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UNIMODULAR WAVELETS AND SCALING FUNCTIONS

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ABSTRACT. We consider unimodular wavelets and scaling functions whose Fourier transforms are supported in a finite disjoint union of closed intervals. In particular, we characterize those unimodular wavelets which can be associated with multiresolution analysis. As an application we have a criterion to determine whether a wavelet from a class of unimodular wavelets of Ha et al. can be associated with multiresolution analysis or not.

1. Introduction

A wavelet for $L^2(R)$ is a function $\psi \in L^2(R)$ for which the family $\{\psi_{j,k}\}_{j,k\in Z}$, where

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k), \quad j,k \in Z,$$

forms an orthonormal basis for $L^2(R)$. It is known in [5] and proved explicitly in [3] that $\psi \in L^2(R)$ is a wavelet if and only if ψ satisfies the following "wavelet equations":

$(W1)$ $\sum_{k\in Z} |\hat{\psi}(\zeta + 2k\pi)|^2 = 1,$

$(W2)$ $\sum_{k\in Z} \hat{\psi}(\zeta + 2k\pi)\overline{\hat{\psi}(2^j(\zeta + 2k\pi))} = 0 \quad \text{for} \quad j \geq 1,$

$(W3)$ $\sum_{j\in Z} |\hat{\psi}(2^{-j}\zeta)|^2 = 1,$ and

$(W4)$ $\sum_{l\geq 0} \hat{\psi}(2^l\zeta)\overline{\hat{\psi}(2^l(\zeta + 2p_0\pi))} = 0 \quad \text{for} \quad p_0 \in 2Z + 1.$

Here and throughout the paper the equality between functions are almost everywhere sense and $\overline{\hat{\psi}}$ is the complex conjugate of the Fourier transform $\hat{\psi}$ of ψ. Since it is apparently hard to solve the equations (W1) - (W4),

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one puts some a priori assumptions on ψ so as to reduce the equations to easily solvable ones.

For example, Y. Meyer assumed that $\hat{\psi}$ has its support in $[-\frac{8\pi}{3}, -\frac{2\pi}{3}] \cup [\frac{2\pi}{3}, \frac{8\pi}{3}]$ so that the infinite sums in (W1) - (W4) reduces to two sums and solved the equations to get the Meyer's wavelet. The Meyer's wavelet is given by

$$\hat{\psi}(\zeta) = e^{-i\zeta/2}\theta_1(\zeta)$$

where θ_1 is positive, even, $C_0^\infty(R)$, supported in $[-\frac{8\pi}{3}, -\frac{2\pi}{3}] \cup [\frac{2\pi}{3}, \frac{8\pi}{3}]$, and satisfies

$$\theta_1^2(\zeta) + \theta_1^2(2\zeta) = 1 \qquad \text{if} \quad \zeta \in [\frac{2\pi}{3}, \frac{4\pi}{3}],$$
$$\theta_1^2(\zeta) + \theta_1^2(2\pi - \zeta) = 1 \qquad \text{if} \quad \zeta \in [\frac{2\pi}{3}, \frac{8\pi}{3}].$$

On the other hand, P. G. Lemarié observed that all sums in the wavelet equations are made of 2π-translations and dyadic dilations and assumed that $\hat{\psi}(\zeta) = \zeta^{-n}\Omega(\zeta)$ where n is a positive even integer and Ω is a 4π-periodic function. Then (W1) - (W4) come down to simple algebraic equations and by solving them, one gets the Battle-Lemarié's wavelet. For example, if $n = 2$, then the Battle-Lemarié's wavelet is given by

$$\hat{\psi}(\zeta) = e^{-i\zeta/2}\sin^2\frac{\zeta}{4}(\frac{\sin\zeta/4}{\zeta/4})^2(\frac{1-\frac{2}{3}\cos^2\zeta/4}{1-\frac{2}{3}\sin^2\zeta/4})^{\frac{1}{2}}(1-\frac{2}{3}\sin^2\frac{\zeta}{2})^{-\frac{1}{2}}.$$

Both Meyer's wavelet and Battle-Lemarié's wavelet can be constructed from multiresolution analysis(MRA).

An increasing sequence $\{V_j\}_{j\in Z}$ of the closed subspaces of $L^2(R)$ is called an MRA if and only if the following hold :

(M1) (density and separation) $\overline{\bigcup V_j} = L^2(R)$ and $\bigcap V_j = \{0\}$;

(M2) (scaling) $f(x) \in V_j$ iff $f(2^{-j}x) \in V_0$;

(M3) (orthonormality) there exists a function φ in V_0 such that $\{\varphi(x-k)\}_{k\in Z}$ is an orthonormal basis for V_0.

The function φ in (M3) is called a scaling function for the MRA $\{V_j\}$. It follows easily from (M2) and (M3) that the family $\{\varphi_{j,k}\}_{k\in Z}$ where

$$\varphi_{j,k}(x) = 2^{j/2}\varphi(2^j x - k)$$

forms an orthonormal basis for V_j. We can associate a wavelet ψ to an MRA with its scaling function φ as follows. Since $\varphi \in V_0 \subset V_1$, there is a unique l^2-sequence $\{p_k\}$ that describes the dilation equation,

$$\varphi(x) = \sum_{k \in Z} p_k \varphi(2x - k).$$

In terms of Fourier transform, it can be written as

$$\hat{\varphi}(\zeta) = m_0(\zeta/2)\hat{\varphi}(\zeta/2)$$

where m_0 is 2π-periodic function in $L^2([0, 2\pi])$ defined by

$$m_0(\zeta) = \frac{1}{2} \sum p_k e^{-ik\zeta}.$$

Then ψ defined by

$$\hat{\psi}(\zeta) = e^{i\zeta/2}\overline{m_0(\zeta/2 + \pi)}\hat{\varphi}(\zeta/2)$$

becomes a wavelet for $L^2(R)$. This association of wavelet with a multiresolution analysis and an algorithm to construct wavelets from multiresolution analysis were obtained by Y. Meyer and S. Mallat [1, 2, 7, 8].

