Comm. Korean Math. Soc. **13** (1998), No. 2, pp. 243–249

EXTENDING SEMIRING HOMOMORPHISMS TO RING HOMOMORPHISMS

SANG HAN LEE

ABSTRACT. In this paper, we introduce a congruence relation on a semiring and a quotient semiring, and prove some homomorphism theorems.

1. Introduction

Semiring was first introduced by H. S. Vandiver in 1934. In order that a semiring S to be embedded in a ring, it is necessary that S should be cancellative. We call a cancellative semiring a *halfring*. In [3], L. Dale extended halfring homomorphisms to ring homomorphisms and studied some halfring homomorphism properties. In this paper, we introduce a congruence relation on a semiring and prove some homomorphism theorems.

2. Preliminaries

There are many different definitions of a semiring appeared in the literature. Throughout this paper, a semiring will be defined as follows:

DEFINITION 2.1 ([1]). A *semiring* is a set R together with two binary operations called addition $(+)$ and multiplication $(\cdot)$ such that $(R, +)$ is an abelian semigroup with zero, $(R, \cdot)$ is a semigroup, and multiplication distributes over addition from both the left and the right. We call additively cancellative semiring a *halfring*.

Received May 30, 1997. Revised February 14, 1998.
1991 Mathematics Subject Classification: Primary 16Y60.
Key words and phrases: semiring, halfring, congruence relation.

DEFINITION 2.2 ([1]). Let R and S be semirings. A function $\phi : R \to S$ is called a *semiring homomorphism* provided:

(1) $\phi(a+b) = \phi(a) + \phi(b)$ for all $a, b \in R$,
(2) $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in R$.

An *isomorphism* is one-to-one homomorphism. The semirings R and S will be called *isomorphic* (denoted by $R \cong S$) if there exists an isomorphism from R onto S.

DEFINITION 2.3. Let R be a semiring and let ρ be a binary relation on R. Then ρ is *compatible* if $(a, b) \in \rho, (c, d) \in \rho$ imply $(a+c, b+d) \in \rho$ and $(ac, bd) \in \rho$ for all $a, b, c, d \in R$.

A compatible equivalence relation is called a *congruence relation.*

For a binary relation ρ on a semiring R, we denote

$$a\rho = \{b \in R : (a, b) \in \rho\} \text{ and } R/\rho = \{a\rho : a \in R\}.$$

The following results are well known.

PROPOSITION 2.4 ([5]). *Let R be a semiring and ρ be a congruence relation on R. Then R/ρ is a semiring under the operations $a\rho + b\rho = (a+b)\rho$ and $a\rho \cdot b\rho = (ab)\rho$. And $\rho^{\#} : R \to R/\rho$ defined by $\rho^{\#}(a) = a\rho$ is a semiring homomorphism.*

THEOREM 2.5 ([5]). *Let R and S be semirings and $\phi : R \to S$ be a semiring homomorphism. Then $Ker\phi = \{(a, b) \in R \times R : \phi(a) = \phi(b)\}$ is a congruence relation on R and there exists a unique one to one semiring homomorphism $\overline{\phi} : R/Ker\phi \to S$ such that $\overline{\phi} \circ (Ker\phi)^{\#} = \phi$.*

COROLLARY 2.6 ([5]). *Let R be a semiring and ρ, σ be congruence relations on R with $\rho \subset \sigma$. Then $\sigma/\rho = \{(a\rho, b\rho) \in R/\rho \times R/\rho : (a, b) \in \sigma\}$ is a congruence relation on R/ρ and $(R/\rho)/(\sigma/\rho) \cong R/\sigma$.*

3. Quotient halfrings and homomorphisms

Let R be a semiring and ρ_R be a binary relation on R defined by

$$(*) \qquad (a, b) \in \rho_R \iff \exists\, u \in R \text{ such that } a + u = b + u.$$

Clearly, ρ_R is reflexive and symmetric. Let $(a,b),(b,c) \in \rho_R$. Then there exist $u,v \in R$ such that $a+u = b+u$ and $b+v = c+v$. These imply $a+(b+u+v) = (a+u)+(b+v) = (b+u)+(c+v) = c+(b+u+v)$, whence ρ_R is transitive. Thus ρ_R is an equivalence relation on R.

