Comm. Korean Math. Soc. 13 (1998), No. 1, pp. 181–193

ON THE GIBBS PHENOMENON FOR THE SHANNON SAMPLING SERIES IN WAVELET SUBSPACES AND A WAY TO GO AROUND

HONG-TAE SHIM

ABSTRACT. The Shannon sampling series is the prototype of an interpolating series or sampling series. Also the Shannon wavelet is one of the protypes of wavelets. But the coefficients of the Shannon sampling series are different from those of the Shannon wavelet expansions. By giving different function values at the point of discontinuity, we analyze the Gibbs phenomenon for the Shannon sampling series. We also find a way to go around this overshoot effect.

1. Introduction

The Gibbs phenomenon [2] in trigonometric series has been well known. This phenomenon involves the overshoot of the partial sums of a series approximation to a function with a jump discontinuity. It also exists for other series and integral approximants, where the approximation is given by a projection [1], [4], [5]. Shim and Volkmer [7] has shown that this overshoot always exists for wavelet expansion if the scaling function is differentiable and sufficiently rapidly decreasing. Helmberg [3] has shown that the Gibbs phenomenon exists for Fourier inetrpolation. It was also recently shown by Shim and Kim [6] that it also exists for inetrpolating series in some wavelet subspaces. In [6], for a function with discontinuity at point $x = x_0$, the function value was defined by $f(x_0) = f(x_0^+)$. But we can assign different function values at the point of discontinuity. In this work we shall concentrate on the Shannon sampling series in wavelet subspaces. So this work is an extension of the previous work [6].

Received April 1, 1997. Revised November 22, 1997.
1991 Mathematics Subject Classification: 40A05, 41A30, 41A58.
Key words and phrases: scaling function, wavelet, sampling interpolation, the Gibbs phenomenon.

2. Background

The prototype of an interpolating or sampling series is the Shannon series

$$f_0(t) = \sum_{n=-\infty}^{\infty} f(n) \frac{\sin \pi(t-n)}{\pi(t-n)}.$$

It interpolates exactly a function $f \in L^2(\mathbb{R})$ which is bandlimited. Such functions belong to the Paley-Wiener space B_π of π band limited functions. The function $\phi(t) = \sin \pi t / \pi t$ may also be considered as a scaling function for a wavelet system, i.e.,

(i) $\phi(t-n)$ is an orthonormal sequence,
(ii) $\phi(t) = \sum_{k=-\infty}^{\infty} c_k \phi(2t-k)$ for some $c_k \in l^2$,
(iii) the closed linear span of $\{\phi(2^m t - n)\}_{n,m \in Z}$ is $L^2(\mathbb{R})$.

Most wavelet systems lead to similar sampling series[8]. However in such cases a distinction must be made between the sampling function $S(t)$ and the scaling function $\phi(t)$. In the Shannon case above, they are the same. Each wavelet system has an associate "multiresolution analysis" consisting of a nested sequence $\{V_m\}$ of subspaces of $L^2(\mathbb{R})$ where the space V_m is the closed linear span of $\{\phi(2^m t - n)\}_{n \in Z}$. A continuous function in $L^2(\mathbb{R})$ may be approximated by its projection onto V_m or by its sampling (i.e. interpolating) series in V_m. These are not the same even for the Shannon system. The former may exhibit the Gibbs phenomenon while the other may not. We shall be concerned only with the latter, whose properties are not, however, so well known.

Under the assumption that $\phi(t)$ is a continuous orthonormal scaling function such that

$$\text{(2.1)} \qquad \begin{aligned} &(i) \quad \phi(t) = O(|t|^{-\beta}) \quad \text{as} \quad t \to \pm\infty, \beta > 5/2, \\ &(ii) \quad \hat{\phi}^*(\omega) = \sum_{n=-\infty}^{\infty} \phi(n) e^{-i\omega n} \neq 0, \omega \in \mathbb{R}, \end{aligned}$$

it was shown in [8] there is a sampling function $S(t) \in V_0$ such that for each $f \in V_0$

$$f(t) = \sum_{n=-\infty}^{\infty} f(n)S(t-n), \quad t \in \mathbb{R}.$$

These sampling series can be used to obtain an approximation in V_m for each continuous $f \in L^2(\mathbb{R})$. It is given by

$$f_m(t) = \sum_{n=-\infty}^{\infty} f(2^{-m}n)S(2^m t - n). \tag{2.2}$$

It was shown in [6] that $f_m(t) \to f(t)$ uniformly for $f \in H^\alpha$(the Sobolev space) for $\alpha > \frac{1}{2}$. This required an additional hypothese on ϕ, that it belongs to Z_λ, which can be shown to be true for ϕ which satisfies (i) and (ii) above [8]. Here a function belongs to Z_λ if

