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SOJOURN TIME DISTRIBUTIONS FOR $M/M/c$ G-QUEUE

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ABSTRACT. We consider an $M/M/c$ queue with two types of customers, positive customers and negative customers. Positive customers are ordinary ones who upon arrival, join a queue with the intention of getting served and each arrival of negative customer removes a positive customer in the system, if any presents, and then is disappeared immediately. The Laplace-Stieltjes transforms (LST's) of the sojourn time distributions of a tagged customer, joinly with the probability that the tagged customer completes his service without being removed are derived under the combinations of various service disciplines; FCFS, LCFS and PS and removal strategies; RCE, RCH and RCR.

1. Introduction

Since Gelenbe [7] introduced the notion of negative customers to represent inhibitator signal in neural network, there has been a growing interest not only in networks of queue (Gelenbe [7], Chao [4], Chao and Pinedo [5,6], Handerson [9]) but also in single node queue with negative customers (Bayer and Boxma [1] and Boucherie and Boxma [2], Gelenbe et al. [8], Harrison and Pitel [10, 11]). Positive customers are ordinary ones who upon arrival, join the queue with the intention of getting served and then leaving the system. In contrast with the ordinary customers, upon arrival to a system, a negative customer removes one ordinary customer, if any presents, from the system and then leaves the system immediately. The queue with negative customers has been

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called G-queue. Negative customers, for example, can be interpreted as removal signals in production networks, as inhibitor signals in neural networks and as commands to delete some transactions in database.

In this paper, we consider an $M/M/c$ G-queue. The purpose of this paper is to derive the LST's of the sojourn time distributions of a tagged customer, jointly with the probability that the tagged customer completes his service without being removed under the combinations of various service disciplines and removal strategies. Service disciplines such as FCFS (first come first served), LCFS (last come first served) and PS (processor sharing) for positive customers and the removal strategies RCH (removal of one among the customers in service), RCE (removal of the customer at the end of queue) and RCR (removal of a customer selected randomly in the system) are considered. In markovian G-queue, the queue length distribution is not affected by the service disciplines and the removal strategies because of the memoryless property of exponential distribution of service time. However this is not the case when one considers the sojourn time distribution of a positive customer.

LST's of sojourn time distributions for $M/M/1$ G-queue with FCFS, LCFS and PS service disciplines and the removal strategies RCE, RCH and RCR were presented in Harrison and Pitel [10]. For the $M/M/c$ queue with PS discipline without negative customers, Braband [3] derived the LST of sojourn time distribution conditioned on the service requirement of a tagged customer in the form of system of integral equations. The LST of the probability distribution of the end-to-end delay in two node tandem networks was investigated by Harrison and Pitel [12]. Boucherie and Boxma [2] showed that the workload distribution in the $M/G/1$ queue in which an arrival of negative customer removes the random amount of workload equals the waiting time distribution in a $GI/G/1$ with ordinary customers only and derived the sojourn time distributions with FIFO discipline and RCH strategy. Our results generalize those of Harrison and Pitel [10] and Braband [3].

This paper organized as follows. In Section 2, queue length distribution in equilibrium is considered and some notations are introduced. In Sections 3 and 4, we derive the LST's of sojourn times for the FCFS and LCFS disciplines with RCH and RCE removal strategies, respectively. Section 5 deals with the PS discipline with RCR removal strategy.

2. Model Description and Queue Length Distribution

We consider a multi server Markovian queue with two types of customers, positive (or usual) customers and negative customers. Positive and negative customers arrive in the system according to two independent Poisson processes with rates λ^+ and λ^- respectively. We define $\lambda = \lambda^+ + \lambda^-$. There are c parallel servers in the system and the service time distribution of each server is exponential with rate μ. Note that, as far as the queue length is concerned, the removal strategy and service discipline are immaterial in $M/M/c$ queue because of the memoryless property of the exponential distribution of service time.