PROPOSITION 3.1. *Let R be a semiring and ρ_R be a binary relation on R defined by (*). Then ρ_R is a congruence relation on R, and R/ρ_R is a halfring.*

PROOF. Let $(a,b),(c,d) \in \rho_R$. Then there exist $u,v \in R$ such that $a+u = b+u$ and $c+v = d+v$. These imply $(a+c)+(u+v) = (a+u)+(c+v) = (b+u)+(d+v) = (b+d)+(u+v)$ and $ac+(uc+bv) = (ac+uc)+bv = (a+u)c+bv = (b+u)c+bv = (bc+uc)+bv = b(c+v)+uc = b(d+v)+uc = bd+(bv+uc)$, whence $(a+c,b+d) \in \rho_R$ and $(ac,bd) \in \rho_R$. Thus ρ_R is a congruence relation on R. Let $a\rho_R, b\rho_R, c\rho_R \in R/\rho_R$ such that $a\rho_R + c\rho_R = b\rho_R + c\rho_R$. Then $(a+c)\rho_R = (b+c)\rho_R$ in R/ρ_R, whence $(a+c,b+c) \in \rho_R$. By definition of ρ_R, there exists $u \in R$ such that $(a+c)+u = (b+c)+u$. Thus $a+(c+u) = b+(c+u)$, so $(a,b) \in \rho_R$ and $a\rho_R = b\rho_R$. This shows that R/ρ_R is a halfring $\square$

Let R be a semiring with halfring R/ρ_R. Then $(\rho_R)^\# : R \to R/\rho_R$ defined by $(\rho_R)^\#(a) = a\rho_R$ is a surjective semiring homomorphism.

PROPOSITION 3.2. *Let R and S be semirings with their halfrings R/ρ_R and S/ρ_S respectively, and $\phi : R \to S$ be a semiring homomorphism. Then there exists a unique homomorphism $\phi/\rho : R/\rho_R \to S/\rho_S$ such that $\phi/\rho \circ (\rho_R)^\# = (\rho_S)^\# \circ \phi$.*

PROOF. Define $\phi/\rho : R/\rho_R \to S/\rho_S$ by $\phi/\rho\,(a\rho_R) = \phi(a)\rho_S$. If $a\rho_R = b\rho_R$, then there exists $u \in R$ such that $a+u = b+u$. Thus $\phi(a)+\phi(u) = \phi(b)+\phi(u)$ and $(\phi(a),\phi(b)) \in \rho_S$, so $\phi(a)\rho_S = \phi(b)\rho_S$. Therefore ϕ/ρ is well-defined. And ϕ/ρ is a halfring homomorphism. In fact, $\phi/\rho\,(a\rho_R + b\rho_R) = \phi/\rho\,((a+b)\rho_R) = \phi(a+b)\rho_S = (\phi(a)+\phi(b))\rho_S = \phi(a)\rho_S + \phi(b)\rho_S = \phi/\rho\,(a\rho_R) + \phi/\rho\,(b\rho_R)$ and $\phi/\rho\,(a\rho_R \cdot b\rho_R) = \phi/\rho\,((ab)\rho_R) = \phi(ab)\rho_S = (\phi(a)\cdot\phi(b))\rho_S = \phi(a)\rho_S \cdot \phi(b)\rho_S = \phi/\rho\,(a\rho_R)\cdot\phi/\rho\,(b\rho_R)$. For any $a \in R$, we have $(\phi/\rho \circ (\rho_R)^\#)(a) = \phi/\rho\,((\rho_R)^\#(a)) = \phi/\rho\,(a\rho_R) = \phi(a)\rho_S = (\rho_S)^\#(\phi(a)) = ((\rho_S)^\# \circ \phi)(a)$. Thus $\phi/\rho \circ (\rho_R)^\# = (\rho_S)^\# \circ \phi$. Finally, if there exists a halfring homomorphism $g : R/\rho_R \to S/\rho_S$ such that $g\circ(\rho_R)^\# = (\rho_S)^\#\circ\phi$, then $g(a\rho_R) = g((\rho_R)^\#(a)) = (g\circ(\rho_R)^\#)(a) =$

$((\rho_S)^{\#} \circ \phi)(a) = (\rho_S)^{\#}(\phi(a)) = \phi(a)\rho_S = \phi/\rho\,(a\rho_R)$. Thus $g = \phi/\rho$ and ϕ/ρ is unique. □

It is clear that $Hom(R,S)$ is a semigroup under addition defined by $(\phi_1+\phi_2)(a) = \phi_1(a)+\phi_2(a)$. Likewise $Hom(R/\rho_R, S/\rho_S)$ is a cancellative semigroup. By Proposition 3.2, we can define a mapping

$$\Phi : Hom(R,S) \to Hom(R/\rho_R, S/\rho_S)$$

by $\Phi(\phi) = \phi/\rho$. Then we have the following theorem.

