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A GEOMETRIC REALIZATION OF (7/3)-RATIONAL KNOT

D. A. DEREVNIN AND YANGKOK KIM

ABSTRACT. Let $(p/q, n)$ denote the orbifold with its underlying space S^3 and a rational knot or link p/q as its singular set with a cyclic isotropy group of order n. In this paper we shall show the geometrical realization for the case $(7/3, n)$ for all $n \geq 3$.

1. Introduction

A well-known fact shows that the n-fold cyclic branched covering of the 3-sphere over a knot K has the same geometric structure over the orbifold (K, n). In general, by Thurston's hyperbolic surgery theorem(see [1], [7] for the manifold version which generalizes orbifolds), $(p/q, n)$ is a hyperbolic 3-orbifold for large n. But only few explicit geometric realizations are known(for case $(8/3, n)$ see [6]). In this paper we shall show the hyperbolic realization for the case $(7/3, n)\,, n \geq 3$.

2. Geometric Preliminary

We refer to [2] and [5] for more detailed information about geometrical properties of $\mathbb{H}^3$. We have to remember here a metric relations between a triple of lines in $\mathbb{H}^3$. Let L and M be an ordered oriented pair of lines in $\mathbb{H}^3$ and N is a common normal of them. Let ρ be a hyperbolic distance between L and M along N. We can canonically orient N from L to M if $\rho \neq 0$ and such that L, M, N is a right triple in other case. We shall consider the angle ϕ between L and M along N as positive if it correspond

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a clockwise rotation from first to second when you see along N according to orientation. The complex number

$$d(L,M) = \rho + i\phi$$

is called complex distance between L and M. Consider a configuration $(L,M;N)$ in the case when N has an arbitrary orientation. With $(L,M;N)$ we associate an complex number

$$\mu\,(L,M;N)\,,$$

called *width* of $(L,M;N)$, defined by

$$\mu = \pm d(L,M).$$

Here we have the sign plus if orientation of N is canonical and minus in other case. Consider now triple of oriented lines L_1, L_3, L_5 in $\mathbb{H}^3$. Since every pair of hyperbolic lines has the unique common normal we can consider a configuration $L_1, L_2, ..., L_6$ of oriented six lines in $\mathbb{H}^3$ with property: $L_i (i \ mod \ 6)$ is a common normal of L_{i-1} and L_{i+1}. Let

$$\sigma_i = \mu(L_{i-1}, L_{i+1}; L_i).$$

Then we have the following "Low of Cosines"(see [5]):

(1) $$\cosh\sigma_i = \cosh\sigma_{i-2}\cosh\sigma_{i+2} + \sinh\sigma_{i-2}\sinh\sigma_{i+2}\cosh\sigma_{i+3}.$$

It is famous(see [2] for instance) that in $\mathbb{H}^2$ there exist quadrilaterals with angles $\pi/2, \pi/2, \pi/2, \varphi$ if and only if $0 \le \varphi < \pi/2$. This quadrilateral is known as *Lambert quadrilateral.* Consider a quadrilateral with described property in $\mathbb{H}^3$. Let points A, B, C, D in $\mathbb{H}^3$ be vertices of the quadrilateral and $\varphi = \angle BCD$. Assume that there is no hyperbolic plane containing our quadrilateral. Let a, b, c and d are lengths of edges AB, BC, CD and DA respectively. Let also α, β, γ and δ are dihedral angles of the tetrahedron $ABCD$ in edges AB, BC, CD and DA respectively.

PROPOSITION 2.1.
(*i*) $\tanh a \tan\delta = \tanh d \tan\alpha$;
(*ii*) $\tanh a \cosh d = \tanh c \cos\delta$;
(*iii*) $\tanh c \sin\delta = \tan\alpha \sinh d$;
(*iv*) $\tanh b \sin\alpha = \tan\delta \sinh a$;
(*v*) $\cos\varphi = \frac{\sinh a \cos\alpha}{\sinh d \cos\delta}\left(\cosh^2 d - \cos^2\delta\right)$.

