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DENJOY-TYPE INTEGRALS OF BANACH-VALUED FUNCTIONS

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ABSTRACT. In this paper Denjoy$_*$-Dunford, Denjoy$_*$-Pettis, Denjoy$_*$-McShane and Denjoy$_*$-Bochner integrals of functions which map an interval $[a, b]$ into a Banach space X are defined. And we give the relations among the integrals.

1. Introduction

The Denjoy integral of a real-valued function is, in a descriptive sense, a natural extension of the Lebesgue integral of a real-valued function. The Bochner, McShane, Pettis and Dunford integrals are generalizations of the Lebesgue integral to Banach-valued functions. Gordon [5] studied the Denjoy extension of the Bochner, Pettis and Dunford integrals. And Park and Lee [8] defined the Denjoy-McShane integral which is an extension of the McShane integral. It is known [6] that the Denjoy$_*$-integrable function on an interval I is Denjoy-integrable on I but the converse is not true in general. In this paper we define Denjoy$_*$-Bochner, Denjoy$_*$-McShane, Denjoy$_*$-Pettis, and Denjoy$_*$-Dunford integrals which are extensions of the Bochner, McShane Pettis and Dunford integrals respectively. And we give the relations among the integrals.

2. Definitions

Throughout this paper X will denote a real Banach space with a norm

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$\| \cdot \|$ and X^* its dual.

DEFINITION 2.1 [9]. Let $F : [a, b] \to X$ be a function and E a subset of $[a, b]$.

(a) A function F is AC (absolutely continuous) (resp. AC_* (absolutely continuous in the restricted sense)) on E if for each $\epsilon > 0$ there exists $\delta > 0$ such that $\sum_i \| F(d_i) - F(c_i) \| < \epsilon$ (resp. $\sum_i O(F; I_i) < \epsilon$, where $O(F; I_i)$ is the oscillation of F on $I_i = [c_i, d_i]$) whenever $\{[c_i, d_i]\}$ is a finite collection of nonoverlapping intervals that have endpoints in E and satisfy $\sum_i (d_i - c_i) < \delta$.

(b) A function F is ACG (generalized absolutely continuous) (resp. ACG_* (generalized absolutely continuous in the restricted sense)) on E if F is continuous on E and E can be expressed as a countable union of sets on each of which F is AC (resp. AC_*).

DEFINITION 2.2 [3]. (a) A McShane partition of $[a, b]$ is a finite sequence $< ([a_i, b_i], t_i) >_{i \leq n}$ such that $< [a_i, b_i] >_{i \leq n}$ is a non-overlapping family of intervals covering $[a, b]$ and $t_i \in [a, b]$ for each i. A gauge on $[a, b]$ is a function $\delta : [a, b] \to (0, \infty)$. A McShane partition $< ([a_i, b_i], t_i) >_{i \leq n}$ is a subordinate to a gauge δ if $t_i - \delta(t_i) \leq a_i \leq t_i + \delta(t_i)$ for every $i \leq n$.

A function $\phi : [a, b] \to X$ is McShane integrable, with McShane integral ω, if for every $\epsilon > 0$ there is a gauge $\delta : [a, b] \to (0, \infty)$ such that

$$\| \omega - \sum_{i \leq n} (b_i - a_i) \phi(t_i) \| < \epsilon$$

for every McShane partition $< ([a_i, b_i], t_i) >_{i \leq n}$ of $[a, b]$ subordinate to δ.

(b) A Henstock partition of $[a, b]$ is a McShane partition $< ([a_i, b_i], t_i) >_{i \leq n}$ of $[a, b]$ such that $t_i \in [a_i, b_i]$ for every $i \leq n$. A function $\phi : [a, b] \to X$ is Henstock integrable, with Henstock integral w, if for every $\epsilon > 0$ there is a gauge $\delta : [a, b] \to (0, \infty)$ such that

$$\| \omega - \sum_{i \leq n} (b_i - a_i) \phi(t_i) \| < \epsilon$$

for every Henstock partition $< ([a_i, b_i], t_i) >_{i \leq n}$ of $[a, b]$ subordinate to δ.