THEOREM 3.3. *Let R and S be semirings with their halfrings R/ρ_R and S/ρ_S respectively. Then the above mappping Φ given by $\Phi(\phi) = \phi/\rho$ is a semigroup homomorphism.*

PROOF. Let $\phi_1, \phi_2 \in Hom(R,S)$ and $a\rho_R \in R/\rho_R$. Then $((\phi_1 + \phi_2)/\rho)(a\rho_R) = ((\phi_1 + \phi_2)(a))\rho_S = (\phi_1(a) + \phi_2(a))\rho_S = \phi_1(a)\,\rho_S + \phi_2(a)\,\rho_S = \phi_1/\rho\,(a\rho_R) + \phi_2/\rho\,(a\rho_R) = (\phi_1/\rho + \phi_2/\rho)\,(a\rho_R)$. Consequently, $(\phi_1 + \phi_2)/\rho = \phi_1/\rho + \phi_2/\rho$. Thus the map

$$\Phi : Hom(R,S) \to Hom(R/\rho_R, S/\rho_S)$$

given by $\Phi(\phi) = \phi/\rho$ is a semigroup homomorphism. □

Let R be a semiring and S be a halfring. If $\phi : R \to S$ is a semiring homomorphism, then $\rho_R \subset Ker\phi$.

By Corollary 2.6, we have the following:

COROLLARY 3.4. *Let R be a semiring with halfring R/ρ_R and S be a halfring. If $\phi : R \to S$ is a semiring homomorphism, then*

$$Ker\phi/\rho_R = \{(a\rho_R, b\rho_R) \in R/\rho_R \times R/\rho_R : (a,b) \in Ker\phi\}$$

is a congruence relation on R/ρ_R and $(R/\rho_R)/(Ker\phi/\rho_R) \cong R/Ker\phi$.

Let R be a semiring and S be a halfring. We denote

$$KH(R,S) = \cap_{\phi \in Hom(R,S)} Ker\phi.$$

Then $KH(R,S)$ is a congruence relation on R such that $\rho_R \subset KH(R,S)$.

By Corollary 2.6, we have the following:

COROLLARY 3.5. *Let R be a semiring with halfring R/ρ_R and S be a halfring. Then*

$$KH(R,S)/\rho_R = \{(a\rho_R, b\rho_R) \in R/\rho_R \times R/\rho_R : (a,b) \in KH(R,S)\}$$

is a congruence relation on R/ρ_R and

$$(R/\rho_R)/(KH(R,S)/\rho_R) \cong R/KH(R,S).$$

EXAMPLE 3.6. Let N be the set of all natural numbers, and let $e_1 = (1,0)$, $e_2 = (0,1)$. Let $A_1 = \{ne_1 : n \in N\}$ and $A_2 = \{ne_2 : n \in N\}$, and let $R = A_1 \cup A_2 \cup \{0\}$. Define $ne_i + me_j = (n+m)e_2$ if $i \neq j$, $ne_i + me_j = (n+m)e_i$ if $i = j$ and $0 + ne_i = ne_i + 0 = ne_i$. And define $ne_i \cdot me_j = (n \cdot m)e_1$ if $i \neq j$, $ne_i \cdot me_j = (n \cdot m)e_i$ if $i = j$ and $0 \cdot ne_i = ne_i \cdot 0 = 0$. Then $(R, +, \cdot)$ is a semiring. Also, $(ne_1)\rho_R = \{ne_1,\ ne_2\}$ and $R/\rho_R = \{(ne_1)\rho_R : n \in N\} \cup \{\{0\}\}$ is a halfring. We have $(\rho_R)^{\#}(ne_i) = (ne_1)\rho_R$.

We know that $N^* = N \cup \{0\}$ is a semiring with usual addition and multiplication. Moreover, N^* is a halfring, whence $N^*/\rho_{N^*} \cong N^*$. Let $\phi : R \to N^*$ be a semiring homomorphism defined by $\phi(ne_i) = n$. Then $\phi/\rho\,((ne_i)\rho_R) = n$. Clearly, $Hom(R, N^*) = \{k\phi : k \in N^*\}$ and $Hom(R/\rho_R, N^*/\rho_{N^*}) = \{k\phi/\rho : k \in N^*\}$. Also,

$$\Phi : Hom(R, N^*) \to Hom(R/\rho_R, N^*/\rho_{N^*})$$

given by $\Phi(k\phi) = k\phi/\rho$ is a homomorphism. We have $\rho_R = Ker\phi = KH(R, N^*)$.