For example, the Meyer's wavelet ψ can be constructed from MRA with its scaling function φ defined by $\hat{\varphi}(\zeta) = \theta(\zeta)$ where θ is positive, even, $C_0^\infty(R)$, supported in $[-\frac{4\pi}{3}, \frac{4\pi}{3}]$, equals to 1 on $[-\frac{2\pi}{3}, \frac{2\pi}{3}]$, and satisfies $\theta^2(\zeta) + \theta^2(2\zeta - \pi) = 1$ if $0 \leq \zeta \leq 2\pi$. The Battle-Lemarié's wavelet can also be constructed from MRA with its scaling function φ with

$$\hat{\varphi}(\zeta) = (\frac{\sin \zeta/2}{\zeta/2})^2(1 - \frac{2}{3}\sin^2\frac{\zeta}{2})^{-\frac{1}{2}}$$

for $n = 2$, for example. But it is known that not every wavelet can be associated with a multiresolution analysis. One of such wavelets is the Journé-Meyer wavelet ψ whose Fourier transform is given by $\hat{\psi}(\zeta) = \chi_{[\frac{4\pi}{7},\pi]}(|\zeta|) + \chi_{[4\pi,4\pi+\frac{4\pi}{7}]}(|\zeta|)$.

Recently, Ha et al. [3, 4] constructed a new class of wavelets, called unimodular wavelets, based on wavelet equations. They showed that a function $\psi \in L^2(R)$ for which $|\hat{\psi}(\zeta)| = 1$ on the support of $\hat{\psi}$ is a wavelet if and only if it satisfies only (W1) and (W3). These two equations force that 2π-translation and dyadic dilation of the support of $\hat{\psi}$ should match nicely and give a severe restriction on the size and the location of the support of $\hat{\psi}$. They were able to produce a large class of wavelets whose Fourier transforms are the characteristic functions of finite disjoint unions of closed intervals. These unimodular wavelets may or may not be associated with multiresolution analysis. For instance, their example ψ_j with

$$\hat{\psi}_j(\zeta) = \chi_{[\frac{2^j\pi}{2^{j+1}-1},\pi]}(|\zeta|) + \chi_{[2^j\pi, 2^j\pi+\frac{2^j\pi}{2^{j+1}-1}]}(|\zeta|), \quad j = 1, 2, 3, \cdots$$

is associated with an MRA if $j = 1$, and is not associated with an MRA if $j \geq 2$. If $j = 2$, it is the well known Journé-Meyer's wavelet.

In this paper, we consider unimodular scaling functions and unimodular wavelets whose Fourier transforms are supported in a finite disjoint union of closed intervals. In Section 2 we characterize unimodular scaling functions and as an application we give another method to produce unimodular wavelets in Section 3. In Section 4, we characterize those unimodular wavelets which can be associated with MRA. As an application, we have a criterion to determine whether wavelets from a class of unimodular wavelets of Ha et al. [3] can be associated with MRA or not in Section 5. This paper represents an improved version of the second author's Master's thesis in KAIST [9] under the direction of the first author.

2. Unimodular Scaling Functions

We recall more on MRA with its scaling function φ. The orthonormality of $\{\varphi(x-k)\}_{k\in Z}$ is equivalent to

$$\sum_{k\in Z} |\hat{\varphi}(\zeta + 2k\pi)|^2 = 1 \quad \text{a.e.} \tag{2.1}$$

The closed subspace V_j is obviously generated by $\{\varphi_{j,k}(x)\}_{k\in Z}$. Let $P_j f$ be the projection of f on V_j for $f \in L^2(R)$. That is,

$$P_j f = \sum_{k\in Z} < f, \varphi_{jk} > \varphi_{jk}, \quad f \in L^2(R). \tag{2.2}$$

Then the density $\overline{\bigcup V_j} = L^2(R)$ is equivalent to

$$P_j f \to f \quad \text{in } L^2(R) \quad \text{as } j \to \infty \quad \text{for all } f \in L^2(R), \tag{2.3}$$

and the separation $\bigcap V_j = \{0\}$ is equivalent to

$$P_j f \to 0 \quad \text{in } L^2(R) \quad \text{as } j \to -\infty \quad \text{for all } f \in L^2(R). \tag{2.4}$$

We now note that

$$\hat{\varphi_{jk}}(\zeta) = 2^{-j/2}\hat{\varphi}(2^{-j}\zeta)e^{-i(2^{-j}k)\zeta}$$

and recall the Poisson summation formula (see [Theorem 2.25, 1])

$$\sum_{k\in Z} f(\zeta+2k\pi) = \frac{1}{2\pi}\sum_{k\in Z}\left(\int_{-\infty}^{\infty} f(\eta)e^{ik\eta}d\eta\right)e^{-ik\zeta}, \quad f\in C_o^\infty(R).$$

An easy calculation using the Plancherel theorem and for any function $f \in C_o^\infty(R)$, a dense subspace of $L^2(R)$, the Poisson summation formula shows that

$$\begin{aligned}
\hat{P_j f} &= \frac{1}{2\pi}\sum_{k\in Z} < \hat{f}, \hat{\varphi}_{jk} > \hat{\varphi}_{jk} \\
&= \frac{1}{2\pi}\sum_{k\in Z} 2^{-j}\left(\int_{-\infty}^{\infty} \hat{f}(\zeta)\overline{\hat{\varphi}(2^{-j}\zeta)}e^{i(2^{-j}k)\zeta}d\zeta\right)e^{-i(2^{-j}k)\zeta}\hat{\varphi}(2^{-j}\zeta) \\
&= \frac{1}{2\pi}\sum_{k\in Z}\left(\int_{-\infty}^{\infty} \hat{f}(2^j\eta)\overline{\hat{\varphi}(\eta)}e^{ik\eta}d\eta\right)e^{-i(2^{-j}k)\zeta}\hat{\varphi}(2^{-j}\zeta) \\
&= \sum_{k\in Z}\hat{f}(2^j(2^{-j}\zeta+2k\pi))\overline{\hat{\varphi}(2^{-j}\zeta+2k\pi)}\hat{\varphi}(2^{-j}\zeta). \qquad (2.5)
\end{aligned}$$

