

A NOTE ON THE WEAK LAW OF LARGE NUMBERS FOR EXCHANGEABLE RANDOM VARIABLES*

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ABSTRACT. In this note, we study a weak law of large numbers for sequences of exchangeable random variables. As a special case, we have an extension of Kolmogorov's generalization of Khintchine's weak law of large numbers to i.i.d. random variables.

1. Introduction and preliminaries

The classical weak law of large numbers due to Khintchine asserts that if $\{X_n\}$ is a sequence of independent identically distributed (i.i.d.) random variables, then $S_n/n \rightarrow EX_1$ in probability whenever $E|X_1| < \infty$ where $S_n = \sum_{i=1}^n X_i$. Of course, this result is itself an immediate consequence of the Kolmogorov's strong law of large numbers. But the Khintchine's weak law of large numbers can hold, in slightly modified form, even if $E|X_1| = \infty$. The weak law of large numbers is sometimes called Kolmogorov's generalization of Khintchine's weak law of large numbers. It was stated in a theorem of Kolmogorov (1929) as follows. If X_1, X_2, \dots , are i.i.d., then there exist constants a_n with $S_n/n - a_n \rightarrow 0$ in probability if and only if $nP(|X_1| > n) \rightarrow 0$ as $n \rightarrow \infty$ (Revesz(1968, p.51). In the theorem we can take $a_n = E(X_1 I_{\{|X_1| \leq n\}})$ (Chow and Teicher(1988, p.128)). However, there appears to have been no discussion on the classical weak law of large numbers of this form for exchangeable random variables. Hence we address this problem in this paper and as a special case we have an extension of Kolmogorov's generalization of Khintchine's weak law of large numbers to i.i.d. random variables.

Let $X_1, X_2, \dots, X_n, \dots$ be a sequence of random variables. We say that it is exchangeable if the joint distribution of (X_1, X_2, \dots, X_n) is the

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same as that of $(X_{\pi(1)}, \dots, X_{\pi(n)})$ for each $n \geq 1$ where $\pi(1), \dots, \pi(n)$ is a permutation of $\{1, 2, \dots, n\}$. Let \mathcal{F} be the class of one-dimensional distribution functions and \mathcal{U} be the σ -field generated by the topology of weak convergence of the distribution functions. Then, de Finetti's theorem asserts that for an infinite sequence of exchangeable random variables $\{X_n\}$ there exists a probability measure μ on $(\mathcal{F}, \mathcal{U})$ such that

$$P\{g(X_1, \dots, X_n) \in B\} = \int_{\mathcal{F}} P_F\{g(X_1, \dots, X_n) \in B\}d\mu(F)$$

for any Borel set B and any Borel function $g : \mathcal{R}^n \rightarrow \mathcal{R}, n \geq 1$. Moreover, $P_F\{g(X_1, \dots, X_n) \in B\}$ is computed under the assumption that the random variables $\{X_n\}$ are i.i.d. with common distribution F . We define $E_F g(X_1, \dots, X_n) = \int g(X_1, \dots, X_n)dP_F$. Blum, Chernoff, Rosenblatt, and Teicher (1958) showed that for a sequence of exchangeable random variables $\{X_n\}$ such that $EX_1 = 0$ and $EX_1^2 < \infty, n^{-\frac{1}{2}} \sum_{j=1}^n X_j \rightarrow N(0, \sigma^2)$ in distribution if and only if $E_F X_1 = 0$ μ -a.s. and $E_F X_1^2 = \sigma^2$ μ -a.s., where $E_F X_1 = 0$ μ -a.s. and $E_F X_1^2 = \sigma^2$ are equivalent to $EX_1 X_2 = 0$ and $E(X_1^2 - \sigma^2)(X_2^2 - \sigma^2) = 0$, respectively. Taylor and Hu (1987) showed that for a sequence of exchangeable random variables $\{X_n\}$ such that $E_F |X_1| < \infty$ μ -a.s., $E_F X_1 = 0$ μ -a.s. if and only if $n^{-1} \sum_{j=1}^n X_j \rightarrow 0$ a.s. Hong and Kwon (1993) and Zang and Taylor (1995) showed that for a sequence of exchangeable random variables $\{X_n\}$ and for a constant $0 < \sigma < \infty, \lim_{n \rightarrow \infty} \sup \sum_{j=1}^n X_j / (2n \log \log n)^{\frac{1}{2}} = \sigma$ a.s. if and only if $E_F X_1 = 0$ μ -a.s. and $\sigma_F^2 = E_F(X_1 - E_F X_1)^2 = \sigma^2$ μ -a.s.

2. The weak law of large numbers

In this section, we study a weak law of large numbers for sequences of exchangeable random variables.