(i) $\hat{f}(\omega) = 1 + o(|\omega|^\lambda)$ as $\omega \to 0$
(ii) $(Zf)(t,\omega) := \sum_{k=-\infty}^{\infty} e^{-i\omega k} f(t-k) = e^{-i\omega t}(1 + o(|\omega|^\lambda))$ uniformly as $\omega \to 0$.

In order to study the Gibbs phenomenon, we require that f is piecewise continuous and in $L^2(\mathbb{R})$. We shall also suppose that a jump discontinuity is at a dyadic rational number, so that by traslation we can take it to zero. The spaces V_m are not translation invariant for irrational translations in general. We shall also assume the jump is in the positive direction, i.e. that $f(0^+) > f(0^-)$. If there is a sequence $t_m \downarrow 0$ such that

$$f_m(t_m) \to \gamma^+ > f(0^+) \tag{2.3}$$

then the sampling series exhibits the Gibs phenomenon on the right hand side of 0 for the function f (and similarly on the left hand side). We shall simply say "Gibbs right" and "Gibbs left" for these two cases if they hold for any function with such a jump at 0. We shall see later that these are independent of the particular function.

There is possible source of ambiguity in our series(2.2) at points of discontinuity. By changing the value of $f(0)$, we could change Gibbs right to Gibbs left and vice versa. This was avoided in [6] by assuming that $f(t) = f(t^+)$ for all $t \in \mathbb{R}$. However this assumption is unnecessarily restrictive. By eliminating it, we can sometimes avoid Gibbs left and right simultaneously. We shall however always suppose that

$$f(0^-) \le f(0) \le f(0^+)$$

to avoid pathological behavior.

The sampling function $S(t)$ is related to the scaling function ϕ by

$$\hat{S}(\omega) = \frac{\hat{\phi}(\omega)}{\sum_{k=-\infty}^{\infty} \hat{\phi}(\omega + 2\pi k)}, \quad \omega \in \mathbb{R} \tag{2.4}$$

where $\hat{f}$ denotes the Fourier transform of f. The denominator in (2.4) is assumed not to vanish for all ω. All such $S(t)$ have the properties [9]

$$(i)\ \int_{-\infty}^{\infty} S(t)dt = 1$$

$$(ii)\ \sum_{k=-\infty}^{\infty} S(t-k) = 1$$

$$(iii)\ \sum_{k=-\infty}^{\infty} \hat{S}(\omega + 2\pi k) = 1$$

$$(iv)\ \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{S}(\omega)d\omega = 1$$

$$(v)\ S(t) = \mathcal{O}(|t|^{-2}).$$

The last property is obtained from the fact that the second derivative of

$$\sum_{k=-\infty}^{\infty} \hat{\phi}(\omega + 2\pi k) = \sum_{n} \phi(n) e^{-i\omega n}$$

is in $L^2(0,2\pi) \cap C(0,2\pi)$ and so is its reciprocal. Hence

$$S(t) = \sum_{n=-\infty}^{\infty} a_n \phi(t-n)$$

has coefficients such that $\{n^2 a_n\} \in l^2$ (see [8]).

3. The Gibbs phenomenon for the Shannon sampling series

The Shannon system, although it serves as a prototype, does not satisfy the hypotheses of the theorems about the Gibbs phenomenon in [6, 7]. The Shannon system is rather simple and may be used to show directly that the Gibbs occurs for both sampling series and orthogonal series. In this particular case the sampling function $S(t)$ is given by $S(t) = \frac{\sin \pi t}{\pi t} = \phi(t)$,the orthogonal scaling function. However, the sampling approximation to a continuous function is not the same as the orthogonal projection since the coefficients need not be the same. Nonetheless both cases can lead to the Gibbs phenomenon for functions with jump discontinuities at 0 and the overshoot can be calculated. Indeed in [7] it was shown that the overshoot is exactly the same as for the Fourier series in the case of orthogonal approximations. We can also calculate it for sampling series. We shall use the function h given by

$$h(t) = \begin{cases} \text{sgn}t - t, & 0 < |t| \leq 1 \\ \alpha, & t = 0 \\ 0, & 1 < |t| \end{cases}$$

to investigate the Gibbs at $t = 0$. Its sampling approximation is given by

$$h_m^s(t) = \alpha S(2^m t) + \sum_{n=1}^{2^m-1} (1 - n2^{-m})[S(2^m t - n) - S(2^m t + n)]. \tag{3.1}$$