Let $X(t)$ be the number of positive customers in the system at time t. Then $\{X(t), t \geq 0\}$ is the birth and death process with birth rates $\lambda_n = \lambda^+$, $n \geq 0$ and death rates

$$\mu_n = \begin{cases} \lambda^- + n\mu, & 1 \leq n \leq c \\ \lambda^- + c\mu, & n \geq c. \end{cases}$$

For the notational simplicity, we let $\tilde{\mu} = \lambda^- + c\mu$. We assume

$$\rho = \frac{\lambda^+}{\tilde{\mu}} < 1$$

for the existence of stationary distribution of $\{X(t), t \geq 0\}$. It is well-known (e.g., Ross [13]) that the stationary distirbution $\{\pi_n, n = 0, 1, \cdots\}$ of $\{X(t), t \geq 0\}$ is given by

$$\pi_n = \begin{cases} \pi_0 \prod_{k=1}^{n} \left(\frac{\lambda^+}{\mu_k}\right), & 1 \leq n \leq c-1 \\ \pi_0 \prod_{k=1}^{c-1} \left(\frac{\lambda^+}{\mu_k}\right) \rho^{n-c+1}, & n \geq c \end{cases} \tag{2.1}$$

with

$$\pi_0 = \left[1 + \sum_{k=1}^{c-1} \prod_{i=1}^{k} \left(\frac{\lambda^+}{\mu_i}\right) + \frac{1}{1-\rho} \prod_{i=1}^{c} \left(\frac{\lambda^+}{\mu_i}\right)\right]^{-1}.$$

Throughout this paper, we consider the system in equilibrium. For later use we introduce some notations. We call the monitored customer whose sojourn time distribution we seek the tagged customer. Like all positive customers the tagged customer obeys the specified service discipline and removal strategy after his arrival.

Let W_t be the random variable representing the length of time required to complete his service for a positive customer at time t, if he leaves the system without being removed. We let $W_t = \infty$ if he is removed during his sojourn in the system. Let A_t and B_t denote the number of customers ahead of and behind the tagged customer at time t respectively when the tagged customer is in queue. When the tagged customer is in service, we denote A_t the the number of customers in service other than the tagged customer and B_t the number of customers in queue at time t. Let $N_t = A_t + B_t$ denote the number of customers in the system at time t other than the tagged customer. We sometimes drop the parameter t in A_t, B_t, N_t and W_t just write A, B, N and W, respectively, to denote the generic random variables. Due to the memoryless property of service time, all the customers in service are in the same line, that is, no one is ahead the other in service. Thus upon arrival of a negative customer, if he has to remove one among the customers in service, he randomly chooses one and removes it.

We define the conditional probability distributions in equilibrium

$$F_n(t) = P(W \leq t|\ N = n \text{ and the tagged customer is in service }),\ n \geq 0$$
$$G_{n,k}(t) = P(W \leq t|A = n, B = k),\ n \geq c-1, k \geq 0$$

and their LST's $F_n^*(s)$ and $G_{n,k}^*(s)$ and the generating functions

$$\tilde{F}(x;s) = \sum_{n=c-1}^{\infty} x^{n-c+1} F_n^*(s),\ |x| < 1,$$
$$\tilde{G}(x,y;s) = \sum_{n=c}^{\infty}\sum_{k=0}^{\infty} x^{n-c} y^k G_{n,k}^*(s),\ |x| < 1,\ |y| < 1.$$

Noting that if $A = c-1$ and $B = k$, then the tagged customer is in service and there are $c-1+k$ customers in the system other than the tagged customer, we obtain

$$G_{c-1,k}^*(s) = F_{c-1+k}^*(s),\ k = 0, 1, \cdots$$

and hence

$$\sum_{k=0}^{\infty} y^k G_{c-1,k}^*(s) = \tilde{F}(y;s). \tag{2.2}$$

3. FCFS Discipline

Under the FCFS discipline, upon arrival of a tagged customer if the number of customers in the system is less than c, he immediately seizes a service channel and starts service, otherwise, he joins at the end of queue and all the customers are ahead of him and no customers are behind him. Thus we have the distribution function $W(t)$ of W

