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SOME DYNAMICAL PROPERTIES OF A WEAKLY ALMOST PERIODIC FLOW

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ABSTRACT. In this paper, we study some dynamical properties of a weakly almost periodic flow. In particular we get, in a weakly almost periodic flow (X,T), the groups I and $A(I)$ of all automorphisms of I are isomorphic, where $E(X)$ is the enveloping semigroup of (X,T) and I is the minimal right ideal in $E(X)$.

1. Introduction

In this paper we consider weakly almost periodic (or simply w.a.p.) flows introduced by R. Ellis and M. Nerukar (see [6]). The fundamental concepts of topological dynamics, such as proximality, distality, almost periodicity, weak almost periodicity, etc., are frequently used. These concepts which have proved extremely fruitful for abstract topological dynamics are used to study a number of dynamical properties of a w.a.p. flow (X,T). In a w.a.p. flow (X,T), we find the groups I and $A(I)$ of all automorphisms of I are isomorphic, where I is the minimal right ideal in the enveloping semigroup $E(X)$. Also, we give a sufficient condition for the Ellis group to be trivial.

2. Preliminaries

Let (X,T,π) (or simply (X,T)) be a flow with compact Hausdorff phase space X. The enveloping semigroup $E(X)$ (or simply E) of a flow is a kind of compactification of the acting group. It is defined to be the closure of $\{\pi^t \mid t \in T\}$, where the closure is in the product topology on

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X^X. A pair of points (x, y), $x, y \in X$ is *proximal* if $xp = yp$ for some $p \in E(X)$. The proximal pairs will be denoted $P(X,T)$ (or simply P). It is a reflexive, symmetric, T-invariant relation, but is not in general transitive. A necessary and sufficient condition for $P(X,T)$ to be an equivalance relation on X is that there is only one minimal right ideal in $E(X)$. Even if $P(X,T)$ is an equivalance relation on X, it need not be closed. A flow is said to be *almost periodic* if the family of maps $\{\pi^t \mid t \in T\}$is equicontinuous. A flow is *distal* if it has no nontrivial proximal pairs. It is well known that the flow (X,T) is almost periodic iff $E(X)$ is a compact topological group and the elements of $E(X)$ are continuous maps. Also (X,T) is distal iff $E(X)$ is a group. Therefore it is trivial that if (X,T) is almost periodic, then it is distal.

LEMMA 2.1. ([5]) *Let I be a minimal right ideal in E and J be the set of idempotents in I. Then :*

1. *The set J is not empty.*
2. *If $v \in I$ and $p \in I$, then $vp = p$.*
3. *Iv is a subgroup of I with identity v ($v \in J$).*
4. *$\{Iv \mid v \in J\}$ is a partition of I.*

Note that $x \in X$ is an almost periodic point of (X,T) iff $\overline{xT}$ is a minimal subset of X. Also $\overline{xT} = xE$ for each $x \in X$. (X,T) is *pointwise almost periodic* if x is an almost periodic point for each $x \in X$.

LEMMA 2.2. ([5]) *Let I be a minimal right ideal in E and $x \in X$. Then the following are equivalent :*

1. *x is an almost periodic point of (X,T).*
2. *$\overline{xT} = xI$.*
3. *There exists an idempotent $u \in I$with $xu = x$.*

LEMMA 2.3. ([2]) *Suppose $P(X,T)$ is an equivalence relation on X. Then :*

1. *$X = \cup_\alpha N_\alpha$, where the N_α are pairwise disjoint, $N_\alpha T \subset N_\alpha$, and each N_α contains precisely one minimal set M_α.*
2. *If $x \in N_\alpha$, then x is proximal to a point $y \in M_\alpha$.*
3. *If $P(X,T)$ is closed in $X \times X$, then the sets N_α are closed.*

3. Main Results

A flow (X,T) is w.a.p. iff each element of $E(X)$ is continuous. Note that products and factors of w.a.p. flows are again w.a.p.. Also note that if (X,T) is w.a.p. and S is a subgroup of T, then (X,S) is w.a.p. (see [6]).

LEMMA 3.1. ([6]) *Let (X,T) be w.a.p. and let I be a minimal right ideal in $E(X)$. Then :*

1. *(I,T) is almost periodic.*
2. *I has a unique idempotent u.*
3. *I is a compact topological group with identity u.*
4. *I is a minimal left ideal.*
5. *I is the only minimal right ideal in $E(X)$.*
6. *Let $p \in E(X)$; then $pu = up$.*

LEMMA 3.2. *Let (X,T) be w.a.p.. Then :*

1. *$P(X,T)$ is an equivalence relation on X.*
2. *$P(X,T)$ is an equivalence relation iff $P(X,T)$ is closed in $X \times X$.*

PROOF. 1. This follows from 5 of Lemma 3.1.