PROOF. Consider the six oriented lines $L_1, L_2, ..., L_6$ in $\mathbb{H}^3$ defined as follows. Let

$$L_2 = AB, \qquad L_3 = CB, \qquad L_5 = CD, \qquad L_6 = AD$$

and

$$L_1 \perp L_2, L_6, \qquad L_4 \perp L_3, L_5$$

with canonical orientation. Then

$$\begin{aligned}
\sigma_1 &= 0 + \frac{\pi}{2} i, \\
\sigma_2 &= a + \left(\alpha - \frac{\pi}{2}\right) i, \\
\sigma_4 &= 0 + \varphi i, \\
\sigma_6 &= d + \left(\frac{\pi}{2} - \delta\right) i.
\end{aligned}$$

Noticing that $\cosh \sigma_1 = 0$ from (1) we have

$$\begin{aligned}
\cos \varphi &= \cosh \sigma_4 = -i \sinh(x + i\gamma) i \sinh(y - i\delta) \\
&= (\sinh x \sinh y \cos \gamma \cos \delta + \cosh x \cosh y \sin \gamma \sin \delta) + \\
&\qquad i (\cosh x \sinh y \sin \gamma \cos \delta - \sinh x \cosh y \cos \gamma \sin \delta).
\end{aligned}$$

Comparing the real and imaginary parts we have

(2) $$\cos \varphi = \sinh a \sinh d \cos \alpha \cos \delta + \cosh a \cosh d \sin \alpha \sin \delta$$

and

(3) $$0 = \cosh a \sinh d \sin \alpha \cos \delta - \sinh a \cosh d \cos \alpha \sin \delta.$$

It is easy to see that (3) is equivalent to (i). We can rewrite (3) in the following form:

(4) $$\frac{\cosh a \sin \alpha}{\cosh d \sin \delta} = \frac{\sinh a \cos \alpha}{\sinh d \cos \delta}.$$

Substituting expression for $\cosh a \sin \alpha$ from (4) in (2) we have

(5) $$\begin{aligned}
\cos \varphi &= \frac{\sinh a \cos \alpha}{\sinh d \cos \delta} \left(\cosh^2 d \sin^2 \delta + \sinh^2 d \cos^2 \delta\right) \\
&= \frac{\sinh a \cos \alpha}{\sinh d \cos \delta} \left(\cosh^2 d - \cos^2 \delta\right).
\end{aligned}$$

It gives (v).

Let $\nu = \angle CAD$. From the right-angled hyperbolic triangle ACD we obtain(see [2] and [5] for hyperbolic and spherical trigonometry)

(6) $$\tan \nu = \frac{\tanh c}{\sinh d}.$$

From the other side we can consider the spherical triangle $B'C'D'$, where B', C' and D' are intersection of AB, AC and AD respectively with sphere of a sufficiently small radius about A. It gives

$$\tan \nu = \frac{\tan \alpha}{\sin \delta}. \tag{7}$$

Elimination of ν now gives (iii). Elimination of α from (i) and (iii) yields (ii). The equality (iv) follows from (iii) by symmetry consideration. □

COROLLARY 2.1. *Let we have a quadrilateral in* $\mathbb{H}^3$ *(not necessary plane) with plane angles* $\pi/2$, $\pi/2$, $\pi/2$, φ . *Then*

$$0 \leq \varphi < \pi/2.$$

PROOF. If a quadrilateral lies in a plane, the result is famous. If not, we notice that

$$a, d > 0 \tag{8}$$

and

$$0 < \alpha, \delta < \pi. \tag{9}$$

From (4) we obtain

$$\frac{\sinh a \cos \alpha}{\sinh d \cos \delta} = \frac{\cosh a \sin \alpha}{\cosh d \sin \delta} > 0$$

With (5) it gives

$$\cos \varphi > 0.$$

Thus

$$\varphi < \frac{\pi}{2}.$$

□

3. Main Results

The fundamental group Γ_n of $(7/3, n)$ has the following presentation(see [3], for instance)

$$\Gamma_n = \langle s, t; s^n = t^n = ws^{-1}w^{-1}t = 1, w = sts^{-1}t^{-1}st \rangle. \tag{10}$$

Let $r = sts^{-1}t^{-1}$ then we have

$$\Gamma_n = \langle s, t; r = sts^{-1}t^{-1}, s^n = t^n = rrs^{-1}r^{-1}t = 1 \rangle. \tag{11}$$

This presentation will be useful for our purposes.