THEOREM 2.1. *Let $\{X_n, n \geq 1\}$ be a sequence of exchangeable random variables such that*

$$(2.1) \quad nP\{|X_1|^p > n\} \rightarrow 0$$

for some $0 < p < 2$ and

$$(2.2) \quad \lim_{n \rightarrow \infty} n^{(2-\frac{2}{p})} \sigma_{\mu}^2(E_F X_1 I_{\{|X_1|^p \leq n\}}) = 0$$

where $\sigma_\mu^2(E_F X_1 I_{\{|X_1|^p \leq n\}}) = \int_{\mathcal{F}} \{E_F X_1 I_{\{|X_1|^p \leq n\}} - E X_1 I_{\{|X_1|^p \leq n\}}\}^2 d\mu(F)$. Then

$$(2.3) \quad \frac{S_n - nE X_1 I_{\{|X_1|^p \leq n\}}}{n^{1/p}} \xrightarrow{P} 0.$$

PROOF. Set $X'_j = X_j I_{\{|X_j|^p \leq n\}}$ for $1 \leq j \leq n$ and $S'_n = \sum_{j=1}^n X'_j$. Then, for each $n \geq 2$ and for $\epsilon > 0$, $P\{|(S_n/n^{1/p}) - (S'_n/n^{1/p})| \geq \epsilon\} \leq P\{S_n \neq S'_n\} \leq P\{\cup_{j=1}^n [X_j \neq X'_j]\} \leq nP\{|X_1|^p > n\}$, so that (2.1) entails $(S'_n/n^{1/p}) - (S_n/n^{1/p}) \xrightarrow{P} 0$. Thus to prove (2.3) it suffices to verify that

$$(2.4) \quad \frac{S'_n - ES'_n}{n^{1/p}} \xrightarrow{P} 0.$$

By de Finetti's theorem,

$$\begin{aligned} E(S'_n - ES'_n)^2 &= \int_{\mathcal{F}} E_F(S'_n - ES'_n)^2 d\mu(F) \\ &= \int_{\mathcal{F}} E_F[(S'_n - E_F S'_n) + (E_F S'_n - ES'_n)]^2 d\mu(F) \\ &= \int_{\mathcal{F}} [E_F(S'_n - E_F S'_n)^2 + (E_F S'_n - ES'_n)^2] d\mu(F) \\ &= \int_{\mathcal{F}} \sum_{i=1}^n \sigma_F^2(X_i') d\mu(F) + \int_{\mathcal{F}} (E_F S'_n - ES'_n)^2 d\mu(F) \\ &\leq \int_{\mathcal{F}} \sum_{i=1}^n E_F(X_i')^2 d\mu(F) + n^2 \int_{\mathcal{F}} (E_F X_1' - E X_1')^2 d\mu(F) \\ &= n \int_{\mathcal{F}} E_F(X_1')^2 d\mu(F) + n^2 \sigma_\mu^2(E_F X_1') \\ &= nE(X_1')^2 + n^2 \sigma_\mu^2(E_F X_1'). \end{aligned}$$

From (2.2), $n^2 \sigma_\mu^2(E_F X_1') = o(n^{2/p})$ and hence it remains to show that

$nE(X_1')^2 = o(n^{\frac{2}{p}})$. Using summation by parts,

$$\begin{aligned} nE(X_1')^2 &= n \sum_{j=1}^n \int_{\{j-1 < |X_1|^p \leq j\}} X_1^2 dP \\ &\leq n \sum_{j=1}^n j^{2/p} [P\{|X_1|^p > j-1\} - P\{|X_1|^p > j\}] \\ &= n[P\{|X_1|^p > 0\} - n^{2/p}P\{|X_1|^p > n\}] \\ &\quad + \sum_{j=1}^{n-1} ((j+1)^{2/p} - j^{2/p})P\{|X_1|^p > j\}] \\ &\leq n[1 + c \sum_{j=1}^{n-1} ((j+1)^{\frac{2}{p}-1} - j^{\frac{2}{p}-1})jP\{|X_1|^p > j\}], \end{aligned}$$

where c is a constant independent of n . By the hypothesis (2.1), $jP\{|X_1|^p > j\}$ goes to zero as $j \rightarrow \infty$ and $\sum_{j=1}^n ((j+1)^{\frac{2}{p}-1} - j^{\frac{2}{p}-1}) = (n+1)^{\frac{2}{p}-1} - 1$. Thus, by Toeplitz Lemma (see Ash(1972, p.270), $nE(X_1')^2 = o(n^{2/p})$, which implies (2.4) and hence establishes (2.3).

If X_1, X_2, \dots , are i.i.d., $(EX_1)^2 = E(X_1X_2) = \int_{\mathcal{F}} E_F(X_1X_2)d\mu(F) = \int_{\mathcal{F}} (E_F X_1)^2 d\mu(F)$ and hence $\sigma_{\mu}^2(E_F X_1 I_{\{|X_1|^p \leq n\}}) = \int_{\mathcal{F}} (E_F X_1 I_{\{|X_1|^p \leq n\}})^2 d\mu(F) - (EX_1 I_{\{|X_1|^p \leq n\}})^2 = 0$ for all n . Thus (2.2) holds and we have the following result for i.i.d. case. □

COROLLARY 2.1. *Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables such that $nP\{|X_1|^p > n\} \rightarrow 0$ for some $0 < p < 2$. Then*

$$\frac{S_n - nEX_1 I_{\{|X_1|^p \leq n\}}}{n^{1/p}} \xrightarrow{P} 0$$

as $n \rightarrow \infty$.