If $h_m^s(t_m) \to \gamma^+ > h(0^+)$ (or $h_m^s(-t_m) \to \gamma^- < h(0^-)$) where $t_m \downarrow 0$, we have Gibbs right (or Gibbs left).

LEMMA 3.1. *The Shannon sampling approximation* (3.1) *exihibits the Gibbs on right and left for all* $\alpha \in \mathbb{R}$.

PROOF. By taking $t_m = 2^{-m-1}$, we find that

(3.2)

$$\begin{aligned}
&h^s_m(2^{-m-1})\\
&=\alpha S(\frac{1}{2}) + \sum_{n=1}^{2^m-1}(1-n2^{-m})[S(\frac{1}{2}-n) - S(\frac{1}{2}+n)]\\
&=\alpha\frac{2}{\pi} + \sum_{n=1}^{2^m-1}(1-n2^{-m})\frac{(-1)^n}{\pi}\left(\frac{1}{\frac{1}{2}-n} - \frac{1}{\frac{1}{2}+n}\right)\\
&=\frac{2\alpha}{\pi} + \frac{2}{\pi}\sum_{n=1}^{2^m-1}(-1)^n\left(\frac{1}{1-2n} - \frac{1}{1+2n}\right) + \frac{2}{2^m\pi}\sum_{n=1}^{2^m-1}\frac{(-1)^n n^2}{n^2-\frac{1}{4}}\\
&=\frac{2\alpha}{\pi} + \frac{2}{\pi}\left[\frac{-1}{1-2} - \frac{(1-)^{2^m-1}}{1+2^{m+1}-2}\right] + \frac{2}{2^m\pi}\left[-1+\frac{1}{2}\sum_{n=1}^{2^m-1}\frac{1}{n^2-\frac{1}{4}}\right]\\
&\to\frac{2}{\pi}(\alpha+1) \quad \text{as} \quad m\to\infty.
\end{aligned}$$

Similarly we have

$$h^s_m(-2^{-m-1}) \longrightarrow \frac{2}{\pi}(\alpha-1) \quad \text{as} \quad m \longrightarrow \infty.$$

Thus we have Gibbs right whenever $\alpha > \frac{\pi}{2} - 1$; and in particular for h continuous on the right ($\alpha = 1$). In this case the overshoot is $\frac{4}{\pi} - 1$, which is greater than that for the orthogonal approximation.

This does not however imply that the Gibbs phenomenon fails to exist on the right for $\alpha \leq \frac{\pi}{2} - 1$. In order to show that it does, we consider other sequences of the form $t_m = a2^{-m}$ for some $a > 0$. Then by calculations similar to (3.1) we find that

$$\begin{aligned}
h^s_m(a2^{-m}) &\longrightarrow \alpha S(a) + \sum_{n=1}^{\infty}[S(a-n) - S(a+n)]\\
&= \alpha S(a) - S(a) + 1 - 2\sum_{n=1}^{\infty} S(a+n)
\end{aligned} \tag{3.3}$$

Thus we have Gibbs right if the last expression in (3.2) is greater than 1, i.e., if

$$(\alpha - 1)S(a) > 2\sum_{n=1}^{\infty} S(a+n).$$

For intervals in which $S(a)$ is positive we find that

$$\alpha > 1 + 2\sum_{n=1}^{\infty} \frac{S(a+n)}{S(a)} = \xi(a) \tag{3.4}$$

is sufficient for Gibbs right, with the oppisite inequality giving it for negative $S(a)$. If $S(a) = 0$, then a is a positive integer, and (3.3) is equal to 1, so that Gibbs right does not occur. The right hand side of (3.4) may be expressed, for $S(a) \neq 0$, as