Consider the group $\langle S, T, \rangle < PSL(2, C)$ generated by the following matrices:

$$S = \begin{pmatrix} \cos(\pi/n) + i\sin(\pi/n)\cosh z & i\sin(\pi/n)\sinh z \\ -i\sin(\pi/n)\sinh z & \cos(\pi/n) - i\sin(\pi/n)\cosh z \end{pmatrix} \text{ and}$$

$$T = \begin{pmatrix} \cos(\pi/n) + i\sin(\pi/n)\cosh z & -i\sin(\pi/n)\sinh z \\ i\sin(\pi/n)\sinh z & \cos(\pi/n) - i\sin(\pi/n)\cosh z \end{pmatrix},$$

where n is a positive integer and z is a complex number. We use the Poincare model of $\mathbb{H}^3$ to describe geometrically the action of the generators S and T on $\mathbb{H}^3$ (we refer to [5] for more detailed information about geometrical properties of matrix action on $\mathbb{H}^3$). The elements S and T are both rotations of order n. Axis of S coincides with imaginary axis j. The hyperbolic line with end points $[-i, i]$ is a mutual perpendicular of axes of T and S. The real part of z is the distance from axis of S to axis of T, and the imagine part is the angle from one to another in obvious sense.

We need the following lemma.

LEMMA 3.1. *Let* $z = \cosh^{-1}((2\cos^2(\pi/n) - w)/(2\sin^2(\pi/n))$, *where* w *is a nonreal solution of cubic equation*

(12)
$$x^3 - (4\cos^2(\pi/n) + 1)x^2 + (12\cos^2(\pi/n) - 2)x - 8\cos^2(\pi/n) + 1 = 0$$

and $R = STS^{-1}T^{-1}$.

Then

$$S^n = T^n = RRS^{-1}R^{-1}T = I.$$

PROOF. From our definitions follows that

$$S^n = T^n = I.$$ □

We claim that $TR - RSR^{-1} = O$, that is equivalent to $RRS^{-1}R^{-1}T = I$. Let $\alpha = \cos(2\pi/n)$ and $\beta = \cosh z$. Then by the choice of w, we note that $w = 2\cos^2(\pi/n) - 2\sin^2(\pi/n)\cosh z = 1 + \alpha - \beta - \alpha\beta$ satisfies the cubic equation (12). Thus in terms of α and β, we have a relation

$$(1-\alpha)^3\beta^3 + (1-\alpha)^2\beta^2 + (\alpha^2 - 2\alpha - 1)(\alpha - 1)\beta + (-1 + \alpha + 2\alpha^2 - \alpha^3) = 0.$$

Let E be the left hand side of the above relation. Then a direct calculation shows that $TR - RSR^{-1}$ has as an (1,1)-component

$$\frac{i}{2}\sqrt{1-\alpha}\sqrt{1+\beta}(1+\alpha-\beta-\alpha\beta)E.$$

Similarly we can show that the other components of $TR - RSR^{-1}$ are also multiples of E. Thus $TR - RSR^{-1} = O$.

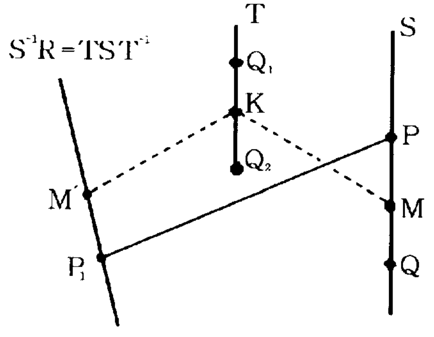

FIGURE 1

If the action of a matrix $U \in PSL(2, C)$ on $\mathbb{H}^3$ has an axis we shall use further notation L_U for the axis of U in $\mathbb{H}^3$. Consider axes of rotations S,T and $T^{-1}ST$ (see figure 1). Let MK,$M'K$ and PP_1 be a mutual perpendiculars for pairs of axes (L_S, L_T), $(L_{TST^{-1}}, L_T)$ and $(L_S, L_{TST^{-1}})$ respectively and Q a point on L_S such that M is a middle point of the segment $[P, Q]$. Let z be as in the lemma 3.1. Denote

$$P_2 = RS^{-1}(P),$$

$$P_3 = S(P_1),$$

$$Q_2 = RSR^{-1}(P_3),$$

$$Q_3 = T^{-1}(P_2),$$

$$Q_1 = S^{-1}(Q_3).$$

As an easy sequence of the point definitions and lemma 3.1 we obtain the following.

