagram

$$\begin{array}{ccc} \pi_X & \xrightarrow{f_\pi, \tilde{g}_\pi} & \pi_Y \\ {\scriptstyle\theta_X}\downarrow & & \downarrow{\scriptstyle\theta_Y} \\ H_1(X) & \xrightarrow{f_{1*}, g_{1*}} & H_1(Y) \end{array}$$

and the composition $\eta \circ \theta_Y$

$$\pi_Y \xrightarrow{\theta_Y} H_1(Y) \xrightarrow{\eta} \mathrm{Coker}(f_{1*} - g_{1*} : H_1(X) \longrightarrow H_1(Y))$$

where θ_X, θ_Y are the abelianizations and η is the natural projection. Then we have

$$\mathrm{Coker}(f_{1*} - g_{1*}) \cong \pi_Y/\mathrm{ker}(\eta \circ \theta_Y).$$

Observe that

$$\begin{aligned}\alpha \in \mathrm{ker}(\eta \circ \theta_Y) &\Longleftrightarrow \theta_Y(\alpha) \in (f_{1*} - g_{1*})(\theta_Y(\pi_Y)) \\ &\Longleftrightarrow \exists \gamma \in \pi_X \ni \theta_Y(\alpha) = f_{1*} \circ \theta_Y(\gamma) - g_{1*} \circ \theta_Y(\gamma) \\ &\qquad = \theta_Y \circ f_\pi(\gamma) - \theta_Y \circ \tilde{g}_\pi(\gamma) = \theta_Y(f_\pi(\gamma)\tilde{g}_\pi(\gamma^{-1})) \\ &\Longleftrightarrow \exists \gamma \in \pi_X \ni \alpha(f_\pi(\gamma)\tilde{g}_\pi(\gamma^{-1}))^{-1} \in \mathrm{ker}\theta_Y = D(\pi_Y) \\ &\Longleftrightarrow \exists \gamma \in \pi_X \ni \alpha \in D(\pi_Y) f_\pi(\gamma)\tilde{g}_\pi(\gamma^{-1}) \\ &\Longleftrightarrow \alpha \in D(\pi_Y)T(\langle e_Y \rangle, \tilde{f}, \tilde{g}).\end{aligned}$$

Hence we have

$$\mathrm{Coker}(f_{1*} - g_{1*}) \cong \pi_Y/D(\pi_Y)T(\langle e_Y \rangle, \tilde{f}, \tilde{g}).$$

□

Recall that $x \in X$ is called a *coincidence of* (f, g) if $f(x) = g(x)$.

THEOREM 3. *If x_0 is a coincidence of (f, g), then*

$$D(\pi_Y)T(\langle e_Y \rangle, \tilde{f}, \tilde{g}) = D(\pi_Y)T(\langle e_Y \rangle, f, g)$$

where $T(\langle e_Y \rangle, f, g) = \{f_\pi(\gamma)g_\pi(\gamma^{-1}) : \gamma \in \pi_X\}$.

PROOF. If $u \in D(\pi_Y)T(\langle e_Y \rangle, \tilde{f}, \tilde{g})$, then there exist $d \in D(\pi_Y)$ and $\gamma \in \pi_X$ such that

$$\begin{aligned}u &= d f_\pi(\gamma)\tilde{g}_\pi(\gamma^{-1}) \\ &= d f_\pi(\gamma)\langle r \rangle g_\pi(\gamma^{-1})\langle r \rangle^{-1} \\ &= d[f_\pi(\gamma), \langle r \rangle]\langle r \rangle f_\pi(\gamma) g_\pi(\gamma^{-1})\langle r \rangle^{-1} \\ &= d[f_\pi(\gamma), \langle r \rangle][\langle r \rangle f_\pi(\gamma) g_\pi(\gamma^{-1}), \langle r \rangle^{-1}] f_\pi(\gamma) g_\pi(\gamma^{-1}).\end{aligned}$$

Hence $D(\pi_Y)T(\langle e_Y\rangle, \tilde{f}, \tilde{g}) \subset D(\pi_Y)T(\langle e_Y\rangle, f, g)$.

Conversely, if $u \in D(\pi_Y)T(\langle e_Y\rangle, f, g)$, then there exist $d \in D(\pi_Y)$ and $\gamma \in \pi_X$ such that

$$\begin{aligned} u &= df_\pi(\gamma)g_\pi(\gamma^{-1}) \\ &= df_\pi(\gamma)\langle r\rangle^{-1}\tilde{g}_\pi(\gamma^{-1})\langle r\rangle \\ &= d[f_\pi(\gamma)\langle c\rangle^{-1}][\langle r\rangle^{-1}f_\pi(\gamma)\tilde{g}_\pi(\gamma^{-1}), \langle r\rangle]f_\pi(\gamma)\tilde{g}_\pi(\gamma^{-1}). \end{aligned}$$

Hence $D(\pi_Y)T(\langle e_Y\rangle, f, g) \subset D(\pi_Y)T(\langle e_Y\rangle, \tilde{f}, \tilde{g})$. Thus

$$D(\pi_Y)T(\langle e_Y\rangle, \tilde{f}, \tilde{g}) = D(\pi_Y)T(\langle e_Y\rangle, f, g). \qquad \square$$

COROLLARY 2. *If x_0 is a coincidence of (f, g), then*

$$|\pi_Y/D(\pi_Y)T(\langle e_Y\rangle, f, g)| \le R(f,g) \le |D(\pi_Y)||\pi_Y/D(\pi_Y)T(\langle e_Y\rangle, f, g)|.$$

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