DEFINITION 2.3 [9]. Let $F : [a,b] \to X$ be a function and $t \in (a,b)$. A vector z in X is the approximate derivative of F at t if there exists a measurable set $E \subset [a,b]$ that has t as a point of density such that $\lim_{\substack{s\to t\\ s\in E}} \frac{F(s)-F(t)}{s-t} = z$. We will write $F'_{ap}(t) = z$.

A function $f : [a,b] \to \mathbb{R}$ is Denjoy (resp. Denjoy$_*$) integrable on $[a,b]$ if there exists an ACG (resp. ACG_*) function $F : [a,b] \to \mathbb{R}$ such that $F'_{ap} = f$ (resp. $F' = f$) almost everywhere on $[a,b]$. A function f is Denjoy (resp. Denjoy$_*$) integrable on a set $E \subset [a,b]$ if $f\chi_E$ is Denjoy (resp. Denjoy$_*$) integrable on $[a,b]$.

DEFINITION 2.4 [5]. (a) A function $f : [a,b] \to X$ is Denjoy-Dunford integrable on $[a,b]$ if for each x^* in X^* the function x^*f is Denjoy integrable on $[a,b]$ and if for every interval I in $[a,b]$ there exists a vector x^{**} in X^{**} such that $x^{**}(x^*) = \int_I x^*f$ for all x^* in X^*.

(b) A function $f : [a,b] \to X$ is Denjoy-Pettis integrable on $[a,b]$ if f is Denjoy-Dunford integrable on $[a,b]$ and $x^{**} \in X$ for every interval I in $[a,b]$.

(c) A function $f : [a,b] \to X$ is Denjoy-Bochner integrable on $[a,b]$ if there exists an ACG function $F : [a,b] \to X$ such that F is approximately differentiable almost everywhere on $[a,b]$ and $F'_{ap} = f$ almost everywhere on $[a,b]$.

A function f is integrable in one of the above senses on a set $E \subset [a,b]$ if the function $f\chi_E$ is integrable in that sense on $[a,b]$.

3. Denjoy-type integrals

In this section, we consider Denjoy$_*$-Dunford, Denjoy$_*$-Pettis, Denjoy$_*$-McShane and Denjoy$_*$-Bochner integrals of functions which map an interval $[a,b]$ into a Banach space X.

DEFINITION 3.1. (a) A function $f : [a,b] \to X$ is Denjoy$_*$-Dunford integrable on $[a,b]$ if for each x^* in X^* the function x^*f is Denjoy$_*$ integrable on $[a,b]$ and if for every interval I in $[a,b]$ there exists a vector x^{**} in X^{**} such that $x^{**}(x^*) = \int_I x^*f$ for all x^* in X^*.

(b) A function $f : [a,b] \to X$ is Denjoy$_*$-Pettis integrable on $[a,b]$ if f is Denjoy$_*$-Dunford integrable on $[a,b]$ and $x^{**} \in X$ for every interval I in $[a,b]$.

(c) A function $f : [a,b] \to X$ is Denjoy$_*$-Bochner integrable on $[a,b]$ if there exists an ACG_* function $F : [a,b] \to X$ such that F is differentiable almost everywhere on $[a,b]$ and $F' = f$ almost everywhere on $[a,b]$.

A function f is integrable in one of the above senses on a set $E \subset [a,b]$ if the function $f\chi_E$ is integrable in that sense on $[a,b]$.

REMARK. Let $X = \mathbb{R}$. Then the Denjoy$_*$-Bochner (resp. Denjoy-Bochner) integrability is equivalent to the Denjoy$_*$ (resp. Denjoy) integrability. Example 6.20 (c) [6] shows that the Denjoy$_*$ integrability is not equal to the Denjoy integrability. So we can take this example as an example which the Denjoy$_*$-Bochner integrability is not equal to the Denjoy-Bochner integrability.