We now consider a converse of Theorem 2.1. In this case we need stronger conditions.

THEOREM 2.2. *Let $\{X_n, n \geq 1\}$ be a sequence of exchangeable random variables such that*

$$(2.5) \quad \frac{S_n - nEX_1 I_{\{|X_1|^p \leq n\}}}{n^{1/p}} \xrightarrow{P_F} 0 \quad \mu - a.s.$$

for some $0 < p < 2$ and

$$(2.6) \quad \int_{\mathcal{F}} \sup_n \{nP_F\{|X_1|^p > n\}\} d\mu(F) < \infty.$$

The we have

$$(2.7) \quad nP\{|X_1|^p > n\} \longrightarrow 0$$

as $n \rightarrow \infty$.

PROOF. Suppose (2.5) and (2.6) holds. If we set $C_n = nEX_1I_{\{|X_1|^p \leq n\}}$ and $d_n = C_n - C_{n-1}$, $n \geq 1$, $C_0 = 0$,

$$\frac{X_n - d_n}{n^{1/p}} = \frac{S_n - C_n}{n^{1/p}} - \frac{(n-1)^{1/p}}{n^{1/p}} \left(\frac{S_{n-1} - C_{n-1}}{(n-1)^{1/p}} \right) \xrightarrow{P_F} 0 \quad \mu - a.s.,$$

whence $(X_n - d_n)/n^{1/p} \xrightarrow{P_F} 0$, necessitating $d_n = o(n^{1/p})$. By Lévy's inequality (see Chow and Teicher(1988), Lemma 3.3.5), for any $\epsilon > 0$ and $\mu - a.s. F$

$$(2.8) \quad \begin{aligned} P_F\{ \max_{1 \leq j \leq n} |S_j - C_j - m_F(S_j - C_j - S_n + C_n)| \geq \frac{1}{2}n^{1/p}\epsilon \} \\ \leq 2P_F\{|S_n - C_n| \geq \frac{1}{2}n^{1/p}\epsilon\} = o(1) \end{aligned}$$

where $m_F(X)$ is a median of X with respect to P_F .

But, taking $X_j = S_j - C_j$ in Exercise 3.3.7 of Chow and Teicher(1988),

$$(2.9) \quad \max_{1 \leq j \leq n} |m_F(S_j - C_j - S_n + C_n)| = o(n^{1/p}) \quad \mu - a.s.$$

Thus, from (2.8) and (2.9), for all $\epsilon > 0$

$$(2.10) \quad \lim_{n \rightarrow \infty} P_F\{ \max_{1 \leq j \leq n} |S_j - C_j| < n^{1/p}\epsilon \} = 1 \quad \mu - a.s.$$

Moreover, for $\max_{1 \leq j \leq n} |d_j| < n^{1/p}\epsilon$ and hence for all large n and for $\mu - a.s. F$,

$$\begin{aligned} P_F\{ \max_{1 \leq j \leq n} |S_j - C_j| < n^{1/p}\epsilon \} &\leq P_F\{ \max_{1 \leq j \leq n} |X_j - d_j| < 2n^{1/p}\epsilon \} \\ &\leq P_F\{ \max_{1 \leq j \leq n} |X_j| < 3n^{1/p}\epsilon \}, \end{aligned}$$

which, in conjunction with (2.10), yields

$$P_F^n\{|X_1| < 3n^{1/p}\epsilon\} = P_F\{\max_{1 \leq j \leq n} |X_j| < 3n^{1/p}\epsilon\} \longrightarrow 1 \quad \mu - a.s.$$

equivalently, for all $\epsilon > 0$

$$(2.11) \quad n \log[1 - P_F\{|X_1| \geq 3n^{1/p}\epsilon\}] \rightarrow 0 \quad \mu - a.s.$$

as $n \rightarrow \infty$. Since $\log(1 - x) = -x + o(x)$ as $x \rightarrow 0$, (2.11) entails

$$(2.12) \quad nP_F\{|X_1|^p > n\} \rightarrow 0 \quad \mu - a.s.$$

as $n \rightarrow \infty$. Hence, by the Dominated Convergence Theorem using (2.6) and (2.12), we have the desired result. \square

COROLLARY 2.2. *Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables such that $(S_n - nEX_1 I_{\{|X_1|^p \leq n\}})/n^{1/p} \xrightarrow{P} 0$ for some $0 < p < 2$. Then we have $nP\{|X_1|^p > n\} \rightarrow 0$ as $n \rightarrow \infty$.*

Combining Corollary 2.1 and 2.2, we have an extension of the Kolmogorov's generalization of Khintchine's weak law of large numbers to i.i.d. random variables.

COROLLARY 2.3. *If $\{X_n, n \geq 1\}$ is a sequence of i.i.d. random variables and $0 < p < 2$, then $(S_n - nEX_1 I_{\{|X_1|^p \leq n\}})/n^{1/p} \xrightarrow{P} 0$ iff $nP\{|X_1|^p > n\} \rightarrow 0$, as $n \rightarrow \infty$.*

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