LEMMA 3.2.

$$P, Q \in L_S,$$

$$P_1 \in L_{TST^{-1}},$$

$$P_2 \in L_{RSR^{-1}},$$

$$P_3 \in L_{R^{-1}TR},$$

$$Q_1, Q_2 \in L_T,$$

$$Q_3 \in L_{STS^{-1}}.$$

Let F be a polyhedron in $\mathbb{H}^3$ with vertices P, P_1, P_2, P_3 and Q, Q_1, Q_2, Q_3 (see figure 2).

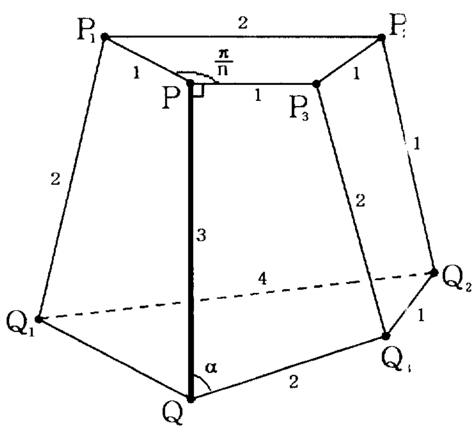

FIGURE 2

LEMMA 3.3.

$$(13)\qquad \begin{array}{llllll} S: & P \to P & T: & Q_1 \to Q_1 & R: & P \to P_2 \\ & Q \to Q & & Q_2 \to Q_2 & & P_1 \to P_3 \\ & P_1 \to P_3 & & Q \to P_1 & & P_2 \to Q_3 \\ & Q_1 \to Q_3 & & Q_3 \to P_2 & & P_3 \to Q_2 \end{array}$$

PROOF. The first column follows immediately from definitions of the points P, Q, P_3, Q_3. From definition of the point Q_3 follows that $T(P_2) = Q_3$. The equality $T(Q) = P_1$ follows from geometrical definitions of the points P_1 and Q. To see that $T(Q_1) = Q_1$ and $T(Q_2) = Q_2$ we shall check that $Q_1, Q_2 \in L_T$. We have

$$Q_1 = S^{-1}(Q_3) = S^{-1}T^{-1}(P_2) = S^{-1}T^{-1}RS^{-1}(P).$$

The point $P \in L_S$. Thus $Q_1 \in L_{VSV^{-1}}$, where $V = S^{-1}T^{-1}RS^{-1}$. From the lemma 3.1 and (10) follows that

$$\left(STS^{-1}T^{-1}ST\right) S^{-1} \left(STS^{-1}T^{-1}ST\right)^{-1} T = 1$$

or, in other form

$$S^{-1}T^{-1}STS^{-1}T^{-1}S^{-1}TST^{-1}S^{-1}TST = 1.$$

One can see easily that it is equivalent to

$$VS^{-1}V^{-1}T = 1$$

or, the same

$$VSV^{-1} = T.$$

It means that $Q_1 \in L_T$. The analogous considerations for the point

$$Q_2 = RSR^{-1}S(P_1)$$

shows that $Q_2 \in L_T$. Consider action of R. Since $S(P) = P$ we have

$$P_2 = RS^{-1}(P) = R(P).$$

The scheme

$$\begin{array}{ccccccccc} & S & & T & & S^{-1} & & T^{-1} & \\ P_3 & \leftarrow & P_1 & \leftarrow & Q & \leftarrow & Q & \leftarrow & P_1 \\ Q_3 & \leftarrow & Q_1 & \leftarrow & Q_1 & \leftarrow & Q_3 & \leftarrow & P_2 \end{array}$$

gives the equalities $R(P_1) = P_3$ and $R(P_2) = Q_3$. The explicit chain of equalities

$$Q_2 = RSR^{-1}(P_3) = RS(P_1) = R(P_3)$$

completes the proof. □

As it follows from the lemma 3.3, there are following four classes of equivalencies for edges of F:

$$\{1\} = \{PP_1, PP_3, P_2P_3, P_2Q_2, Q_2Q_3\},$$

$$\{2\} = \{QQ_1, QQ_3, P_1P_2, P_3Q_3, P_1Q_1\},$$

$$\{3\} = \{PQ\},$$

$$\{4\} = \{Q_1Q_2\}.$$

LEMMA 3.4. *Let $z = r + i\delta$ be as in the* lemma 3.1. *Let also r_1, r_2 and r_3 be the length of the edges from the first, second and third class of equivalency. Then the polyhedron F has the following metrical properties:*