LEMMA 3.2 [5]. *Let $F : [a,b] \to \mathbb{R}$ be ACG on $[a,b]$. If $F'_{ap} = \theta$ almost everywhere on $[a,b]$, then F is constant on $[a,b]$.*

THEOREM 3.3. *Let $F : [a,b] \to X$ be ACG_* on $[a,b]$ and suppose that F is differentiable on $[a,b]$. If $F' = \theta$ almost everywhere on $[a,b]$, then F is constant on $[a,b]$.*

PROOF. Suppose that F is not constant on $[a,b]$. Then there exist $t_1, t_2 \in [a,b]$ such that $F(t_1) \neq F(t_2)$. Choose $x^* \in X^*$ such that $x^*F(t_1) \neq x^*F(t_2)$. Since x^*F is ACG_* on $[a,b]$, x^*F is ACG on $[a,b]$ and $(x^*F)' = (x^*F)'_{ap} = 0$ almost everywhere on $[a,b]$. Thus x^*F is constant by Lemma 3.2. This is a contradiction. □

REMARK. The above theorem guarantees the uniqueness of the Denjoy$_*$-Bochner integral.

THEOREM 3.4. *If $f : [a,b] \to X$ is Bochner integrable, then f is Denjoy$_*$-Bochner integrable.*

PROOF. Let f be Bochner integrable. Then there exists an AC function $F : [a,b] \to X$ such that F is differentiable almost everywhere on $[a,b]$ and $F' = f$ almost everywhere on $[a,b]$. Clearly F is ACG_* and hence f is Denjoy$_*$-Bochner integrable. □

THEOREM 3.5. If $f : [a,b] \to X$ is Pettis integrable, then f is Denjoy$_*$-Pettis integrable.

PROOF. Let f be Pettis integrable. Then f is Dunford integrable on $[a,b]$ and $x_E^{**} \in X$ for every measurable set E in $[a,b]$. Then x^*f is Lebesgue integrable and thus x^*f is Denjoy$_*$ integrable for all $x^* \in X^*$. Hence f is Denjoy$_*$-Pettis integrable. □

THEOREM 3.6. *If $f : [a,b] \to X$ is Dunford integrable on $[a,b]$, then f is Denjoy$_*$-Dunford integrable.*

PROOF. Let f be Dunford integrable on $[a,b]$. Then x^*f is Lebesgue integrable on $[a,b]$ for each $x^* \in X^*$. Since x^*f is Denjoy$_*$ integrable, f is Denjoy$_*$-Dunford integrable. □

Since Denjoy$_*$ integrable functions are Denjoy integrable, we can obtain the following theorem.

THEOREM 3.7. *Let $f : [a,b] \to X$ be a function. Then*

(a) if f is Denjoy$_*$-Bochner integrable, then f is Denjoy-Bochner integrable,

(b) if f is Denjoy$_*$-Pettis integrable, then f is Denjoy-Pettis integrable,

(c) if f is Denjoy$_*$-Dunford integrable, then f is Denjoy-Dunford integrable.

DEFINITION 3.8 [8]. A function $f : [a,b] \to X$ is Denjoy-McShane integrable on $[a,b]$ if there exists a continuous function $F : [a,b] \to X$ such that

(i) for each $x^* \in X^*$, x^*F is ACG and

(ii) for each $x^* \in X^*$, x^*F is approximately differentiable almost everywhere on $[a,b]$ and $(x^*F)'_{ap} = x^*f$ almost everywhere on $[a,b]$.

DEFINITION 3.9. A function $f : [a,b] \to X$ is Denjoy$_*$-McShane integrable on $[a,b]$ if there exists a continuous function $F : [a,b] \to X$ such that

(i) for each $x^* \in X^*$ x^*F is ACG_* and

(ii) for each $x^* \in X^*$ x^*F is differentiable almost everywhere on $[a,b]$ and $(x^*F)' = x^*f$ almost everywhere on $[a,b]$.

Clearly if f is Denjoy$_*$-McShane integrable, the f is Denjoy-McShane integrable.