4. Extending semiring homomorphisms to ring homomorphisms

In order that a semiring R to be embedded in a ring it is necessary and sufficient that R is a halfring. To embed a halfring R in a ring we proceed as follows: Let $R^* = \{(a,b) : a, b \in R\}$. In R^* define $(a,b) = (a',b')$ if and only if $a + b' = a' + b$. This gives an equivalence relation on R^*. Let $\overline{R}$ be the set of all equivalence classes in R^*. In R^* define $(a,b) + (a',b') = (a+a', b+b')$ and $(a,b)(a',b') = (aa'+bb', ab'+ba')$, then $\overline{R}$ is a ring with respect to these operations. The map $\psi_R : R \to \overline{R}$ given by $\psi_R(a) = (a, 0)$ is well defined and one-one, and it follows that R can be embedded in $\overline{R}$. We identify the ordered pair (a, b) with $a - b$. Then $\overline{R} = \{a - b : a, b \in R\}$ is called the *ring of difference* of R and is the smallest ring containing R (See. Dale [3]). Since $\overline{R}$ is the smallest ring containing R, we will call $\overline{R}$ the *closure* of R. Let R and S be halfrings with their closures $\overline{R}$ and $\overline{S}$ respectively, and $\phi : R \to S$ be a halfring homomorphism. Define $\overline{\phi} : \overline{R} \to \overline{S}$ by $\overline{\phi}(a - b) = \phi(a) - \phi(b)$, where $a - b \in \overline{R}$. Then the following proposition is clear.

PROPOSITION 4.1. *Let R and S be halfrings with their closures $\overline{R}$ and $\overline{S}$ respectively, and $\phi : R \to S$ be a halfring homomorphism. Then there exists a unique ring homomorhism $\overline{\phi} : \overline{R} \to \overline{S}$ such that $\overline{\phi} \circ \psi_R = \psi_S \circ \phi$.*

THEOREM 4.2 ([3]). *Let R and S be halfrings with their closures $\overline{R}$ and $\overline{S}$ respectively. Then the map*

$$\Phi : Hom(R, S) \to Hom(\overline{R}, \overline{S})$$

given by $\Phi(\phi) = \overline{\phi}$ is an isomorphism.

From Proposition 3.2 and Proposition 4.1, we have the following:

THEOREM 4.3. *Let R and S be semirings and $\phi : R \to S$ be a semiring homomorphism. Then there exists a unique ring homomorhism*

$$\overline{\phi/\rho} : \overline{R/\rho_R} \to \overline{S/\rho_S}$$

such that the following diagram is commuative:

$$\begin{array}{ccccc}
R & \xrightarrow{(\rho_R)^\#} & R/\rho_R & \xrightarrow{\psi_{R/\rho_R}} & \overline{R/\rho_R} \\
\phi\downarrow & & \phi/\rho\downarrow & & \overline{\phi/\rho}\downarrow \\
S & \xrightarrow{(\rho_S)^\#} & S/\rho_S & \xrightarrow{\psi_{S/\rho_S}} & \overline{S/\rho_S}
\end{array}$$

From Theorem 3.3 and Theorem 4.2, we have the following:

THEOREM 4.4. *Let R and S be semirings. Then the map*

$$\Phi : Hom(R, S) \to Hom(\overline{R/\rho_R}, \overline{S/\rho_S})$$

given by $\Phi(\phi) = \overline{\phi/\rho}$ *is a homomorphism.*

ACKNOWLEDGEMENT. The author is grateful to Professor Hee Sik Kim for his valuable suggestions to improve this paper.

References

[1] P. J. Allen, *A fundamental theorem of homomorphisms for semirings*, Proc. Amer. Math. Soc. **21** (1969), 412-416.
[2] L. Dale, *Standard halfrings and standard ideals*, Kyungpook Math. J. **21** (1981), 281-288.
[3] L. Dale, *Extending certain semiring homomorphisms to ring homomorphisms*, Kyungpook Math. J. **23** (1983), 13-18.
[4] Hee Sik Kim, *On quotient semiring and extension of quotient halfring*, Comm. Korean Math. Soc. **4** (1989), 17-22.
[5] W. J. Bae, H. S. Kim and Y. S. Park, *On isomorphism theorems of seminear-rings*, Youngnam Math. J. **2** (1995), 51-56.

Okchon College
Okchon, Chungbuk 373-800, Korea