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ON THE DECOMPOSITION OF CS-MODULE AND ITS APPLICATIONS

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ABSTRACT. In this short note, we try to apply the decomposition theorem of CS-module to reprove the direct sum of CS-modules be CS under certain condition.

Throughout this paper R will denote a ring with identity and all R-modules will be unital right R-modules. For two R-modules X and Y, we use $X \subseteq_e Y$ and $X \subset_\oplus Y$ to mean that X is an essential submodule of Y and X is isomorphic to a direct summand of Y.

An R-module M is called a CS-module (or an extending module) if it satisfies

(C_1) : For any submodule X of M, there exists a direct summand X^* of M such that $X \subseteq_e X^*$

M is called continuous if it satisfies (C_1) and

(C_2): Every submodule of M which is isomorphic to a direct summand of M is a direct summand.

M is called a quasi-continuous module if it satisfies (C_1) and

(C_3) : If X and Y are direct summands of M with $X \cap Y = 0$, then $X \oplus Y$ is a direct summand.

The following lemma was introduced to the author by Professor Oshiro in 1994. The usefulness is approved in various papers (cf. [4], [5] and [6]).

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In this short note, we try to reprove some well-known results, i.e., when the direct sum of two CS-modules (quasi-continuous, continuous modules resp.) becomes CS-module (quasi-continuous, continuous module resp.)

LEMMA 1. (cf. [4], Lemma 2.1) *Let P be an R-module with a decomposition $P = M_1 \oplus M_2$ such that each M_i is CS-module. For any submodule X of P, then there are submodules $T(i) \subseteq_e T(i)^* \subset_\oplus M_i$, decompositions $M_i = T(i)^* \oplus N_i$ for $i = 1, 2$ and a submodule $X(1) \oplus X(2) \subseteq_e X$ for which the following properties hold:*
1) $X(1) = T(1) \subseteq_e T(1)^*$
2) $X(2) \subseteq T(2) \oplus N_1$
3) $\sigma(X(i)) = T(i) \subseteq_e T(i)^*$ *for* $i = 1, 2$
$X(i) \simeq \sigma(X(i))$ *(by* $\sigma|X(i)$*)*
where σ is the projection

$$\sigma : T(1)^* \oplus T(2)^* \oplus N_1 \oplus N_2 \to T(1)^* \oplus T(2)^*$$

4) $X \simeq \sigma(X)$ *(by* $\sigma|X$*)*

We use the following well-known fact(cf. [5]) in the proof of propositions. We list this as lemma.

LEMMA 2. *For any module $P = M_1 \oplus M_2$ and for $X \subseteq P$, let $\sigma_X : P \to M_1$ be the projection map with $X \cong \sigma(X)$ such that $\beta_1 : \sigma(X) \oplus K$ for some K, then $M_1 = X \oplus K$ and $P = X \oplus K \oplus M_2$.*

PROOF. The proof can be found in [8]. □

PROPOSITION 3. *Let $P = M_1 \oplus M_2$ with each M_i is CS-module and if M_i is M_j-injective for $i \neq j$. Then P is a CS-module.*

PROOF. For any submodule X of P, by the Lemma 1, we have the following

$$X(1) \oplus X(2) \subseteq_e X$$

$$P = T(1)^* \oplus T(2)^* \oplus N_1 \oplus N_2.$$

Consider two projection homomorphism σ and τ

$$\sigma : P \to T(1)^* \oplus T(2)^* \text{ and } \tau : P \to N_1 \oplus N_2$$

Define a map $\varphi : \sigma(X) \to \tau(X)$ by $\sigma(x) \mapsto \tau(x)$ for all $x \in X$. We can check easily that φ is well-defined homomorphism.

