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ON WEAKLY PRIME IDEALS OF ORDERED Γ-SEMIGROUPS

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ABSTRACT. We introduce the concept of weakly prime ideals in $po-\Gamma-$semigroup and give some characterizations of weakly prime ideals.

A $po-\Gamma-$semigroup([2]) is an ordered set M at the same time a $\Gamma-$semigroup such that:

$$a \leq b \Longrightarrow a\gamma x \leq b\gamma x \quad \text{and} \quad x\mu a \leq x\mu b$$

$\forall a, b, x \in M$ and $\forall \gamma, \mu \in \Gamma$.

For $A, B \subseteq M$, let $A\Gamma B := \{a\gamma b | a \in A, b \in B, \gamma \in \Gamma\}$.

In [1], N. Kehayopulu defined the ideal and weakly prime in a $po-$semigroup and M. K. Sen and N. K. Saha([3], [4] and [5]) introduced the concepts of ideals in a $\Gamma-$semigroup. We now introduce the ideals and weakly prime ideals in $po-\Gamma-$semigroup.

DEFINITION 1. Let M be a $po-\Gamma-$semigroup and A a nonempty subset of M. A is called a *right*(resp. *left*) *ideal* of M if

(1) $A\Gamma M \subseteq A$(resp. $M\Gamma A \subseteq A$).

(2) $a \in A, b \leq a$ for $b \in M \Longrightarrow b \in A$.

A is called an *ideal* of M if it is a right and left ideal of M.

DEFINITION 2. Let M be a $po-\Gamma-$semigroup and T a nonempty subset of M. T is called *weakly prime* if for all ideals A, B of M such that

$$A\Gamma B \subseteq T \Longrightarrow A \subseteq T \quad \text{or} \quad B \subseteq T.$$

T is called a *weakly prime ideal* if T is an ideal which is weakly prime.

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NOTATION([1]). For $H \subseteq M$,

$$(H] = \{a \in M : a \leq h \quad \text{for some} \quad h \in H\}.$$

We write $(a]$ instead of $(\{a\}](a \in M)$.

We can easily prove the following lemma.

LEMMA. *Let M be a $po - \Gamma -$semigroup. Then we have*
(1) *$A \subseteq (A]$ for any subset A of M.*
(2) *If $A \subseteq B \subseteq M$, then $(A] \subseteq (B]$.*
(3) *$(A]\Gamma(B] \subseteq (A\Gamma B]$ for all $A, B \subseteq M$.*
(4) *$((A]] \subseteq (A]$ for all $A \subseteq M$.*
(5) *For every left(resp. right, two-sided) ideal T of M, $(T] = T$.*
(6) *If A, B are ideals of M, then $(A\Gamma B]$ and $A \cup B$ are ideals of M.*
(7) *$(M\Gamma a\Gamma M]$ is an ideal of M for every $a \in M$.*

REMARK. If A is a left ideal of M and B is a right ideal of M, then $(A\Gamma B]$ is an ideal of M.

We denote by $I(a), R(a), L(a)$ the ideal, right ideal, left ideal of M, respectively, generated by $a(a \in M)$.
One can easily prove that:

$$I(a) = (a \cup M\Gamma a \cup a\Gamma M \cup M\Gamma a\Gamma M],$$

$$L(a) = (a \cup M\Gamma a], \quad R(a) = (a \cup a\Gamma M].$$

Recently , N. Kehayopulu showed the following Theorem.

THEOREM([1]). *Let S be a $po-$semigroup and T an ideal of S. The following are equivalent:*
(1) *T is weakly prime.*
(2) *If $a, b \in S$ such that $(aSb] \subseteq T$, then $a \in T$ or $b \in T$.*
(3) *If $a, b \in S$ such that $I(a)I(b) \subseteq T$, then $a \in T$ or $b \in T$.*
(4) *If A, B are right ideals of S such that $AB \subseteq T$, then $A \subseteq T$ or $B \subseteq T$.*
(5) *If A and B are left ideals of S such that $AB \subseteq T$, then $A \subseteq T$ or $B \subseteq T$.*
(6) *If A is a right ideal, B a left ideal of S such that $AB \subseteq T$, then $A \subseteq T$ or $B \subseteq T$.*

THEOREM 3. *Let M be a po $-\Gamma-$semigroup and T an ideal of M. The following are equivalent:*