(14) $$(i) \quad PQ = Q_1Q_2\ ;$$

$$(ii) \quad r_1 = Re(d_1), \quad r_3 = Re(d_3),$$

where

$$\cosh d_1 = \cosh^2 z - \sinh^2 z \cos\frac{2\pi}{n},$$

$$\cosh\frac{d_3}{2} = \frac{i\cot\frac{\pi}{n}}{\cosh z};$$

$$(iii) \quad \tanh\frac{r_2}{2} = \frac{\tan\delta\sinh a}{\sin\alpha},$$

where a and α are determined by

$$\tan\alpha = \frac{\sin\delta\tanh r_3/2}{\sinh r},$$

$$\tanh a = \frac{\cos\delta\tanh r_3/2}{\cosh r};$$

(iv)

(15) $$\angle QPP_1 = \angle QPP_3 = \angle Q_1Q_2P_2 = \angle Q_1Q_2Q_3 = \frac{\pi}{2},$$

(16) $$\angle P_1PP_3 = \angle PP_3P_2 = \angle P_3P_2Q_2 = \angle P_2Q_2Q_3 = \frac{2\pi}{n},$$

$$(17) \qquad \varphi = \angle PQQ_1 = \angle PQQ_3 = \angle Q_2Q_1P_1 = \angle Q_2Q_1Q < \frac{\pi}{2}.$$

PROOF. The edge PP_1 is a mutual perpendicular of axes L_S and $L_{TST^{-1}}$. Remark that as a sequence all edges from the first class of equality are common perpendiculars of correspondent axes(see the lemma 3.2).The edge $PP_3 = S(PP_1)$. It implies that

$$\angle QPP_1 = \angle QPP_3 = \frac{\pi}{2},$$

and

$$\angle P_1PP_3 = \frac{2\pi}{n}.$$

Analogously we obtain all other equalities from (15) and (16).

Consider such rotation of order two U that $S = UTU^{-1}$. It determines an automorphism of $\langle S, T\rangle$. Notice that L_U pass through the middle of segment $[M, K]$ and perpendicular to h-line MK. It is easy to see that

$$(18) \qquad U(L_S) = L_T$$

and

$$U(L_{R^{-1}TR}) = L_{RSR^{-1}}.$$

The common perpendicular PP_3 of L_S and $L_{R^{-1}TR}$ comes to common perpendicular Q_2P_2 of L_T and $L_{RSR^{-1}}$. Analogously we obtain $U(PP_1) = Q_2Q_3$. We have

$$(19) \qquad U(P) = Q_2, U(P_3) = P_2, U(Q_3) = P_1.$$

With (18) it means that the tetrahedron PP_3Q_3Q comes to the tetrahedron $Q_2P_2P_1Q_1$. Thus

$$PQ = Q_1Q_2$$

and

$$(20) \qquad \angle PQQ_3 = \angle Q_2Q_1P_1.$$

From the lemma 3.3 and (20) it follows now that equalities in (16) are true.

To find the length of edges consider first the six of oriented lines

$$\begin{aligned} L_1 &= L_T, \qquad L_3 = L_S, \qquad L_5 = L_{TST^{-1}}, \\ L_2 &= KM, \qquad L_4 = PP_1, \qquad L_6 = M'K. \end{aligned}$$

Notice that $\sigma_1 = 2\pi/n$, $\sigma_2 = \sigma_4 = z$ and $\sigma_3 = \sigma_5 = d_3/2$, where $Re(d_3) = PQ = r_3$. Application of (1) for different combinations indices gives (ii). Consider the quadrilateral Q_1KVW, where the points Q_1,K are as above and V, W are the middle points of segments KM', Q_1P_1 respectively. It is easy to see that we obtain the Lambert quadrilateral and (iii) follows from the proposition 2.1. It remains to show now that $\varphi < \pi/2$. As it follows directly from corollary 2.1