From the hypothesis we know that $N_1 \oplus N_2$ is $T(1)^* \oplus T(2)^*$-injective, thus we have an extension $\varphi^* : T(1)^* \oplus T(2)^* \to N_1 \oplus N_2$ such that $\varphi^*|_{\sigma(X)} = \varphi$. Put $X^* = \{\alpha + \varphi^*(\alpha) \,|\, \alpha \in T(1)^* \oplus T(2)^*\}$, then $X^* \oplus N_1 \oplus N_2 = P$. And we know that

$$\begin{aligned}
&\{\alpha + \varphi(\alpha) \,|\, \alpha \in \sigma(X)\} \subseteq_e X^* \\
&= \{\sigma(\alpha) + \tau(\alpha) \,|\, \alpha \in (X)\} \\
&= X.
\end{aligned}$$

Thus $X \subseteq_e X^* \subset_\oplus P$. So we can say that P is a CS-module. □

REMARK. Recently Professor Oshiro try to construct concrete form of X^* for the given $X \subset P = M_1 \oplus M_2$ situation. He presented some result in case each $M_i = Z$ and for arbitrary $X \subseteq Z \oplus Z$ (cf. [3]).

LEMMA 4. *Let P be a CS-module with a decomposition $P = M_1 \oplus M_2$, if $X \subset_\oplus P$ then we have a decomposition $P = X \oplus N_1 \oplus N_2$ where N_1 and N_2 appeared in* Lemma 1. *(i.e. we can see what is the direct complement of X in P.)*

PROOF. By Lemma 1, we have $P = T(1)^* \oplus T(2)^* \oplus N_1 \oplus N_2$ and $X(1) \oplus X(2) \subseteq_e X$. Since X is a direct summand of a CS-module P, X is CS-module. For each $X(i) \subseteq X$, by the definition of CS-module, there exist direct summands $X(i)^\sharp$ of X such that

$$X(i) \subseteq_e X(i)^\sharp.$$

Thus we have $X(1) \oplus X(2) \subseteq_e X(i)^\sharp \oplus X(2)^\sharp \subseteq_e X$.
For the projection $\varphi : P \to T(1)^* \oplus T(2)^*$, let $\varphi_i : P \to T(i)^*$ for $i = 1, 2$. Let $K \subseteq T(1)^* \oplus T(2)^*$ such that

$$\varphi_i : X(i)^\sharp \cong K(i) \subseteq T(i)^* \text{ and } T(i) \subseteq_e K, K(i) \cong T(i)^*.$$

Thus we have $X(i)^\sharp \cong T(i)^*$. So we can write

$$\begin{aligned}
X \cong \varphi(X) &\subseteq_e T(1)^* \oplus T(2)^* \\
&\subseteq_e X(1)^\sharp \oplus X(2)^\sharp \\
&\subseteq_e X
\end{aligned}$$

Thus $\varphi(X) = T(1)^* \oplus T(2)^*$ so $X \cong T(1)^* \oplus T(2)^*$.

$$\begin{aligned} P &= T(1)^* \oplus T(2)^* \oplus N_1 \oplus N_2 \quad \text{(by Lemma 1)} \\ &= X \oplus N_1 \oplus N_2 \quad \text{(by Lemma 2)} \end{aligned}$$

□

PROPOSITION 5. *Let $P = M_1 \oplus M_2$, each M_i is quasi-continuous and M_i is M_i-injective for $i \neq j$. Then P is a quasi-continuous module.*

PROOF. Let $A, B \subset_{\oplus} P$ and $A \cap B = 0$. Since $A \subset_{\oplus} P$, we can write $P = A \oplus Q$. Also by the Lemma 4, we can rewrite $P = A \oplus N_1 \oplus N_2$. Consider two projections $\sigma_A : P \to A$ and $\sigma_Q : P \to Q$. We can check that $\sigma_Q(B) \cong B$ (since $A \cap B = 0$). Thus $A \oplus B \cong A \oplus \sigma_Q(B)$. $\sigma_Q(B)$ is a submodule of Q, which is a CS-module. There exists a direct summand $[\sigma_Q(B)]^*$ such that

$$\sigma_Q(B) \subseteq_e [\sigma_Q(B)]^* \oplus Q.$$

Consider a homomorphism $\varphi : \sigma_Q(B) \to \sigma_A(B)$ which is well-defined by $\sigma_Q(b) \mapsto \sigma_A(b)$. Since $[\sigma_Q(B)]^*$ is A-injective. We have an extension $\varphi^* : [\sigma_Q(B)]^* \to A$.
Note that

$$A \oplus [\sigma_Q(B)]^* \subset_{\oplus} P \text{ and } [\sigma_Q(B)]^* \subset_{\oplus} Q).$$