From now on we assume $\varphi \in L^2(R)$ is unimodular, namely, $|\hat{\varphi}(\zeta)| = 1$ on the support of $\hat{\varphi}$ and let $M = \text{supp}(\hat{\varphi})$. Then $\hat{\varphi}(\zeta) = \theta(\zeta)\chi_M(\zeta)$ where $|\theta(\zeta)| = 1$. We can extend θ to be a 2π periodic function, because of the condition (D1) of Theorem 2.1 below. Hence the closure of the span of $\{\varphi(x-k)\}_{k\in Z}$ is the same no matter what θ is. Therefore it is enough to consider $\hat{\varphi} = \chi_M$. Throughout this section we assume that M is a finite disjoint union of closed intervals. We can prove

THEOREM 2.1. *φ is a scaling function of an MRA if and only if M satisfies*

(D1) $\bigcup_{k\in Z}(M+2k\pi) = R$ *(disjoint union),*
(D2) $\frac{1}{2}M \subset M$,
(D3) M contains a neighborhood of zero.

In this case, the associated wavelet ψ is given by $|\hat{\psi}(\zeta)| = \chi_K$ with $K = 2M - M$.

REMARK. Here and throughout the paper the containment and equality between sets are almost everywhere sense.

PROOF. The simplicity on the unimodular functions comes from the fact that the infinite summations on the conditions practically reduces to only one term. Since $|\hat{\varphi}| = \chi_M$, the condition (D1) is equivalent to the equation (2.1). This means that (D1) is equivalent to (M3), the orthonormality of the translates of the scaling function φ. Equation (2.5) reduces to

$$\hat{P_j f} = \hat{f}(\zeta)|\hat{\varphi}(2^{-j}\zeta)|^2 = \hat{f}(\zeta)\chi_{2^jM}(\zeta). \tag{2.6}$$

By the Plancherel theorem, Equations (2.3) and (2.4) are equivalent to $\hat{P_j f} \to \hat{f}$ as $j \to \infty$ and $\hat{P_j f} \to 0$ as $j \to -\infty$ for all $f \in L^2(R)$ respectively, since it holds on the dense subspace $C_o^\infty(R)$. Therefore, the density and separation of $\{V_j\}_{j\in Z}$ are equivalent to the condition that M contains a neighborhood of zero and has a finite Lebesgue measure. Now, if φ is a scaling function for an MRA, we have $\hat{\varphi}(2\zeta) = m_0(\zeta)\hat{\varphi}(\zeta)$ for some 2π-periodic function m_0. This implies $\frac{1}{2}M \subset M$ for $\hat{\varphi} = \chi_M$. Conversely, if $\frac{1}{2}M \subset M$ then we can define m_0 by

$$m_0(\zeta) = \frac{\chi_{\frac{1}{2}M}(\zeta)}{\chi_M(\zeta)} \qquad \text{for} \quad \zeta \in M$$

and extend $m_0(\zeta)$ as a 2π-periodic function because the set M satisfies D(1). Then the dilation equation, $\hat{\varphi}(2\zeta) = m_0(\zeta)\hat{\varphi}(\zeta)$, is satisfied which implies (M2).

For the second part of the theorem, we recall that the associated wavelet ψ with the scaling function φ is given by

$$\hat{\psi}(\zeta) = e^{i\zeta/2}\overline{m_0(\zeta/2+\pi)}\hat{\varphi}(\zeta/2). \tag{2.7}$$

We also recall that one of essential equations which m_0 must satisfy is

$$|m_0(\zeta)|^2 + |m_0(\zeta+\pi)|^2 = 1. \tag{2.8}$$

It then follows from (2.7) and (2.8) that

$$\begin{aligned} |\hat{\psi}(\zeta)|^2 &= |m_0(\zeta/2+\pi)|^2|\hat{\varphi}(\zeta/2)|^2 \\ &= (1-|m_0(\zeta/2)|^2)|\hat{\varphi}(\zeta/2)|^2 \\ &= |\hat{\varphi}(\zeta/2)|^2 - |\hat{\varphi}(\zeta)|^2 \\ &= \chi_{2M} - \chi_M \\ &= \chi_{2M-M} \end{aligned}$$

since $M \subset 2M$. That is, $|\hat{\psi}| = \chi_{2M-M}$. This completes the proof. □

REMARK. The theorem is false if M is an infinite disjoint union of closed intervals. The counterexample is given in [6] as

$$M=\bigcup_{k=1}^{\infty}[-\frac{1}{2^k}(2-\frac{1}{2^k})\pi,-\frac{1}{2^k}\pi]\cup[0,\frac{5}{4}\pi]\cup\bigcup_{k=1}^{\infty}[(2-\frac{1}{2^k})\pi,(2-\frac{1}{2^{k+1}}(2-\frac{1}{2^{k+1}}))\pi],$$

which does not contain an open neighborhood of zero, violating condition (D3).

3. Examples of Unimodular Scaling Functions

In this section, we consider those scaling functions φ with $\hat{\varphi}=\chi_M$ where M consists of a few disjoint closed intervals. The associated wavelet ψ is given by $|\hat{\psi}|=\chi_K$ where $K=2M-M$. In this way we can generate many new unimodular wavelets associated with MRA. For simplicity, we choose the all parameters, a,b,c,d,e and f in the following examples to be positive.