(1) *T is weakly prime.*

(2) *If $a, b \in M$ such that $(a\Gamma M\Gamma b] \subseteq T$, then $a \in T$ or $b \in T$.*

(3) *If $a, b \in M$ such that $I(a)\Gamma I(b) \subset T$, then $a \in T$ or $b \in T$.*

(4) *If A and B are right ideals of M such that $A\Gamma B \subseteq T$, then $A \subseteq T$ or $B \subseteq T$.*

(5) *If A and B are left ideals of M such that $A\Gamma B \subseteq T$, then $A \subseteq T$ or $B \subseteq T$.*

(6) *If A is a right ideal, B a left ideal of M such that $A\Gamma B \subseteq T$, then $A \subseteq T$ or $B \subseteq T$.*

PROOF. (1) $\Longrightarrow$ (2). Let $a, b \in M, (a\Gamma M\Gamma b] \subseteq T$. Then, by (3), (2), (1) and (5) of the Lemma, we have

$$\begin{aligned}(M\Gamma a\Gamma M]\Gamma(M\Gamma b\Gamma M] &\subseteq (M\Gamma a\Gamma M\Gamma M\Gamma b\Gamma M]\\ &\subseteq (M\Gamma(a\Gamma M\Gamma b)\Gamma M]\\ &\subseteq (M\Gamma(a\Gamma M\Gamma b]\Gamma M]\\ &\subseteq (M\Gamma T\Gamma M] \subseteq (T] = T.\end{aligned}$$

Since $(M\Gamma a\Gamma M]$ and $(M\Gamma b\Gamma M]$ are ideals of M and T is weakly prime, $(M\Gamma a\Gamma M] \subseteq T$ or $(M\Gamma b\Gamma M] \subseteq T$. Let $(M\Gamma a\Gamma M] \subseteq T$. Then by (3) and (2) of the Lemma, we get

$$\begin{aligned}&I(a)\Gamma I(a)\Gamma I(a)\\ &\quad=(a \cup M\Gamma a \cup a\Gamma M \cup M\Gamma a\Gamma M]\Gamma(a \cup M\Gamma a \cup a\Gamma M \cup M\Gamma a\Gamma M]\\ &\quad\quad\Gamma(a \cup M\Gamma a \cup a\Gamma M \cup M\Gamma a\Gamma M]\\ &\quad\subseteq(M\Gamma a \cup M\Gamma a\Gamma M]\Gamma(a \cup M\Gamma a \cup a\Gamma M \cup M\Gamma a\Gamma M]\\ &\quad\subseteq((M\Gamma a \cup M\Gamma a\Gamma M)\Gamma(a \cup M\Gamma a \cup a\Gamma M \cup M\Gamma a\Gamma M)]\\ &\quad\subseteq(M\Gamma a\Gamma M] \subseteq T.\end{aligned}$$

□

And then, by (5), (3) and (2) of the Lemma,

$$(I(a)\Gamma I(a)]\Gamma I(a) = (I(a)\Gamma I(a)]\Gamma(I(a)] \subseteq (I(a)\Gamma I(a)\Gamma I(a)] \subseteq (T] = T.$$

Since T is weakly prime and $(I(a)\Gamma I(a)]$ is an ideal of M, we have

$$(I(a)\Gamma I(a)] \subseteq T \quad \text{or} \quad I(a) \subseteq T.$$

If $I(a) \subseteq T$, then $a \in I(a) \subset T$. And if $(I(a)\Gamma I(a)] \subseteq T$, then $I(a)\Gamma I(a) \subseteq (I(a)\Gamma I(a)] \subseteq T$. Since T is weakly prime, $I(a) \subseteq T$ and so $a \in T$. Similarly, from $(M\Gamma b\Gamma M] \subseteq T$, we have $b \in T$.

(2) $\Longrightarrow$ (3). Let $a, b \in M$ and $I(a)\Gamma I(b) \subseteq T$. Then, by (2) and (5) of the Lemma,

$$(a]\Gamma(M\Gamma b] \subseteq (a \cup M\Gamma a \cup a\Gamma M \cup M\Gamma a\Gamma M]\Gamma(b \cup M\Gamma b \cup b\Gamma M \cup M\Gamma b\Gamma M] \subseteq T,$$

and so

$$(a\Gamma M\Gamma b] \subseteq ((a]\Gamma(M\Gamma b]] \subseteq (T] = T.$$

By (2), we have $a \in T$ or $b \in T$.