$$\begin{aligned}
\xi(a) &= 1 + 2\sum_{n=1}^{\infty} \frac{\sin \pi a(-1)^n}{\pi(a+n)} \cdot \frac{\pi a}{\sin \pi a} \\
&= 1 + 2a\sum_{n=1}^{\infty} \frac{(-1)^n}{a+n} = 1 + 2a\sum_{n=0}^{\infty} \frac{(-1)^n}{a+n} - \frac{2a}{a} \\
&= -1 + 2a\sum_{n=0}^{\infty} \frac{(-1)^n}{a+n} = -1 + 2a\beta(a)
\end{aligned} \tag{3.5}$$

where $\beta(a) = \sum_{n=0}^{\infty} \frac{(-1)^n}{a+n} = \int_0^1 \frac{t^{\alpha-1}}{1+t}dt = \int_0^{\infty} \frac{e^{-a\omega+\omega}}{1+e}d\omega$ for $a > 0$. We use this to find (3.4)

(3.6)

$$\begin{aligned}
\xi(a) &= -1 + 2a\int_0^{\infty} \frac{e^{-a\omega}}{1+e^{-\omega}}d\omega \\
&= -1 + 2a\frac{e^{-a\omega}}{-a}\frac{1}{1+e^{-\omega}}\Big|_0^{\infty} - 2a\int_0^{\infty} \frac{e^{-a\omega}}{-a}\frac{e^{-\omega}}{(1+e^{-\omega})^2}(-d\omega) \\
&= -1 + \frac{2a}{2a} + \frac{2a}{a}\int_0^{\infty} \frac{e^{-a\omega}}{(e^{-\omega/2}+e^{\omega/2})^2}d\omega \\
&= 2\int_0^{\infty} \frac{e^{-a\omega}}{4\cosh \omega/2}d\omega > 0.
\end{aligned}$$

From this expression, it is also clear that $\xi(a)$ converges to zero monotonically as $a \longrightarrow \infty$. From (3.4) and (3.5) we see that $\xi(0) = 1$ and $\xi(1) = 2\log 2 - 1$.

Now $S(a)$ is positive when $a \in (2n, 2n+1), n = 0, 1, \cdots$, and for each $\alpha > 0$ we can find an a such that $S(a) > 0$ and $\alpha > \xi(a)$. Similarly for each $\alpha < 2\log 2 - 1$, we can find an a such that $S(a) < 0$ and $\alpha < \xi(a)$. Hence for all values of α, Gibbs right exists and by a symmetric argument so does Gibbs left. □

We can use these results to obtain similar ones for other functions with a jump discontinuity at 0. Indeed let $f \in C^1[(-\infty, 0) \cup (0, \infty)]$ and suppose both f and f' can be extended to $L^2(\mathbb{R})$ by assigning some value at zero. Then g given by

$$
\begin{aligned}
&g(t) = f(t) - f(0^+)h(t) - th(t)[f'(0^+) - f(0^+)], \quad t > 0\\
&g(t) = f(t) + f(0^-)h(t) + th(t)[f'(0^-) - f(0^-)], \quad t < 0\\
&g(0) = 0,
\end{aligned}
$$

is continuous on all of $\mathbb{R}$ and $g \in L^2(\mathbb{R})$ while g' is continuous near zero and $g' \in L^2(\mathbb{R})$. Thus $g \in H^1(\mathbb{R})$, the Sobolev space.

LEMMA 3.2. *Let $g \in H^1(\mathbb{R})$, then the Shannon sampling expansion of g,*

$$g_m(t) = \sum_{n=-\infty}^{\infty} g(n2^{-m})S(2^m t - n), \quad m \in \mathbb{Z}, \quad t \in \mathbb{R},$$

converges uniformly to $g(t)$ on $\mathbb{R}$ as $m \longrightarrow \infty$.