I. *The case where M is one interval.*

In this case M should be of the form $M=[-a,2\pi-a]$, where $0<a<2\pi$. Then the associated unimodular wavelet ψ is given by $\hat{\psi}(\zeta)=\theta(\zeta)\chi_K(\zeta)$ with $K=[-2a,-a]\cup[2\pi-a,4\pi-2a]$. It is the general form of unimodular wavelet whose Fourier transform has its support consisting of the 2-intervals in [3]. Historically, if $a=\pi$ then it is associated with Littlewood-Paley basis which has the time - frequency properties complementary to the Haar basis [2].

II. *The case where M consists of two intervals does not happen.*

Without loss of generality, we may assume that $M=[-a,b]\cup[c,d]$ where $0<b<c<d$. From (D2) $d\leq 2b$, and from (D1) we should have $b+2k\pi=c$, $-a+2(k+1)\pi=d$, where k is a positive integer. Therefore, $-a+2(k+1)\pi=d\leq 2b=2c-4k\pi$, and so $-a+6k\pi+2\pi\leq 2c<2d=-2a+4k\pi+4\pi$. Hence, we should have $2k\pi-2\pi<-a<0$, which is impossible for a positive integer k.

III. *The case where M is of the form* $M = [-a,-b] \cup [-c,d] \cup [e,f]$.

We know from (D1) that $|M| = 2\pi$ and 2π-translations of M are disjoint. Here, $|\cdot|$ denotes the Lebesgue measure. From (D2), $f \le 2d \le 4\pi$ and $a \le 2c \le 4\pi$. Suppose $f \ge 2\pi$. Then $d \ge \pi$ and so $c \le \pi, a \le 2\pi$. Since 2π-translations of M are disjoint, $[e,f] \subset [2\pi, 4\pi]$ and $[-a,-b] \subset [-2\pi, 0]$. We must be careful for the lengths of intervals and the order such that $-2\pi < -a < -b < -c < 0$ and $\pi < d < 2\pi < e < f < 4\pi$. From (D1), we have only two possibilities;

$$\begin{aligned} \textit{either} \quad (i) \quad & [e,f] - 2\pi \text{ and } [-a,-b] + 2\pi \\ & \qquad \text{make a closed interval} [-c, 2\pi - b], \\ \textit{or} \quad (ii) \quad & [e,f] - 4\pi = [-b,-c]. \end{aligned}$$

In the case (i), $e = d + 2\pi$. Thus $d + 2\pi = e < f \le 2d$; so $2\pi < d$, which is a contradiction. Hence the case (i) does not occur. In the case (ii), $e = 4\pi - b$ and $f = 4\pi - c$. In this case $[e,f] - 4\pi$ should fill the gap between the intervals $[-a,-b]$ and $[-c,d]$. Therefore $d + a = 2\pi$. Thus $4\pi - c = f \le 2d = 4\pi - 2a$; so $2a \le c < a$, which is a contradiction. Hence, the case (ii) does not occur either. Therefore we have $f < 2\pi$. By considering $-M$, we should have $a < 2\pi$. From (D1), we have $[e,f] - 2\pi = [-b,-c]$ and $a + d = 2\pi$. Therefore, $d = 2\pi - a, e = 2\pi - b$ and $f = 2\pi - c$. Hence, M is of the form

$$M = [-a,-b] \cup [-c, 2\pi - a] \cup [2\pi - b, 2\pi - c],$$

where $0 < c < b < a \le 2c < 4\pi$.

The associated wavelet ψ is given by $|\hat{\psi}| = \chi_K$ where

$$\begin{aligned} K = [-2a,-2b] \cup [-2c,-a] \cup [-b,-c] \cup [2\pi - a, 2\pi - b] \\ \cup [2\pi - c, 4\pi - 2a] \cup [4\pi - 2b, 4\pi - 2c]. \end{aligned}$$

If $a + c = 2\pi$ and $b = \pi$ then M and K are symmetric about the origin and K consists of 6 intervals. In particular, if $a = 2c$ then we can see that

$$M = [-\frac{4}{3}\pi, -\pi] \cup [-\frac{2}{3}\pi, \frac{2}{3}\pi] \cup [\pi, \frac{4}{3}\pi].$$

The associated wavelet ψ with $|\hat{\psi}| = \chi_K$ is given by

$$K = [-\frac{8}{3}\pi, -2\pi] \cup [-\pi, -\frac{2}{3}\pi] \cup [\frac{2}{3}\pi, \pi \cup [2\pi, \frac{8}{3}\pi].$$

This example appears in Theorem 4.3 of Ha et al. [3] for $j = 1$.

IV. *The case where M is of the form $M = [-a, b] \cup [c, d] \cup [e, f]$.*

From (D2), $f \leq 2d$ and $d \leq 2b$. If $c - b \geq 2\pi$ then $b + 2\pi \leq c < d \leq 2b$. Hence $b > 2\pi$, which is impossible. Therefore $c - b < 2\pi$. A moment's consideration shows that the gap interval $[b, c]$ should be a 2π-translation of the interval $[e, f]$. Since $|M| = 2\pi$, $a + d = 2\pi$. Thus $f \leq 4\pi - 2a$. Therefore, we have $[-a, b] \cup ([e, f] - 2\pi) \cup [c, d] = [-a, d]$. It follows that $e = b + 2\pi$, $f = c + 2\pi$ and $d = 2\pi - a$. We now have

$$M = [-a, b] \cup [c, 2\pi - a] \cup [b + 2\pi, c + 2\pi],$$

where $0 < b < 2\pi, b < c \leq \pi + b/2$ and $2\pi - 2b < a \leq \pi - c/2$.