(3.1)

$$\begin{aligned} W(t) &= \sum_{n=0}^{\infty} P(W \le t | N = n)\pi_n \\ &= \sum_{n=0}^{c-1} \pi_n P(W \le t | N = n, \text{ the tagged customer is in service}) \\ &\quad + \sum_{n=c}^{\infty} P(W \le t | A = n, B = 0)\pi_n \\ &= \sum_{n=0}^{c-1} \pi_n F_n(t) + \sum_{n=c}^{\infty} \pi_n G_{n,0}(t). \end{aligned}$$

and the LST $W^*(s)$ of $W(t)$

$$W^*(s) = \sum_{n=0}^{c-1} \pi_n F_n^*(s) + \pi_c \tilde{G}(\rho, 0; s). \tag{3.2}$$

The probability that the tagged customer is not removed is given by $W^*(0)$ and the conditional mean sojourn time on the tagged customer of not being removed is $-\frac{W^{*'}(0)}{W^*(0)}$.

3.1 RCE with FCFS Discipline

In this subsection we consider FCFS service discipline for the positive customers with being removed of the last customer if there are any customers in queue or being removed randomly one of the customers in service if all customers in the system are in service at the epoch of negative arrival.

First we derive $F_n^*(s)$ and $\tilde{F}(x;s)$. When $N_0 = n$ and tagged customer is in service, the possible events that can be occurred during $[0, h]$ for infinitesimal $h > 0$ and the resultants are as follows:

(i) an arrival of positive customer with probability $\lambda^+ h + o(h)$ and hence $N_h = n + 1$

(ii) an arrival of negative customer which removes the customer other than the tagged customer with probability $\lambda^- h \frac{n}{n+1} + o(h)$ if $n < c$ and with probability $\lambda^- h + o(h)$ if $n \geq c$ and hence $N_h = n - 1$

(iii) an arrival of negative customer which removes the tagged customer with probability $\lambda^- h \frac{1}{n+1} + o(h)$ if $n < c$ and with probability $o(h)$ if $n \geq c$ and hence $N_h = n - 1$ and $W_h = \infty$

(iv) a departure of one customer in service other than the tagged customer with probability $\min(c-1, n)\mu h + o(h)$ and hence $N_h = n - 1$

(v) departure of the tagged customer with probability $\mu h + o(h)$ and hence $W_h = 0$

(vi) no changes of the number of customers in the system with probability $1 - (\lambda + \min(c, n+1)\mu)h + o(h)$.

Using the Markov property and the law of total probability, we have the following equations for transitions: for $0 \leq n \leq c - 1$

$$
\begin{aligned}
F_n(t+h) &= \lambda^+ h F_{n+1}(t) + \lambda^- \frac{n}{n+1} h F_{n-1}(t) + \mu h \\
&\quad + n\mu h F_{n-1}(t) + (1 - (\lambda + (n+1)\mu)h)F_n(t) + o(h),
\end{aligned} \tag{3.3a}
$$

where $F_{-1}(t) = 0$ and for $n \geq c$,

$$
\begin{aligned}
F_n(t+h) &= \lambda^+ h F_{n+1}(t) + \lambda^- h F_{n-1}(t) + \mu h \\
&\quad + (c-1)\mu h F_{n-1}(t) + (1 - (\lambda^+ + \tilde{\mu})h)F_n(t) + o(h),
\end{aligned} \tag{3.3b}
$$

