Comm. Korean Math. Soc. **13** (1998), No. 1, pp. 131–136

F-PROXIMAL FLOWS

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ABSTRACT. The proximal flow have been introduced and studied by J. Auslander [1] and R. Ellis [2]. S. Glasner showed the various properties of the proximal flow in [7]. S. K. Kaul [3] have introduced the first prolongation set which contains the orbit closure. We define the F-proximal flow and $FP_\pi(y)$ by using the first prolongation set and investigated its properties.

1. Preliminaries and Notations

In this paper, let T be an arbitrary, but fixed, topological group and we consider a flow (X,T) with a compact Hausdorff space X. A closed nonempty subset M of X is said to be a *minimal set* if, for every $x \in M$, the orbit xT is a dense subset of M. If X is itself minimal, we say it is a *minimal flow.* A subset A of T is said to be *syndetic* if there exists a compact subset K of T with $T = AK$.

The points x and y of X in the flow (X,T) are said to be *proximal* if there exists a net $\{t_\alpha\}$ of elements of T such that $\lim xt_\alpha = \lim yt_\alpha$. The flow (X,T) is said to be proximal if every two points of X are proximal. We denote $P(X,T) = \{(x,y) \in X \times X \mid x \text{ and } y \text{ are proximal}\}$ which is called the *proximal relation* on (X,T). The flow (X,T) is said to be *distal* if $x,y \in X$ and $(x,y) \in P(X,T)$ imply $x = y$.

Following [3] we say that $t_i \to \infty$ for a net $\{t_i\}$ in T if the net $\{t_i\}$ is ultimately outside each compact subset of T. For a flow (X,T) we shall define the first prolongation set and the first prolongation limit set of x in X respectively, by $D(x) = \{\, y \mid x_it_i \to y \text{ for some } x_i \to x,\ t_i \in T\}$ and $J(x) = \{\, y \mid x_it_i \to y \text{ for some } x_i \to x, t_i \to \infty\}$.

A point $x \in X$ is said to have property M if whenever there are nets $x_i \to x, y_i \to x$ and $\{t_i\}$ in T such that the nets $\{x_it_i\}$ is convergent, then

Received April 10, 1997. Revised November 22, 1997.
1991 Mathematics Subject Classification: 54H20.
Key words and phrases: F-proximal, first prolongation set.

the net $\{y_i t_i\}$ is also convergent. A flow (X,T) is said to be T-weakly equicontinuous if $J(x,x) \subset \Delta = \{(x,x)|x \in X\}$ and x has property M, for every $x \in X$.

If (Y,T) is also a flow, a *homomorphism* from (X,T) to (Y,T) is continuous map $\phi : X \to Y$ such that $\phi(xt) = \phi(x)t$ $(x \in X, t \in T)$.

In this context, the meaning of epimorphisms, isomorphisms and automorphisms is clear.

An extension $\varphi : (X,T) \to (Y,T)$ is almost one-to-one if there exists a point $y_0 \in Y$ such that $\varphi^{-1}(y_0)$ is a singleton.

As is customary, let X^X denote the set of all functions from X to X, provided with the topology of pointwise convergence, and consider T as the subset $\{t : x \to xt, t \in T\}$ of X^X. The *enveloping semigroup* $E(X)$ of the flow (X,T) is the closure of T in X^X. Then $E(X)$ is a compact Hausdorff space and we may consider $(E(X),T)$ as a flow, whose phase space $E(X)$ admits a semigroup structure. The *minimal right ideals* I of $E(X)$ (that is, the nonempty subsets I of $E(X)$ such that $IE(X) \subset I$, which contains no proper nonempty subsets with the same property) coincide with the minimal sets in the flow $(E(X),T)$ (see 3.4 [2]).

2. F-proximal flows

DEFINITION 2.1. Let (X,T) be a flow and $x, y \in X$. Then x and y are said to be *F-proximal* if $D(x,y) \cap \Delta \neq \emptyset$.

We denote $FP(X,T) = \{(x,y) \in X \times X \mid x \text{ and } y \text{ are F-proximal }\}$. The flow (X,T) is said to be *F-proximal* if every two points of X are F-proximal.

THEOREM 2.2. *If (X,T) is a proximal flow, then (X,T) is an F-proximal flow.*

PROOF. Let (X,T) be a proximal flow, then $\overline{(x,y)T} \cap \Delta \neq \emptyset$ for any $x, y \in X$. Since $D(x,y)$ contains $\overline{(x,y)T}$, $D(x,y) \cap \Delta \neq \emptyset$ for any $x, y \in X$. Hence (X,T) is an F-proximal flow. □

REMARK 2.3. The converse of Theorem 2.2 does not hold.