PROOF. The error is given by

$$
\begin{aligned}
g_m(t) - g(t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} [\hat{g}_m(\omega) - \hat{g}(\omega)]e^{i\omega t}d\omega\\
=& \frac{1}{2\pi}\int_{-2^m\pi}^{2^m\pi} [\hat{g}_m(\omega) - \hat{g}^*(\omega) + \hat{g}^*(\omega) - g(\omega)]e^{i\omega t}dt\\
&- \frac{1}{2\pi}\left\{\int_{-\infty}^{-2^m\pi} + \int_{2^m\pi}^{\infty}\right\}\hat{g}(\omega)e^{i\omega t}d\omega
\end{aligned}
$$

where $\hat{g}^*(\omega) = \sum_{k=-\infty}^{\infty} \hat{g}(\omega + 2^m 2\pi k)$ is the periodic extention of $\hat{g}(\omega)$. Note that $\hat{g}_m(\omega)$ has support in $[-2^m\pi, 2^m\pi]$ and

$$\hat{g}_m(\omega) = \sum_{n=-\infty}^{\infty} g(2^{-m}n)e^{-i\omega 2^{-m}n}\hat{S}(\omega 2^{-m})2^{-m}$$

$$\hat{g}^*(\omega)\hat{S}(\omega 2^{-m}) = \hat{g}^*(\omega), |\omega| < 2^m\pi,$$

by the Poisson summation formula. Hence we have

(3.7)

$$\begin{aligned}
&|g_m(t) - g(t)| \\
&\le \frac{1}{2\pi}\int_{-2^m\pi}^{2^m\pi} |\hat{g}^*(\omega) - \hat{g}(\omega)|d\omega + \frac{1}{2\pi}\left\{\int_{-\infty}^{-2^m\pi} + \int_{2^m\pi}^{\infty}\right\} |\hat{g}(\omega)|d\omega \\
&= \frac{1}{2\pi}\int_{-2^m\pi}^{2^m\pi} |\sum_{k\neq 0} \hat{g}(\omega + 2^m 2\pi k)|d\omega + \frac{1}{2\pi}\left\{\int_{-\infty}^{-2^m\pi} + \int_{2^m\pi}^{\infty}\right\} |\hat{g}(\omega)|d\omega \\
&= \frac{1}{\pi}\left\{\int_{-\infty}^{-2^m\pi} + \int_{2^m\pi}^{\infty}\right\} |\hat{g}(\omega)|d\omega,
\end{aligned}$$

and since $\hat{g} \in L^1(\mathbb{R})$

$$\int |\hat{g}(\omega)|d\omega \le \left\{\int |\hat{g}(\omega)|^2(\omega^2+1)d\omega \quad \int (\omega^2+1)^{-1}d\omega\right\}^{\frac{1}{2}}$$

the last expression in (3.7) $\to 0$ as $m \to \infty$. □

By combining previous two lemmas we can conclude the following theorem.

THEOREM 3.3. *Let f be as above; then the Shannon expansion of f exhibits the Gibbs phenomenon on both the right and the left.*

4. How to avoid the Gibbs phenomenon

The sampling function $S(t)$ which exactly recovers $f \in V_0$ from its sampling expansion

$$f(t) = \sum_{n=-\infty}^{\infty} f(n)S(t-n), \quad f \in V_0$$

is unique for a given multiresolution analysis $\{V_m\}$. If, however, we are interested in finding a sampling series

$$f_0(t) = \sum_{n=-\infty}^{\infty} f(n)u(t-n)$$

which associates with each $f \in L^2(\mathbb{R}) \cap C(\mathbb{R})$ an element $f_0 \in V_0$, then we have more latitude. We still need to check that the dilations

$$f_m(t) = \sum_{n=-\infty}^{\infty} f(2^{-m}n)u(2^m t - n) \tag{4.1}$$

converg to $f(t)$ as $m \to \infty$. If we can find a $u \in V_0$ such that

$$\begin{aligned} &(i) \quad u(t) \geq 0, && t \in \mathbb{R}, \\ &(ii) \quad \sum_{n=-\infty}^{\infty} u(t-n) = 1, && t \in \mathbb{R}, \\ &(iii) \quad u(t) = o(|t|^{-1-\alpha}) \quad as && t \to \infty, \alpha > 0 \end{aligned} \tag{4.2}$$

then we have the desired result

THEOREM 4.1. *Let $u(t) \in V_0$ satisfy (4.2) and let f be a piecewise continuous bounded function in $L^2(\mathbb{R})$. Then f_m given by (4.1) satisfies*

$$f_m(t) \to f(t) \qquad as \qquad m \to \infty$$

at each point of continuity of f and does not exhibit the Gibbs phenomenon.