(3) $\Longrightarrow$ (4). Let A and B be right ideals of M such that $A\Gamma B \subseteq T$ and $A \not\subseteq T$. Let $a \in A, a \notin T$ and $b \in B$. Then we have

$$\begin{aligned} I(a) &= (a \cup M\Gamma a \cup a\Gamma M \cup M\Gamma a\Gamma M] \\ &\subseteq (A \cup M\Gamma A \cup A\Gamma M \cup M\Gamma A\Gamma M] \\ &= (A \cup M\Gamma A]. \end{aligned}$$

and

$$\begin{aligned} I(b) &= (b \cup M\Gamma b \cup b\Gamma M \cup M\Gamma b\Gamma M] \\ &\subseteq (B \cup M\Gamma B \cup B\Gamma M \cup M\Gamma B\Gamma M] \\ &= (B \cup M\Gamma B]. \end{aligned}$$

Then, by (3), (2) and (5) of the Lemma,

$$\begin{aligned} I(a)\Gamma I(b) &\subseteq (A \cup M\Gamma A]\Gamma(B \cup M\Gamma B] \\ &\subseteq ((A \cup M\Gamma A)\Gamma(B \cup M\Gamma B)] \\ &= (A\Gamma B \cup M\Gamma A\Gamma B \cup A\Gamma M\Gamma B \cup M\Gamma A\Gamma M\Gamma B] \\ &= (A\Gamma B \cup M\Gamma A\Gamma B] \\ &\subseteq (T \cup M\Gamma T] = (T] = T. \end{aligned}$$

Since $a \notin T$, by (3), we have $b \in T$ and so $B \subseteq T$.

(3) $\Longrightarrow$ (5). We can prove this by the similar method to the previous case.

(3) $\Longrightarrow$ (6). Let A be a right ideal, B a left ideal and such that $A\Gamma B \subseteq T, A \nsubseteq T$. Let $a \in A, a \notin T$ and $b \in B$. Since $I(a) \subseteq (A \cup M\Gamma A]$ and $I(b) \subseteq (B \cup B\Gamma M]$, we have

$$\begin{aligned} I(a)\Gamma I(b) &\subseteq (A \cup M\Gamma A]\Gamma(B \cup B\Gamma M] \\ &\subseteq (A\Gamma B \cup M\Gamma A\Gamma B \cup A\Gamma B\Gamma M \cup M\Gamma A\Gamma B\Gamma M] \\ &\subseteq (T \cup M\Gamma T \cup T\Gamma M \cup M\Gamma T\Gamma M] \\ &= (T] = T. \end{aligned}$$

Since $a \notin T$, we have $b \in T$ by (3) and so $B \subseteq T$.

(4), (5), (6) $\Longrightarrow$ (1). They are obvious.

REMARK. In Theorem 3, condition (4) is equivalent to the condition:

(4)′ If $a, b \in M$ such that $R(a)\Gamma R(b) \subseteq T$, then $a \in T$ or $b \in T$. In fact, let A, B be right ideals, $A\Gamma B \subseteq T, a \in A, a \notin T$ and $b \in B$. Then

$$\begin{aligned} R(a)\Gamma R(b) &= (a \cup a\Gamma M]\Gamma(b \cup b\Gamma M] \\ &\subseteq (A \cup A\Gamma M]\Gamma(B \cup B\Gamma M] \\ &\subseteq (A\Gamma M \cup A\Gamma M\Gamma B \cup A\Gamma B\Gamma M \cup A\Gamma M\Gamma B\Gamma M] \\ &= (A\Gamma B] \subseteq (T] = T. \end{aligned}$$

Since $a \notin T$, we have $b \in R(b) \subseteq T$ by (4)′. Similarly, the condition (5) and (6) is equivalent respectively, to the following condition:

(5)′ If $a, b \in M$ such that $L(a)\Gamma L(b) \subseteq T$, then $a \in T$ or $b \in T$.

(6)′ If $a, b \in M$ such that $R(a)\Gamma R(b) \subseteq T$, then $a \in T$ or $b \in T$.

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