Let $X = [0,1]$ and let $T = Z$. Now we define $\varphi : X \to X$ by $\varphi(x) = x^2$ and $\Pi : X \times T \to X$ by $\Pi(x,n) = \varphi^n(x)$. Then $P(X,T) = X \times$

$X - \{(0,1),(1,0)\}$. Let $\{(\frac{1}{n},1)|n \in N\}$ and $\{(1,\frac{1}{n})|n \in N\}$ be nets in $X \times X$. Then clearly $\{(\frac{1}{n},1)\}$ and $\{(1,\frac{1}{n})\}$ converge to (0,1) and (1,0) respectively. But

$$\Pi((\frac{1}{n},1),-n) = ((\frac{1}{n})^{2^{-n}},1) \to (1,1),$$
$$\Pi((1,\frac{1}{n}),-n) = ((1,(\frac{1}{n})^{2^{-n}}) \to (1,1).$$

Hence $(1,1) \in D(0,1)$ and $(1,1) \in D(1,0)$. Therefore $(0,1) \in FP(X,T)$, $(1,0) \in FP(X,T)$. Hence (X,T,Π) is not a proximal flow, but is an F-proximal flow.

LEMMA 2.4. *Let X be T-weakly equicontinuous. Then the followings hold.*

(1) *If there are nets $\{x_i\}$ in X and $\{t_i\}$ in T such that $x_i \to x$ and $x_i t_i \to y$, then $x t_i \to y$.*
(2) *If for a net $\{t_i\}$ in T and $x, y \in X$, $x t_i \to y$, then $y t_i^{-1} \to x$.*

PROOF. See Lemma 2.3 in [5] □

PROPOSITION 2.5.

(i) $P(X,T) \subset FP(X,T)$.
(ii) *If (X,T) is T-weakly equicontinuous. Then $FP(X,T) = P(X,T)$.*

PROOF. It follows from the Lemma 2.4. □

PROPOSITION 2.6. *Let $\phi : (X,T) \to (Y,T)$ be the epimorphism and (X,T) is F-proximal. Then (Y,T) is F-proximal.*

PROOF. For any $y_1, y_2 \in Y$, there are $x_1, x_2 \in X$ such that $\phi(x_1) = y_1, \phi(x_2) = y_2$. Since (X,T) is F-proximal, there exists nets $\{x_{1i}\}, \{x_{2i}\}$ in X and $\{t_i\}$ in T such that $x_{1i} \to x_1, x_{2i} \to x_2$ and $\lim x_{1i} t_i = \lim x_{2i} t_i$. Now $\phi(x_{1i}) \to \phi(x_1) = y_1, \phi(x_{2i}) \to \phi(x_2) = y_2$ and $\lim \phi(x_{1i}) t_i = \lim \phi(x_{2i}) t_i$. Therefore (Y,T) is F-proximal. □

PROPOSITION 2.7. *An almost one-to-one extension of a minimal F-proximal flow is F-proximal.*

PROOF. Let $\varphi : (X,T) \to (Y,T)$ be an almost one-to-one extension. Suppose that there exist a point $y_0 \in Y$ such that $\varphi^{-1}(y_0) = \{x_0\}$.

Let x_1, x_2 be points of X and denoted $y_i = \varphi(x_i)$ $(i = 1,2)$. Since (Y,T) is minimal, We can find nets t_μ in T, $y_{1\mu}, y_{2\mu}$ in Y such that $\lim_\mu(y_{1\mu}, y_{2\mu}) = (y_1, y_2) = (\varphi(x_1), \varphi(x_2))$ and $\lim_\mu y_{1\mu}t_\mu = \lim_\mu y_{2\mu}t_\mu = y_0$.