PROOF. Let t be a point of continiuty; then we have
(4.3)

$$\begin{aligned}
|f_m(t) - f(t)| &= |\sum_{n=-\infty}^{\infty} f(2^{-m}n)u(2^m t - n) - f(t) \sum_{n=-\infty}^{\infty} u(2^m t - n)| \\
&\leq \sum_{|t-2^{-m}n|<\delta} |f(2^{-m}n) - f(t)|u(2^m t - n) \\
&+ \sum_{|t-2^{-m}n|\geq\delta} |f(2^{-m}n) - f(t)|u(2^m t - n) \\
&\leq \epsilon \sum_{|t-2^{-m}n|<\delta} u(2^m t - n) \\
&+ 2||f||_\infty \sum_{|2^m t-n|\geq 2^m\delta} u(2^m t - n) \\
&\leq \epsilon + 2||f||_\infty \sum_{|2^m t-n|\geq 2^m\delta} C\left|\frac{1}{2^m t - n}\right|^{1+\alpha} \\
&= \epsilon + 0(1) \quad as \quad m \to \infty,
\end{aligned}$$

where $\epsilon > 0$ is arbitrary and δ is such that $|f(t) - f(s)| < \epsilon$ whenever $|t - s| < \delta$. Thus $f_m(t) \to f(t)$ as $m \to \infty$. To show that the Gibbs phenomenon does not hold, it suffices to show that

$$\sum_{n=0}^{\infty} u(t - n) \leq 1, \qquad t > 0$$

and

$$\sum_{n=1}^{\infty} u(t + n) \geq 0, \qquad t < 0.$$

But both of these inequalities follow from the fact that $\sum_{n=-\infty}^{\infty} u(t-n) = 1$ and $u(t) \geq 0$. □

Now all we need to do is to find such a function for the Shannon wavelet subspaces.

EXAMPLE. : we take

$$u(x) = \frac{1}{2}\phi^2(\frac{x}{2}).$$

Then $u \in V_0$ since $\hat{u}(\omega)$ has support $[-\pi, \pi]$. Furthermore $\hat{u}(\omega)$ satisfies $\hat{u}(0) = 1$ while $\hat{u}(2\pi k) = 0, \quad k \neq 0$. Thus the periodic function is given by its Fourier series

$$\sum_{k=-\infty}^{\infty} u(x-k) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n x}.$$

But the coefficients are

$$\begin{aligned} a_n &= \int_0^1 \sum_{k=-\infty}^{\infty} u(x-k) e^{-i2\pi n x} dx \\ &= \int_{-\infty}^{\infty} u(x) e^{-i2\pi n x} dx \\ &= \hat{u}(2\pi n) = \delta_{0n} \end{aligned}$$

and hence $\sum_{k=-\infty}^{\infty} u(x-k) = 1$.

ACKNOWLEDGMENT. The author thanks G. G. Walter for support in writing this work.

References

[1] H. S. Carslaw, *A historical note on the Gibbs' phenomenon in Fourier series ans integrals*, Bull. Amer. Math. Soc. **31** (1925), 420-424.

[2] J. W. Gibbs, *letter to editor*, Nature(London) **59** (1899), 606.

[3] G. Helmberg, *The Gibbs phenomenon for Fourier interpolation*, J. Approx. Th. **78** (1994), 41-63.

[4] S. Kelly, *Gibbs phenomenon for wavelets*, App. Comp. Harmon. Anal **3** (1996), 72.

[5] F. B. Richard, *A Gibbs phenomenon for spline functions*, J. Approx. Th. **66** (1991), 334-351.

[6] H. T. Shim and H. O. Kim, *On gibbs' phenomenon for sampling series in wavelet subspaces*, Applicable analysis. **61** (1996), 97-109.

[7] H. T. Shim and H. Volkmer, *On the Gibbs phenomenon for wavelet expansions*, J. Approx. Th. **84** (1995), 75-95.

[8] G. G. Walter, *Wavelets and Other Orthogonal Systems with Applications*, CRC Press, Boca Raton, FL, 1994.
[9] ———, *Sampling theorems as part of wavelet theory*, Proc. Conf. Info. Sci. Sys., Johns Hopkins (1991), 907-912.

Department of Mathematics
Sun Moon University
Asan, Choongnam 337-840, Korea