The associated unimodular wavelet ψ is given by $|\hat{\psi}| = \chi_K$, where $K = K^+ \cup K^-$ and

$$\begin{aligned} K^- &= [-2a, -a], \\ K^+ &= [b, c] \cup [2\pi - a, 2b] \cup [2c, b + 2\pi] \\ &\quad \cup [c + 2\pi, 4\pi - 2a] \cup [2b + 4\pi, 2c + 4\pi]. \end{aligned}$$

By the appropriate choice of a, b and c, the six intervals above for K may be reduced to three to five intervals. For example, if the intervals consisting of M are "well matched", that is, if $2b = d, 2c = e$ and $2d = f$, then we have

$$M = [-\frac{2}{7}\pi, \frac{6}{7}\pi] \cup [\frac{10}{7}\pi, \frac{12}{7}\pi] \cup [\frac{20}{7}\pi, \frac{24}{7}\pi].$$

The associated wavelet ψ is given by $|\hat{\psi}| = \chi_K$ where

$$K = [-\frac{4}{7}\pi, -\frac{2}{7}\pi] \cup [\frac{6}{7}\pi, \frac{10}{7}\pi] \cup [\frac{40}{7}\pi, \frac{48}{7}\pi].$$

This example appears in Theorem 4.6 and in Remark 4.8 of Ha et al. [3] as the case of $j = p = 2$.

4. Unimodular Wavelets Associated with MRA

The unimodular wavelets introduced by Ha et al. [3] may or may not be associated with MRA. See [3]. In this section, we give a characterization of unimodular wavelets which are associated with MRA in terms of the support of its Fourier transform. Throughout this section we assume that ψ is a unimodular wavelet with $|\hat{\psi}| = \chi_K$, where K is a finite disjoint union of closed intervals.

PROPOSITION 1. *If a unimodular wavelet ψ is associated with an MRA with its scaling function φ, then φ is also unimodular and*

$$|\hat{\varphi}(\zeta)| = \sum_{j>0} |\hat{\psi}(2^j\zeta)| = \sum_{j<0} \chi_{2^jK}(\zeta) = \chi_M(\zeta)$$

where $M = \dot{\bigcup}_{j<0} 2^j K$ is a disjoint union.

PROOF. We know that ψ and φ are related by

$$\hat{\psi}(\zeta) = \theta(\zeta)\overline{m_0(\zeta/2+\pi)}\hat{\varphi}(\zeta/2),$$

where $|\theta| = 1$ and $m_0(\zeta)$ is a 2π-periodic function which satisfies

$$|m_0(\zeta)|^2 + |m_0(\zeta+\pi)|^2 = 1.$$

Since $\hat{\varphi}(\zeta) = m_0(\zeta/2)\hat{\varphi}(\zeta/2)$, we have

$$\begin{aligned} |\hat{\psi}(\zeta)|^2 &= |m_0(\zeta/2+\pi)|^2|\hat{\varphi}(\zeta/2)|^2 = (1-|m_0(\zeta/2)|^2)|\hat{\varphi}(\zeta/2)|^2 \\ &= |\hat{\varphi}(\zeta/2)|^2 - |\hat{\varphi}(\zeta)|^2. \end{aligned}$$

It follows that

$$|\hat{\varphi}(\zeta)|^2 = \sum_{j>0} |\hat{\psi}(2^j\zeta)|^2 = \sum_{j<0} \chi_{2^jK}(\zeta).$$

Since the unimodular wavelet should satisfy (W3), it follows that $\{2^jK\}_{j<0}$ should be disjoint. Therefore, $|\hat{\varphi}| = \chi_M$ where

$$M = \dot{\bigcup}_{j<0} 2^j K.$$

is a disjoint union. This completes the proof. □

We note that the conditions (W1) and (W3) which $\hat{\psi}$ should satisfy can be respectively written in terms of K as follows :

$(U1)$: $\dot{\bigcup}_{l\in Z}(K+2l\pi) = R$,

$(U3)$: $\dot{\bigcup}_{j\in Z}(2^jK) = R$.

We also recall the functions τ_a and δ_a introduced in [3]. Let a be a real number. For each $x \in R$, there is a unique integer $k(x)$ such that $a \le x + 2k(x)\pi < a+2\pi$. Then $\tau_a : R \to [a, a+2\pi)$ is defined as $\tau_a(x) = x + 2k(x)\pi$. It is known in [3] that for any $a \in R$, τ_a is one to one on K except on a set of measure zero and $|[a, a+2\pi) - \tau_a(K)| = 0$. Let $a > 0$. For each $x > 0$ there is a unique integer $j(x)$ such that $a \le 2^{j(x)}x < 2a$. We define $\delta_a : (0,\infty) \to [a, 2a)$ by $\delta_a(x) = 2^{j(x)}x$. Let $K^+ = K \cap (0,\infty)$ and $K^- = K \cap (-\infty, 0)$. Then it is known in [3] that for any $a > 0$ and $b < 0$, δ_a and δ_b are one to one on K^+ and K^- except on a set of measure zero and $|[a, 2a) - \delta_a(K^+)| = 0$ and $|[2b, b) - \delta_b(K^-)| = 0$, respectively. Now we can prove

THEOREM 4.1. *A unimodular wavelet ψ is associated with an MRA if and only if $\dot\bigcup_{l\in Z} \dot\bigcup_{j<0}(2^jK + 2l\pi) = R$.*

PROOF. To prove the necessity, suppose ψ is associated with a unimodular scaling function φ. Then by Proposition 1, $|\hat\varphi| = \chi_M$ with $M = \dot\bigcup_{j<0} 2^jK$. From (D1), we should have

$$R = \dot\bigcup_{l\in Z}(M + 2l\pi) = \dot\bigcup_{l\in Z} \dot\bigcup_{j<0}(2^jK + 2l\pi).$$