2. It is known that if $P(X,T)$ is closed in $X \times X$, then it is an equivalance relation on X (see [2, Corollary 1]). Now let I be the only minimal right ideal in $E(X)$. For any $(x,y) \in \overline{P(X,T)}$, there exists a net $\langle(x_\alpha, y_\alpha)\rangle$ in $P(X,T)$ such that $(x_\alpha, y_\alpha) \to (x,y)$. Then $x_\alpha q = y_\alpha q$ $(q \in I)$. But since each element of $E(X)$ is continuous, it follows that $xq = yq$ $(q \in I)$. Therefore $(x,y) \in P(X,T)$. This means that $P(X,T)$ is closed in $X \times X$. □

THEOREM 3.1. *Let (X,T) be w.a.p. and let I be the only minimal right ideal, u a unique idempotent. Then the following are pairwise equivalent :*

1. *$xu = x$ $(x \in X)$.*
2. *$I = E(X)$.*
3. *$E(X)$ is a group.*
4. *(X,T) is distal.*
5. *(X,T) is almost periodic.*
6. *(X,T) is pointwise almost periodic.*

PROOF. ($1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 5 \Rightarrow 6 \Rightarrow 1$) Let $p \in E(X)$ and $x \in X$. Then $xp = (xu)p = x(up)$, which implies $p = up \in IE \subset I$. Hence $I = E(X)$ is a group from 3 of Lemma 3.1. This means that (X,T) is distal. But since each element of $E(X)$ is continuous, it follows that (X,T) is almost periodic. Because $E(X)$ is a minimal right ideal and $e \in E(X)$, x is an almost periodic point of (X,T) $(x \in X)$, by Lemma 2.2. Again by Lemma 2.2 this implies that $xu = x$ $(x \in X)$. The proof is completed. □

In general $L_q : p \mapsto qp$ of (I,T) onto (I,T) is an automorphism, but $R_q : p \mapsto pq$ of (I,T) onto (I,T) is not even a homomorphism. The following theorem gives a sufficient condition for R_q to be an automorphism. If the flow (X,T) is w.a.p., then R_u is an identity automorphism, where u is the unique idempotent of minimal right ideal in $E(X)$. However, R_q need not be an automorphism if $q \neq u$.

THEOREM 3.2. *Let (X,T) be a w.a.p. flow, I the minimal right ideal in $E(X)$, and $u^2 = u \in I$. Then :*

1. *$R_u : p \mapsto pu$ of I onto I is an identity automorphism.*
2. *If T is abelian, then $R_q : p \mapsto pq$ of (I,T) onto (I,T) is also an automorphism $(q \in I)$.*

PROOF. 1. Let $q \in I$. Since q is a continuous function on X, it follows that R_q is also continuous. Now let $p \in I$, $t \in T$. Then $R_u(pt) = (pt)u = p(tu) = p(ut) = R_u(p)t$ by Lemma 3.1.6. Since I is a group with identity u, R_u is an identity automorphism.

2. Let $t \in T$ and $q \in I$. Since T is abelian, we have $tq = qt$. Thus $R_q(pt) = (pt)q = p(tq) = p(qt) = (pq)t = R_q(p)t$ $(p \in I)$. By Lemma 3.1.3, there exists $r \in I$ such that $qr = rq = u$. Hence $R_qR_r = R_{rq} = R_u = R_{qr} = R_rR_q$. This means that R_q is an automorphism.

□

THEOREM 3.3. *Let (X,T) be w.a.p.. Then there exists a partition of X consisting of compact Hausdorff invariant subsets and each of this partition elements contains exactly one minimal set*

PROOF. This follows from Lemma 2.3 and Lemma 3.2. □

THEOREM 3.4. *Let (X,T) be w.a.p., and let $\varphi : (X,T) \to (X/P,T)$ be the canonical map, and ψ be the epimorphism of $E(X)$ onto $E(X/P)$ induced by φ. Then :*

1. $\psi(u) = e$.
2. $\psi(I) = E(X/P)$.
3. *The spaces* I *and* $E(X/P)$ *are homeomorphic.*
4. *The flows* I *and* $E(X/P)$ *are isomorphic.*
5. *The groups* I *and* $E(X/P)$ *are isomorphic.*

PROOF. 1. Since P is a closed equivalence relation on X , it follows that $E(X/P)$ is a group. Then $\psi(u) = \psi(u^2) = \psi(u)\psi(u)$ and so $\psi(u) = e$.