Dividing both sides of (3.3) by h and then letting $h \to 0$, we have for $0 \leq n \leq c - 1$,

$$
F_n'(t) = \lambda^+ F_{n+1}(t) + \frac{n}{n+1}\mu_{n+1} F_{n-1}(t) + \mu - (\lambda + (n+1)\mu)F_n(t), \tag{3.4a}
$$

and for $n \geq c$,

$$\text{(3.4b)} \qquad F_n'(t) = \lambda^+ F_{n+1}(t) + \mu_{c-1} F_{n-1}(t) + \mu - (\lambda^+ + \tilde{\mu}) F_n(t),$$

Taking the LST of the equations (3.4) and noting that

$$\int_0^\infty e^{-st}\, dF_k'(t) = -F_k'(0) + sF_k^*(s) = -\mu + sF_k^*(s), k = 0, 1, \cdots$$

we obtain for $0 \leq n \leq c-1$

$$\text{(3.5a)} \quad (\lambda + (n+1)\mu + s) F_n^*(s) = \lambda^+ F_{n+1}^*(s) + \frac{n}{n+1} \mu_{n+1} F_{n-1}^*(s) + \mu$$

and for $n \geq c$,

$$\text{(3.5b)} \qquad (\lambda^+ + \tilde{\mu} + s) F_n^*(s) = \lambda^+ F_{n+1}^*(s) + \mu_{c-1} F_{n-1}^*(s) + \mu.$$

Multiplying the both sides of (3.5) by x^{n-c+1} and summing over n from $c-1$ to ∞ yields

$$\text{(3.6)} \qquad a(x;s)\tilde{F}(x;s) = -\frac{c-1}{c} \tilde{\mu} x F_{c-2}^*(s) + \lambda^+ F_{c-1}^*(s) - \frac{x}{1-x} \mu.$$

where

$$a(x;s) = \mu_{c-1} x^2 - (\lambda^+ + \tilde{\mu} + s)x + \lambda^+.$$

Applying the Rouche's theorem to $a(x;s) = f(x) + g(x)$ with $f(x) = \mu_{c-1}x^2 + \lambda^+$ and $g(x) = -(\lambda^+ + \tilde{\mu} + s)x$, we see that $a(x;s)$ as a function of x has exactly one zero, say $x_0(s)$, in the unit disc and the $x_0(s)$ is given by

$$\text{(3.7)} \qquad x_0(s) = \frac{(\lambda^+ + \tilde{\mu} + s) - \sqrt{(\lambda^+ + \tilde{\mu} + s)^2 - 4\lambda^+ \mu_{c-1}}}{2\mu_{c-1}}.$$

Noting that $a(0;s) = \lambda^+ > 0$ and $a(\rho;s) = -\rho^2\mu - \rho s < 0$ and the intermediate value theorem yield for $s \geq 0$

$$\text{(3.8)} \qquad 0 < x_0(s) < \rho.$$

Since $\tilde{F}(x;s)$ is analytic in $|x| < 1$ for each fixed $s > 0$, the right hand side of (3.6) vanishes at $x_0(s)$ and we have the following relation between $F^*_{c-2}(s)$ and $F^*_{c-1}(s)$:

$$\text{(3.9)} \qquad \frac{c-1}{c}\tilde{\mu}x_0(s)F^*_{c-2}(s) - \lambda^+ F^*_{c-1}(s) = -\frac{x_0(s)}{1-x_0(s)}\mu,$$

Combining (3.5a) for $n \le c-2$ and (3.9), we have the following tridiagonal system for $F^*_0(s), \cdots, F^*_{c-1}(s)$