To prove sufficiency, we set $M = \dot\bigcup_{j<0} 2^jK$ and let $\hat\varphi = \chi_M$. Then M satisfies (D1) and (D2). Since $\{2^jK\}_{j\in Z}$ are disjoint, K^+ and K^- bounded away from the origin. Therefore, $a = \inf(K^+) > 0$ and $b = \sup(K^-) < 0$. As noted before, $(a, 2a) = \delta_a(K^+) \subset \dot\bigcup_{j<1} 2^jK$ and $(2b, b) = \delta_b(K^-) \subset \dot\bigcup_{j<1} 2^jK$. Hence we have $(b, a) \subset M$. Hence M satisfies (D3). Therefore, φ is a unimodular scaling function. According to Theorem 2.1, the associated unimodular wavelet ψ is given by $|\hat\psi| = \chi_K$, since $2M - M = K$. This completes the proof. $\square$

5. Application

The previous theorem gives a simple method to determine whether a unimodular wavelet can be associated with an MRA or not. That is, if $M = \dot\bigcup_{j<0} 2^jK$ satisfies (D1) then the unimodular wavelet can be associated with an MRA. Otherwise, it is not associated with an MRA. We apply this method to an example of Ha et al. [3]. They proved that for

$$K_{j,p} = [-2\left(1 - \frac{2p+1}{2^{j+1}-1}\right)\pi, -\left(1 - \frac{2p+1}{2^{j+1}-1}\right)\pi] \cup$$

$$[\frac{2(p+1)}{2^{j+1}-1}\pi, \frac{2(2p+1)}{2^{j+1}-1}\pi] \cup [\frac{2^{j+1}(2p+1)}{2^{j+1}-1}\pi, \frac{2^{j-2}(p+1)}{2^{j+1}-1}\pi]$$

with $j \geq 2$ and $1 \leq p \leq 2^j - 2$, $\psi \in L^2(R)$ with $|\hat{\psi}| = \chi_{K_{j,p}}$ is a unimodular wavelet. They show that it is not associated with an MRA for odd p and for $p = 2$ and $j = 3$, but it is associated with an MRA for $p = j = 2$. For other j's and p's, it is not proved whether it is associated with an MRA or not. The following theorem gives a criterion to determine whether ψ among them is associated with an MRA or not. Let $M_{j,p} = \dot{\bigcup}_{i<0} 2^i K_{j,p}$. Then we can see that

$$M_{j,p} = [-(1 - \frac{2p+1}{2^{j+1}-1})\pi, \frac{2(p+1)}{2^{j-1}-1}\pi] \cup \bigcup_{1 \leq l \leq j} [\frac{2^l(2p+1)}{2^{j-1}-1}\pi, \frac{2^{l+1}(p+1)}{2^{j-1}-1}\pi]$$

which consists of $j+1$ disjoint closed intervals.

For a fixed j, the p's for which ψ with $|\hat{\psi}| = \chi_{K_{j,p}}$ are associated with MRA are related to the permutations σ's of $\{1, 2, \cdots, j\}$ which satisfy

(1) $\sigma(1) \neq 1$ and $\gcd(\sigma(1), j+1) = 1$,

(2) $\sigma(i) \equiv i\sigma(1), \quad \mod (j+1)$.

More precisely, we have

THEOREM 5.1. *For a fixed j, a unimodular wavelet ψ with $|\hat{\psi}| = K_{j,p}$ is associated with an MRA if and only if*

$$p = \frac{1}{2}(2^{\sigma(1)} + 2^{\sigma(2)} + \cdots + 2^{\sigma(i_0 - 1)}) \tag{5.1}$$

for a permutation σ satisfying (1) *and* (2) *where $\sigma(i_0) = 1$.*

PROOF. Let $a = \frac{(2p+1)}{2^{j+1}-1}\pi$ and $b = \frac{2(p+1)}{2^{j-1}-1}\pi$. Then $M_{j,p}$ is written in the form

$$\begin{aligned} M_{j,p} &= [a - \pi, b] \cup [2a, 2b] \cup \cdots \cup [2^j a, 2^j b], \\ &= \mathrm{I}_0 \cup \mathrm{I}_1 \cup \cdots \cup \mathrm{I}_j, \end{aligned} \tag{5.2}$$

which is a disjoint union of closed intervals and has the Lebesgue measure 2π. Suppose ψ with $|\hat{\psi}| = \chi_{K_{j,p}}$ is associated with an MRA. In view of Theorem 4.1, there must exist some non-negative integers $k_1, k_2, \cdots, k_j$ and a permutation σ of $\{1, 2, \cdots, j\}$ such that

$$\begin{aligned} &[a - \pi, a + \pi] \\ &= \mathrm{I}_0 \cup \{\mathrm{I}_{\sigma(1)} - 2\pi k_1\} \cup \{\mathrm{I}_{\sigma(2)} - 2\pi k_2\} \cup \cdots \cup \{\mathrm{I}_{\sigma(j)} - 2\pi k_j\}, \end{aligned} \tag{5.3}$$

where the intervals on the right are arranged in the order of the left end points, and the adjacent intervals have only end points in common. Since the length between b and $2a$ does not exceed 2π, $\sigma(1) \neq 1$. We will show that the permutation σ satisfies the conditions (1) and (2). From the first two intervals on the right of (5.3), we see that

$$\begin{aligned} b &= 2^{\sigma(1)}a - 2\pi k_1 \\ 2b &= 2^{\sigma(1)+1}a - 2(2\pi k_1) \\ \cdots & \quad \cdots \\ 2^{j-\sigma(1)}b &= 2^j a - 2^{j-\sigma(1)}(2\pi k_1). \end{aligned}$$