THEOREM 3.10. *If $f : [a,b] \to X$ be McShane integrable, then f is Denjoy$_*$-McShane integrable.*

PROOF. Suppose that $f : [a,b] \to X$ is McShane integrable. Then for each $x^* \in X^*$ x^*f is McShane integrable and hence x^*f is Lebesgue integrable. Let $F(t) = (M)\int_a^t f$. Then F is continuous by [7, Theorem 8] and for each $x^* \in X^*$

$$x^*F(t) = (M)\int_a^t x^*f = (L)\int_a^t x^*f.$$

Thus x^*F is AC and thus x^*F is ACG_* and $(x^*F)' = x^*f$ almost everywhere on $[a,b]$. Hence f is Denjoy$_*$-McShane integrable. □

THEOREM 3.11. *If $f : [a,b] \to X$ is Denjoy$_*$-Bochner integrable on $[a,b]$, then f is Denjoy$_*$-McShane integrable.*

PROOF. Let $f : [a,b] \to X$ be Denjoy$_*$-Bochner integrable. Then there exists an ACG$_*$ function F such that $F' = f$ almost everywhere on $[a,b]$. For each $x^* \in X^*$ x^*F is ACG$_*$ and $(x^*F)' = x^*f$ almot everywhere on $[a,b]$. Hence f is Denjoy$_*$-McShane integrable. □

THEOREM 3.12. *If $f : [a,b] \to X$ is Denjoy$_*$-McShane integrable on $[a,b]$, then f is Denjoy$_*$-Pettis integrable.*

PROOF. Suppose f is Denjoy$_*$-McShane integrable and let $F(t)=(D_*$-$M)\int_a^t f$. Since x^*F is ACG_* and $(x^*F)' = x^*f$ almost everywhere on $[a,b]$ for each $x^* \in X^*$, x^*f is Denjoy$_*$ integrable. For every interval $[c,d]$ in $[a,b]$, we have

$$\begin{aligned} x^*(F(d) - F(c)) &= x^*F(d) - x^*F(c) \\ &= (D_*)\int_a^d x^*f - (D_*)\int_a^c x^*f \\ &= (D_*)\int_c^d x^*f. \end{aligned}$$

Since $F(d) - F(c) \in X$, f is Denjoy$_*$-Pettis integrable. □

A portion of a set $E \subset \mathbb{R}$ is a nonempty set P of the form $P = E \bigcap I$ where I is an open interval.

We can obtain the following two theorems from [5].

THEOREM 3.13. *If $f : [a,b] \to X$ is Denjoy$_*$-Bochner integrable on $[a,b]$, then each perfect set in $[a,b]$ contains a portion on which f is Bochner integrable.*

THEOREM 3.14. *Suppose that X contains no copy of c_0 and let $f : [a,b] \to X$ be a function. If f is Denjoy$_*$-Pettis integrable on $[a,b]$, then every perfect set in $[a,b]$ contains a portition on which f is Pettis integrable.*

THEOREM 3.15. *Suppose that X contains no copy of c_0 and let $f : [a,b] \to X$ be Henstock integrable. If f is Denjoy$_*$-McShane integrable on $[a,b]$, then every perfect set in $[a,b]$ contains a portition on which f is McShane integrable.*

PROOF. Let f be Denjoy$_*$-McShane integrable on $[a,b]$. Since f is Denjoy$_*$-Pettis by Theorem 3.12, by the Theorem 3.14 every perfect set in $[a,b]$ contains a portion on which f is Pettis integrable. Since f is Henstock integrable, by [3, Theorem 8] f is McShane integrable. □

The following example is due to Alexiewicz [1].

EXAMPLE 3.16. A Denjoy$_*$-McShane integrable function that is not Denjoy$_*$-Bochner integrable.

Let $\{r_k\}$ be an enumeration of the rational numbers in $[0,1)$ and for each pair of positive integers n and k let

$$I_n^k = (r_k + \frac{1}{n+1}, r_k + \frac{1}{n}).$$

For each k define $f_k : [0,1] \to l_2$ by

$$f_k(t) = \{(n+1)\chi_{I_n^k}(t)\}.$$

Then the series $\sum_k 4^{-k} f_k$ is l_2-valued almost everywhere on $[0,1]$. For each positive integer j let $A_j = \bigcup\{t \in [0,1] : |t - r_k| < 2^{-j-k}\}$ and let $A = \bigcap_j A_j$. Then $\mu(A) = 0$ and $\{r_k\} \subset A$. Define $g : [0,1] \to l_2$ by $g(t) = \sum_k 4^{-k} f_k(t)$ for t in $[0,1] - A$ and $g(t) = \theta$ for t in A. Then g is Pettis integrable and measurable, but not Denjoy-Bochner integrable by [5, Example 42]. Thus g is McShane integrable but not Denjoy$_*$-Bochner integrable. By Theorem 3.10, g is Denjoy$_*$-McShane integrable.