Then there exists nets $x_{1\mu}, x_{2\mu}$ such that $\varphi(x_{1\mu}) = y_{1\mu}$, $\varphi(x_{2\mu}) = y_{2\mu}$. Then

$$\lim_\mu \varphi(x_{1\mu}) = \varphi(x_1),\ \lim_\mu \varphi(x_{2\mu}) = \varphi(x_2)$$

$$\lim_\mu x_{1\mu} = x_1,\ \lim_\mu x_{2\mu} = x_2$$

$$\lim_\mu (x_{1\mu}, x_{2\mu})t_\mu = (x_1', x_2').$$

Then

$$\begin{aligned}\varphi(x_i') = \varphi(\lim_\mu x_{i\mu}t_\mu) &= \lim_\mu \varphi(x_{i\mu}t_\mu)\\ &= \lim_\mu \varphi(x_{i\mu}t_\mu) = \lim_\mu y_{i\mu}t_\mu = y_0\ (i = 1,2)\end{aligned}$$

So $x_1' = x_2' = x_0$. Therefore x_1, x_2 are F-proximal. Hence (X,T) is an F-proximal flow. □

LEMMA 2.8. *Let (X,T) be a flow and S a syndetic subgroup of T. Then $FP(X,T) = FP(X,S)$.*

PROOF. It is clear that $FP(X,S) \subset FP(X,T)$. We show that $FP(X,T) \subset FP(X,S)$. Let $T = SK$, where K is a compact subset of T and $(x,y) \in FP(X,T)$. Then there exist $x_i \to x$, $y_i \to y$ and $t_i \in T$ such that $\lim_i x_it_i = \lim_i y_it_i$. Now $t_i = s_ik_i$ for some k_i in K and for some s_i in S. We can assume that $\lim_i k_i = k$, $\lim_i x_is_i = z_1$, $\lim_i y_is_i = z_2$ and $\lim_i x_it_i = \lim_i y_it_i = z$ exist. So

$$\begin{aligned}(z,z) &= (\lim_i x_it_i,\ \lim_i y_it_i)\\ &= (\lim_i x_is_ik_i,\ \lim_i y_is_ik_i)\\ &= ((\lim_i x_is_i)k,\ (\lim_i y_is_i)k)\\ &= (z_1k,\ z_2k)\end{aligned}$$

Thus for $i = 1, 2,$ $z_i = zk^{-1}$ implies $z_1 = z_2$. This means that x and y are F-proximal under S. Therefore $FP(X, S) = FP(X, T)$. □

THEOREM 2.9. [INHERITANCE THEOREM] *Let (X, T) be a flow, and S a syndetic subgroup of T. Then (X, T) is F-proximal if and only if (X, S) is F-proximal.*

PROOF. This follows from Lemma 2.8. □

PROPOSITION 2.10. *$FP(X, T)$ is a closed invariant reflexive, symmetric relation on X.*

DEFINITION 2.11. Let $\pi : X \to Y$ be an epimorphism. For each $y \in Y$, we define

(1) $FP_\pi(y) = \{(x, x') \in X \times X | \pi(x) = \pi(x') = y, (x, x') \in FP(X, T)\}$.

(2) $P_\pi(y) = \{(x, x') \in X \times X | \pi(x) = \pi(x') = y, (x, x') \in P(X, T)\}$.

PROPOSITION 2.12. *$FP_\pi(y)$ is a reflexive and symmetric relation on $\pi^{-1}(y)$ for each $y \in Y$.*

Let βT denote the Stone-Cěch compactification of T. Then $(\beta T, T)$ is an universal point-transitive flow, and βT is an enveloping semigroup for X, whenever X is a flow with acting group T. Given a point $y \in Y$, let

$$E_y = \{p \in \beta T \mid yp = y\}.$$

Let I be a minimal right ideal in some enveloping semigroup $E(X)$ and $B(I)$ denoted the set of idempotent elements in I.

Let $E(X)$ be an enveloping semigroup for X, with minimal right ideals I, I'. Then given $u \in B(I)$, there exists a unique $u' \in B(I')$ such that $uu' = u$ and $u'u = u'$.

We say that u and u' are *equivalent idempotents* and the relations thus defined is actually an equivalence relation.

PROPOSITION 2.13. *Let $\pi : (X, T) \to (Y, T)$ be an epimorphism, Y a minimal set, and $y \in Y$. If $FP_\pi(y)$ is an equivalence relation on $\pi^{-1}(y)$. Then $(xu, xv) \in FP_\pi(y)$ for all $x \in \pi^{-1}(y)$ and u, v is any equivalent idempotents in E_y.*

PROOF. Suppose that $FP_\pi(y)$ is an equivalence relation and let u, v be any equivalent idempotents in E_y. Then for any $x \in \pi^{-1}(y)$, we have: $\pi(xu) = \pi(x)u = yu = y$ and $\pi(xv) = \pi(x)v = yv = y$. Hence $(x, xu) \in P_\pi(y)$ and $(x, xv) \in P_\pi(y)$. Now since $P_\pi(y) \subset FP_\pi(y), (xu, xv) \in FP_\pi(y)$. □

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