Therefore, the pairs

$$(5.4) \qquad (0,\sigma(1)),(1,\sigma(1)+1),\cdots,(j-\sigma(1),j)$$

are adjacent pairs in the sequence $(0,\sigma(1),\sigma(2),\cdots,\sigma(j))$. Also, from (5.3) $a+\pi = 2^{\sigma(j)}b - 2\pi k_j$. We have

$$\begin{aligned} a &= 2^{\sigma(j)}b - \pi(2k_j+1) \\ 2a &= 2^{\sigma(j)+1}b - 2\pi(2k_j+1) \\ \cdots & \quad \cdots \\ 2^{\sigma(1)-1}a &= 2^{\sigma(j)+\sigma(1)-1}b - 2^{\sigma(1)-1}\pi(2k_j+1). \end{aligned}$$

Therefore, the pairs

$$(5.5) \quad (\sigma(j)+1,1),(\sigma(j)+2,2),\cdots,(\sigma(j)+\sigma(1)-1,\sigma(1)-1)$$

are adjacent pairs in the sequence $(0,\sigma(1),\sigma(2),\cdots,\sigma(j))$. Since the largest index among $\{0,1,\cdots,j-\sigma(1),\sigma(j)+1,\sigma(j)+2,\cdots,\sigma(j)+\sigma(1)-1\}$ in (5.4) and (5.5) must be $\sigma(j)+\sigma(1)-1$, $j=\sigma(j)+\sigma(1)-1$. That is, $\sigma(j)=j+1-\sigma(1)$. We also note that for any $n (0<n\le j)$,

$$\sigma(n) = \begin{cases} \sigma(n-1)-\sigma(j) = \sigma(n-1)+\sigma(1)-(j+1) & (1\le n\le \sigma(1)-1) \\ \sigma(n-1)+\sigma(1) & (\sigma(1)\le n\le j), \end{cases}$$

where $\sigma(0)=0$. Therefore, $\sigma(n) = n\sigma(1) - k(n)(j+1)$ for some non-negative integer $k(n)$. Hence $\sigma(n)\equiv n\sigma(1), \quad \mathrm{mod}(j+1)$. Since σ is a permutation of $\{1,2,\cdots,j\}$, $\sigma(1)$ and $j+1$ must be prime to each other. Now, let $\sigma(i_0)=1$. Then $\bigcup_{i=1}^{i_0-1} I_{\sigma(i)}$ has the Lebesgue measure

$$2a-b = \frac{2p}{2^{j+1}-1}\pi.$$

That is,

$$\sum_{i=1}^{i_0-1} \frac{2^{\sigma(i)}}{2^{j+1}-1}\pi = \frac{2p}{2^{j+1}-1}\pi.$$

Therefore,

$$p = \frac{1}{2}(2^{\sigma(1)} + 2^{\sigma(2)} + \cdots + 2^{\sigma(i_0-1)}).$$

This proves the necessity part.

To prove sufficiency, we let p be given by (5.1) for a permutation σ satisfying (1) and (2) with $\sigma(i_0) = 1$. For the $M_{j,p}$ given by (5.2) we have to show Equation (5.3) for some non-negative integers $k_1, k_2, \cdots, k_j$. Since the Lebesgue measure of $M_{j,p}$ is 2π, we should only show that, for any $n (0 \le n < j)$, there is a non-zero integer k such that $2^{\sigma(n)}b = 2^{\sigma(n+1)}a - 2\pi k$. From the condition (2) of the permutation, we have only two possibilities for any $n(0 \le n < j)$

$$\begin{cases} \text{either} \quad \text{(i)} \quad \sigma(n+1) = \sigma(n) + \sigma(1), \\ \qquad \text{or} \quad \text{(ii)} \quad \sigma(n+1) = \sigma(n) + \sigma(1) - (j+1), \end{cases}$$

where $\sigma(0) = 0$.

In the case (i), we see that $2^{\sigma(n+1)}a - 2^{\sigma(n)}b = 2^{\sigma(n)}\left(2^{\sigma(1)}a - b\right)$. Thus we have to show that there is a positive integer k_1 such that $2^{\sigma(1)}a - b = 2\pi k_1$, i.e.,

$$\text{(5.6)} \qquad \frac{2p(2^{\sigma(1)} - 1) + 2^{\sigma(1)} - 2}{2^{j+1} - 1}\pi = 2\pi k_1.$$

The integer part of the numerator has the form

$$\begin{aligned} &2p(2^{\sigma(1)} - 1) + 2^{\sigma(1)} - 2 \\ &= [2^{\sigma(1)} + 2^{\sigma(2)} + \cdots + 2^{\sigma(i_0-1)}](2^{\sigma(1)} - 1) + 2^{\sigma(1)} - 2 \\ &= 2^{\sigma(1)+\sigma(1)} + 2^{\sigma(2)+\sigma(1)} + \cdots + 2^{\sigma(i_0-1)+\sigma(1)} - 2^{\sigma(1)} - 2^{\sigma(2)} \\ &\qquad - \cdots - 2^{\sigma(i_0-1)} + 2^{\sigma(1)} - 2 \\ &= 2^{\sigma(2)}[2^{\sigma(1)+\sigma(1)-\sigma(2)} - 1] + 2^{\sigma(3)}[2^{\sigma(2)+\sigma(1)-\sigma(3)} - 1] \\ &\qquad + \cdots + 2[2^{\sigma(i_0-1)+\sigma(1)-1} - 1]. \end{aligned}$$

(5.6) follows from the fact that

$$\begin{aligned} \sigma(1) + \sigma(1) - \sigma(2) &= 0 \text{ or } j+1 \\ \sigma(2) + \sigma(1) - \sigma(3) &= 0 \text{ or } j+1 \\ &\cdots \\ \sigma(i_0 - 2) + \sigma(1) - \sigma(i_0 - 1) &= 0 \text{ or } j+1 \\ \sigma(i_0 - 1) + \sigma(1) - 1 &= j+1 \end{aligned}$$

which follows from the condition (2) of the permutation σ, and $\sigma(i_0) = 1$.