EXAMPLE 3.17. A Denjoy$_*$-Pettis integrable function that is not Denjoy$_*$-McShane integrable.

For each $n \in N$ let

$$I_n' = (\frac{1}{n+1}, \frac{n+\frac{1}{2}}{n(n+1)}),\ I_n'' = (\frac{n+\frac{1}{2}}{n(n+1)}, \frac{1}{n})$$

and define $f_n : [0,1] \to R$ by $f_n(t) = 2n(n+1)(\chi_{I_n'}(t) - \chi_{I_n''(t)})$. Define $f : [0,1] \to c_0$ by $f(t) = \{f_n(t)\}$. Then by [5, Example 44] x^*f is Lebesgue integrable on $[a,b]$. Hence x^*f is Denjoy$_*$ integrable on $[a,b]$ for each $x^* \in X^*$ and $\int_E f = \{\int_E f_n\}$ for every measurable set $E \subset [0,1]$. For each interval $I \subset [0,1]$ we have $\int_I f \in c_0$ by the choice of $\{f_n\}$ and it follows that f is Denjoy$_*$-Pettis integrable on $[0,1]$. From [8, Example 3.8] and Theorem 3.7 f is not Denjoy$_*$-McShane integrable.

The following example is due to Gordon [5].

EXAMPLE 3.18. A Denjoy$_*$-Bochner integrable function that is not Bochner integrable.

Let X be an infinite-dimensional Banach space. By the Dvoretsky-Rogers [2] Theorem there exists a series $\sum_n x_n$ in X that converges unconditionally but not absolutely convergent. For each positive n let $I_n = (\frac{1}{n+1}, \frac{1}{n})$ and define $f : [0,1] \to X$ by $f(t) = (\frac{1}{\mu(I_n)})x_n$ for t in I_n and $f(t) = \theta$ for all other values of t. The function f is measurable since it is countably valued, but f is not Bochner integrable on $[0,1]$ since

$$\int_0^1 \| f \| = \sum_n \int_{I_n} \frac{1}{\mu(I_n)} \| x_n \| = \sum_n \| x_n \| = \infty.$$

Define $F : [0,1] \to X$ by

$$F(t) = \frac{t - \frac{1}{n+1}}{\mu(I_n)} x_n + \sum_{k=n+1}^{\infty} x_k$$

for $t \in (\frac{1}{n+1}, \frac{1}{n}]$ and $F(0) = \theta$. Then F is continuous and $F' = f$ almost everywhere on $[0,1]$. Furthermore, the function F is ACG$_*$ on $[0,1]$ since F is AC_* on $\{0\}$ and on each of the intervals $[\frac{1}{n+1}, \frac{1}{n}]$. Hence the function f is Denjoy$_*$-Bochner integrable on $[0,1]$.

As with the Pettis integral, a Denjoy-Pettis integrable function must be weakly measurable.

THEOREM 3.19. *If $f : [a, b] \to X$ be Denjoy-Pettis integrable on $[a, b]$, then f is weakly measurable.*

PROOF. Since f is Denjoy-Pettis integrable on $[a, b]$, x^*f is Denjoy integrable on $[a, b]$ for each $x^*f \in X^*$. By [5, Theorem 12(a)] x^*f is measurable and hence f is weakly measurable. □

COROLLARY 3.20. *If $f : [a, b] \to X$ be Denjoy-McShane integrable on $[a, b]$, then f is weakly measurable.*

COROLLARY 3.21. *If $f : [a, b] \to X$ be Denjoy$_*$-McShane integrable on $[a, b]$, then f is weakly measurable.*

EXAMPLE 3.22. A Denjoy$_*$-McShane integrable function need not be measurable.

Define $f : [0, 1] \to l_\infty[0, 1]$ by $f(t) = \chi_{[0,t]}$, $t \in [0, 1]$. Then f is McShane integrable and not measurable [7]. By Theorem 3.10 f is Denjoy$_*$-McShane integrable.

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