On the other hand,

$$\begin{aligned} [\sigma_Q(B)]^* &= \{\alpha + \varphi^*(\alpha) \mid \alpha \in [\sigma_Q(B)]^*\} \\ &{}_e\supseteq \{\alpha + \varphi(\alpha) \mid \alpha \in \sigma_Q(B)\} \\ &= B \end{aligned}$$

Since $B \subset_{\oplus} P$ and $B \subseteq_e [\sigma_Q(B)]^*$, we have $B = [\sigma_\theta(B)]^*$. Thus we have

$$A \oplus [\sigma_Q(B)]^* = A \oplus \sigma_Q(B) = A \oplus B.$$

Thus $A \oplus B \subset_{\oplus} P$, which means that P satisfies (C_3) condition.
Thus combining with Proposition 3, we have the result. □

Quasi-continuous module P has the following characterization: If $P \subseteq_e E(P) = \sum_I \oplus E_\alpha$, then $P = \sum_I \oplus(E_\alpha \cap P)$. (cf. [1], Theorem 2.8.)

Oshiro and Rizvi generalized the above result as follows:

LEMMA 6. ([4], Prop 1.2) *P is a quasi-continuous module if and only if $P \subseteq_e F$ and $F = \sum_I \oplus F_\alpha$ then $P = \sum_I \oplus (F_\alpha \cap P)$.*

PROPOSITION 7. *Let $P = M_1 \oplus M_2$, each M_i is continuous module and if M_i is M_j-injective for $i \neq j$. Then P is a continuous module.*

PROOF. We need to show the condition (C_2) : $X \subseteq P$ and $X \cong T \subset_\oplus P$ claim that $X \subset_\oplus P$. For any submodules $X \subseteq P$, by the Lemma 1 we have a decomposition $P = T(1)^* \oplus T(2)^* \oplus N_1 \oplus N_2$ and $X_e \supseteq X(1) \oplus X(2)$. For a map $\sigma : P \to T(1)^* \oplus T(2)^*$ $\sigma(X) \simeq X$ and $\sigma(X) \subseteq_e T(1)^* \oplus T(2)^*$ $X(1) \oplus X(2) \simeq T(1) \oplus T(2) \subseteq_e \sigma(X)$.

The summand Y of quasi-continuous module P is quasi-continuous in X. Note that $\sigma(X) \subseteq_e T(1)^* \oplus T(2)^*$ and apply lemma 5 we have $\sigma(X) = T(1)^* \cap \sigma(X) \oplus T(2)^* \oplus \sigma(X)$. Put $T(i)^* \cap \sigma(X) = T(i)^\circ$, then we have $T(i) \subseteq_e T(i)^\circ$. Noting that $X \simeq \sigma(X)$ by σ we write $\sigma^{-1}(T(i)^\circ) = X(i)^\sharp$. So we have $X = X(1)^\sharp \oplus X(2)^\sharp$. Since $X \simeq Y$, we write isomorphic image of $X(i)^\sharp$; $X(1)^\sharp \simeq \overline{Y(1)}$ and $X(2)^\sharp \simeq \overline{Y(2)}$. Note that $\overline{Y(1)} \subset_\oplus M_1$, and $X(1)^\sharp \cong T(1)^\circ \subseteq T(1)^* \subset_\oplus M_1$. We have $T(i)^\circ = T(i)^\sharp$ (using $M_i = M_j$-injective). Thus $T(1)^\circ \oplus T(2)^\circ = T(1)^\sharp \oplus T(2)^\sharp$. $\sigma(X) = T(1)^* \oplus T(2)^*$. Thus $X \oplus N_1 \oplus N_2 = P$. □

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