2. Let $p \in E(X)$. Then pI is a right ideal in $E(X)$ and $pI \subset I$ by Lemma 3.1.4. Since I is a minimal right ideal in $E(X)$, we have $pI = I$. Thus $\psi(p) = \psi(p)\psi(u) = \psi(pu) \in \psi(I)$, which implies that $\psi(I) = E(X/P)$.
3. Since X is a compact Hausdorff space, and P is a closed equivalence relation on X, it follows that X/P is Hausdorff, and so $E(X/P)$ is Hausdorff. To see that the map $p \mapsto \psi(p)$ of I onto $E(X/P)$ is one-to-one, let $\psi(p) = e$ for some $p \in I$ and $x \in X$. Then $\varphi(xp) = \varphi(x)\psi(p) = \varphi(x)$, which implies that $(xp\, , x) \in P$. Hence $xpr = xr$ $(r \in I)$. Then $xp = xpu = xu$, whence $xp^2 = xup = xp$. Therefore $p^2 = p$ and so $p = u$. Then the map $p \mapsto \psi(p)$ of I onto $E(X/P)$ is a continuous bijection. But since I is compact and $E(X/P)$ is Hausdorff, this map is a homeomorphism.
4. Note that $\psi(pt) = \psi(p)t \quad (p \in I,\ t \in T)$.
5. Note that $\psi(pq) = \psi(p)\psi(q) \quad (p,\, q \in I)$.

□

THEOREM 3.5. *If* (X,T) *is a w.a.p. flow, then the flow* $(X/P,T)$ *is distal.*

PROOF. This follows from Theorem 3.4 and the fact that (X,T) is distal iff $E(X)$ is a group. □

THEOREM 3.6. *Let* (X,T) *be w.a.p.. Then* $A(I)$ *and* I *are isomorphic, where* I *is the only minimal right ideal in* $E(X)$, *and* $A(I)$ *is the group of all automorphisms of* I.

PROOF. Define $\Gamma : A(I) \to I$ by $\Gamma(\phi) = p$ if $\phi(u) = p$, where $u^2 = u \in I$ $(\phi \in A(I))$. Let $p \in I$. Since I is a group, it follows that (u,p) is an almost periodic point of $(I \times I,T)$. By [3, Theorem 3], there exists $\phi \in A(I)$ such that $\phi(u) = p$ and so Γ is onto. Now suppose $\phi(u) = \psi(u)$. Since I is minimal, given any point $p \in I$, there exists a net $\langle t_\alpha \rangle$ in T

with $ut_\alpha \to p$. Then $\phi(p) = \phi(\lim ut_\alpha) = \lim \phi(ut_\alpha) = \lim \phi(u)t_\alpha = \lim \psi(u)t_\alpha = \lim \psi(ut_\alpha) = \psi(\lim ut_\alpha) = \psi(p)$, whence $\phi = \psi$. It is immediate to verify that we need only show that $\Gamma(\phi\psi) = \Gamma(\phi)\Gamma(\psi)$. For if $\phi, \psi \in A(I)$, we have $\phi(u)\psi(u) = \phi(u\psi(u)) = \phi\psi(u)$. □

Now let (X,T) be a w.a.p. flow, I the only minimal right ideal, u a unique idempotent, and $x_o \in Xu$. Then $x_oE = \overline{x_oT} = x_oI$, (x_oI, x_o) is a point transitive minimal flow, and the Ellis group is defined as the following. $G(x_oI, x_o) = \{p \in I \mid x_op = x_o\}$. Note that G is a compact topological subgroup of I. Since (G, I, T) is a biflow, it follows that $(I/G, T)$ is isomorphic with (x_oI, x_o) (see [7]).

THEOREM 3.7. *If (X,T) is a w.a.p. flow with T abelian and $\overline{x_oT} = X$, then $G(x_oI, x_o) = \{u\}$.*

PROOF. Let $p \in G(x_oI, x_o)$. For any $y \in X := \overline{x_oT}$, there exists a net $\langle t_\alpha \rangle$ in T such that $x_ot_\alpha \to y$. Since T is abelian, we have $yp = (\lim x_ot_\alpha)p = \lim(x_ot_\alpha)p = \lim x_o(t_\alpha p) = \lim x_o(pt_\alpha) = \lim(x_op)t_\alpha = \lim x_ot_\alpha = y$. For any $y \in X = x_oI$, there exists $q \in I$ such that $y = x_oq$. Then we have $yu = (x_oq)u = x_o(qu) = x_oq = y$. This means that $p = u$. The proof is completed. □

The following theorems are immediate.

THEOREM 3.8. *Let (X,T) be a w.a.p. flow with T abelian and $\overline{x_oT} = X$. Then the flows (I,T) and (x_oI, T) are isomorphic.*

THEOREM 3.9. *Let (X,T) be a w.a.p. flow with T abelian. If (X,T) is minimal, then the flows $(E(X),T)$ and (X,T) are isomorphic.*

PROOF. Note that if (X,T) is w.a.p. minimal, it is almost periodic by [1, Theorem 6 of chapter 4]. □

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