$$\varphi = \angle Q_2Q_1P_1 = \angle KQ_1W < \frac{\pi}{2}.$$

It completes the proof of the lemma. □

Now we have to check the Poincare condition on the dihedral angles of F. Using the lemmas 3.1, 3.3 one can see that the overgluing of edges of F corresponds to the relations:

$$PP_1, PP_3, P_2P_3, P_2Q_2, Q_2Q_3 \rightarrow RRS^{-1}R^{-1}T = 1,$$

$$QQ_1, QQ_3, P_1P_2, P_3Q_3, P_1Q_1 \rightarrow STS^{-1}T^{-1}R^{-1} = 1,$$

$$PQ \rightarrow S^n = 1,$$

$$Q_1Q_2 \rightarrow S^n = 1.$$

It means that the sum of dihedral angles in edges from the same class of equivalency is equal $2k\pi$. We shall show that for all cases $k = 1$. Consider the overgluing of vertex Q of F. As it follows from the lemmas 3.1, 3.3 and 3.4 the picture on the sphere Σ_ε of a small radius with center Q will correspond to figure 3.

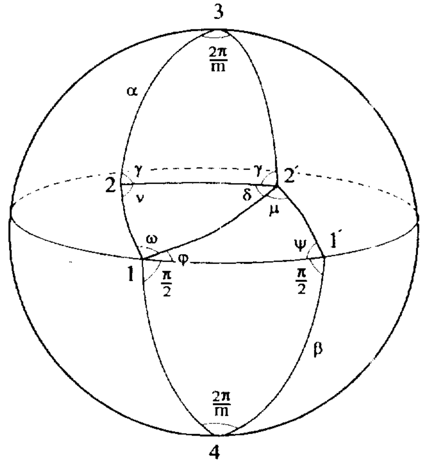

FIGURE 3

Here

$$\begin{aligned} 1 &= \Sigma_\varepsilon \cap QQ_3, \\ 2 &= \Sigma_\varepsilon \cap QP' \left(= T^{-1}S^{-1}R^{-1}(Q_2P_2)\right), \\ 1' &= S^{-1}(1), \qquad 2' = S^{-1}(2), \\ \{3,4\} &= \Sigma_\varepsilon \cap L_S. \end{aligned}$$

Because of rotation symmetry the spherical tessellation is determined completely by position of the points 1 and 2. Since $\beta = \pi/2$ and $\alpha < \pi/2$(the lemma 3.4) the situation is sufficiently fixed to conclude that(we refer to fig. 3 for notations)

$$2\gamma + \delta + \mu + \nu = 2\pi$$

and

$$\varphi + \psi + \omega + \frac{\pi}{2} + \frac{\pi}{2} = 2\pi.$$

Thus the sums of dihedral angles about the edges from the second and first classes of equivalency are equal 2π. The remark that the Poincare condition about axes of rotations L_S and L_T are obviously hold complete the proof of the following theorem.

THEOREM 3.1. *The group*

$$\Gamma_n = \langle s, t; r = sts^{-1}t^{-1}, s^n = t^n = rrs^{-1}r^{-1}t = 1\rangle$$

acts as discontinuous group of hyperbolic motions on hyperbolic 3*-space for any* $n > 2$. *A fundamental domain is the polyhedron* F *described above.*

COROLLARY 3.1. *Let* z *be as in the* lemma 3.1. *Then*

$$\Theta : s \longrightarrow S = \begin{pmatrix} \cos\frac{\pi}{n} + i\sin\frac{\pi}{n} & 0 \\ 0 & \cos\frac{\pi}{n} - i\sin\frac{\pi}{n} \end{pmatrix},$$
$$t \longrightarrow T = \begin{pmatrix} \cos\frac{\pi}{n} + i\sin\frac{\pi}{n}\cosh z & -i\sin\frac{\pi}{n}\sinh z \\ i\sin\frac{\pi}{n}\sinh z & \cos\frac{\pi}{n} - i\sin\frac{\pi}{n}\cosh z \end{pmatrix},$$

gives a faithful representation for Γ_n *in* $PSL(2, C)$.

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D. A. Derevnin
Department of Mathematics
Pusan National University
Pusan 609–735, Korea
E-mail: derek@hyowon.cc.pusan.ac.kr

Yangkok Kim
Department of Mathematics
Dongeui University
Pusan 614–714, Korea
E-mail: ykkim@hyomin.dongeui.ac.kr