In the case (ii), we see that $2^{\sigma(n)}b - 2^{\sigma(n+1)}a = 2^{\sigma(n)}\left(b - 2^{\sigma(1)-(j+1)}a\right)$. As in the case (i), we show that there is a positive integer k such that

$$\frac{2^{\sigma(n)}\{2p(1 - 2^{\sigma(1)-(j+1)}) + 2 - 2^{\sigma(1)-(j+1)}\}}{2^{j+1} - 1}\pi = 2\pi k. \tag{5.7}$$

The integer part of the numerator can be written as

$$\begin{aligned}
&2^{\sigma(n)}\{2p(1 - 2^{\sigma(1)-(j+1)}) + (2 - 2^{\sigma(1)-(j+1)})\} \\
&= 2^{\sigma(n)}\{2^{\sigma(1)} + 2^{\sigma(2)} + \cdots + 2^{\sigma(i_0-1)} + 2 - 2^{\sigma(1)-(j+1)} \\
&\quad -2^{\sigma(1)+\sigma(1)-(j+1)} - 2^{\sigma(2)+\sigma(1)-(j+1)} - \cdots - 2^{\sigma(i_0-1)+\sigma(1)-(j+1)}\} \\
&= 2^{\sigma(n)-\sigma(1)-(j+1)}[2^{j+1} - 1] + 2^{\sigma(n)+\sigma(1)+\sigma(1)-(j+1)}[2^{\sigma(2)-\sigma(1)-\sigma(1)+j+1} - 1] \\
&\quad + \cdots + 2^{\sigma(n)+\sigma(i_0-2)+\sigma(1)-(j+1)}[2^{\sigma(i_0-1)-\sigma(i_0-2)-\sigma(1)+j+1} - 1] \\
&\quad + 2^{\sigma(n)+\sigma(i_0-1)+\sigma(1)-(j+1)}[2^{1-\sigma(i_0-1)-\sigma(1)+j+1} - 1].
\end{aligned}$$

(5.7) follows from the fact that

$$\begin{aligned}
\sigma(2) - \sigma(1) - \sigma(1) + (j+1) &= 0 \text{ or } j+1 \\
\sigma(3) - \sigma(2) - \sigma(1) + (j+1) &= 0 \text{ or } j+1 \\
&\cdots \\
\sigma(i_0 - 1) - \sigma(i_0 - 2) - \sigma(1) + (j+1) &= 0 \text{ or } j+1 \\
1 - \sigma(i_0 - 1) - \sigma(1) + (j+1) &= 0
\end{aligned}$$

which follows from the condition (2) of the permutation σ, and $\sigma(i_0) = 1$. This completes the proof. □

REMARK. For a fixed j, we have an algorithm which determine p for which $K_{j,p}$ is associated with an MRA:

for $k = 2$ to j do
if (gcd $(k, j+1) = 1$)
then make permutation σ by

1. $\sigma(1) = k$,
2. $\sigma(i) \equiv ki, \quad \mathrm{mod}(j+1)$

if $(i_0 k \equiv 1, \quad \mathrm{mod}(j+1))$
we set $p = \frac{1}{2}(2^{\sigma(1)} + 2^{\sigma(2)} + \cdots + 2^{\sigma(i_0-1)})$.

TABLE 1. j and p for which $K_{j,p}$ is associated with MRA.

index j	the permutation σ	i_0	index p
2	2, 1	2	2
3	3, 2, 1	3	6
4	2, 4, 1, 3	3	10
	3, 1, 4, 2	2	4
	4, 3, 2, 1	4	14
5	5, 4, 3, 2, 1	5	30
6	2, 4, 6, 1, 3, 5	4	42
	3, 6, 2, 5, 1, 4	5	54
	4, 1, 5, 2, 6, 3	2	8
	5, 3, 1, 6, 4, 2	3	20
	6, 5, 4, 3, 2, 1	6	62
7	3, 6, 1, 4, 7, 2, 5	3	36
	5, 2, 7, 4, 1, 6, 3	5	90
	7, 6, 5, 4, 3, 2, 1	7	126
8	2, 4, 6, 8, 1, 3, 5, 7	5	170
	4, 8, 3, 7, 2, 6, 1, 5	7	238
	5, 1, 6, 2, 7, 3, 8, 4	2	16
	7, 5, 3, 1, 8, 6, 4, 2	4	84
	8, 7, 6, 5, 4, 3, 2, 1	8	254
9	3, 6, 9, 2, 5, 8, 1, 4, 7	7	438
	7, 4, 1, 8, 5, 2, 9, 6, 3	3	72
	9, 8, 7, 6, 5, 4, 3, 2, 1	9	510
10	2, 4, 6, 8, 10, 1, 3, 5, 7, 9	6	682
	3, 6, 9, 1, 4, 7, 10, 2, 5, 8	4	292
	4, 8, 1, 5, 9, 2, 6, 10, 3, 7	3	136
	5, 10, 4, 9, 3, 8, 2, 7, 1, 6	9	990
	6, 1, 7, 2, 8, 3, 9, 4, 10, 5	2	32
	7, 3, 10, 6, 2, 9, 5, 1, 8, 4	7	886
	8, 5, 2, 10, 7, 4, 1, 9, 6, 3	6	730
	9, 7, 5, 3, 1, 10, 8, 6, 4, 2	5	340
	10, 9, 8, 7, 6, 5, 4, 3, 2, 1	10	1022

By this algorithm, we computed those indices j and p for which $K_{j,p}$ are associated with MRA in the following table.

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