(3.10)

$$\begin{pmatrix} b_0 & c_0 & & & & \\ a_1 & b_1 & c_1 & & O & \\ & a_2 & b_2 & c_2 & & \\ & & \ddots & \ddots & \ddots & \\ & O & & a_{c-2} & b_{c-2} & c_{c-2} \\ & & & & a_{c-1} & b_{c-1} \end{pmatrix} \begin{pmatrix} F^*_0(s) \\ F^*_1(s) \\ F^*_2(s) \\ \vdots \\ F^*_{c-2}(s) \\ F^*_{c-1}(s) \end{pmatrix} = -\mu \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \\ \frac{x_0(s)}{1-x_0(s)} \end{pmatrix},$$

where

$$a_k = \begin{cases} \frac{k}{k+1}\mu_{k+1}, & 1 \le k \le c-2 \\ \frac{c-1}{c}\tilde{\mu}x_0(s), & k = c-1 \end{cases}$$

$$b_k = \begin{cases} -(\lambda + (k+1)\mu + s), & 0 \le k \le c-2 \\ -\lambda^+, & k = c-1 \end{cases}$$

$$c_k = \lambda^+, \ 0 \le k \le c-2.$$

Note that $|b_k| > |a_k| + |c_k|$, $0 \le k \le c-2$ with $a_0 = 0$ and

$$|b_{c-1}| - |a_{c-1}| = \tilde{\mu}(\rho - \frac{c-1}{c}x_0(s)) > 0,$$

where the last inequality comes from (3.8). Hence the coefficient matrix in (3.10) is strictly diagonally dominant and the linear system (3.10) has unique solution.

Now we derive the generating function $\tilde{G}(x,y;s)$. Note that, if $A = n \ge c$ then the tagged customer is in queue and, if $B = k \ge 1$ then the probability that the tagged customer is removed in $[0,h]$ for infinitesimal

$h > 0$ is $o(h)$. Thus when $A = n$, $n \geq c$, and $B = k$, $k \geq 0$, the possible events that can occur during $[0, h]$ and the resultants are as follows:

(i) an arrival of a positive customer with probability $\lambda^+ h + o(h)$ and hence $A_h = n$ and $B_h = k + 1$.

(ii) an arrival of a negative customer with probability $\lambda^- h + o(h)$ and hence $A_h = n$ and $B_h = k - 1$ for $k \geq 1$ and $W_h = \infty$ for $k = 0$

(iii) a departure of one customer in service with probability $c\mu h + o(h)$ and hence $A_h = n - 1$ and $B_h = k$

(iv) no changes of A and B with probability $1 - (\lambda^+ + \tilde{\mu})h + o(h)$.

Using the Markov property and the law of total probability, we have for $n \geq c$, $k \geq 0$,

$$
\begin{aligned}
G_{n,k}(t+h) =& \lambda^+ h G_{n,k+1}(t) + \lambda^- h G_{n,k-1}(t) + c\mu h G_{n-1,k}(t) \\
&+ (1 - (\lambda^+ + \tilde{\mu})h) G_{n,k}(t) + o(h)
\end{aligned} \tag{3.11}
$$

and dividing the both sides of (3.11) by h and letting $h \to 0$ yield

$$
G'_{n,k}(t) = \lambda^+ G_{n,k+1}(t) + \lambda^- G_{n,k-1}(t) + c\mu G_{n-1,k}(t) - (\lambda^+ + \tilde{\mu}) G_{n,k}(t) \tag{3.12}
$$

with $G_{n,-1}(t) = 0$. Taking LST to the both sides of (3.12) with $G'_{n,k}(0) = 0$, we obtain for $n \geq c$, $k \geq 0$,

$$
(\lambda^+ + \tilde{\mu} + s) G^*_{n,k}(s) = \lambda^+ G^*_{n,k+1}(s) + \lambda^- G^*_{n,k-1}(s) + c\mu G^*_{n-1,k}(s). \tag{3.13}
$$

Multiplying $x^{n-c} y^k$ to the both sides of (3.13) and summing over $n \geq c$ and $k \geq 0$, we have from the relation (2.2) that

$$
R(x, y; s) \tilde{G}(x, y; s) = \lambda^+ \tilde{G}(x, 0; s) - c\mu y \tilde{F}(y; s). \tag{3.14}
$$

where $R(x, y; s) = \lambda^- y^2 - (\lambda^+ + \tilde{\mu}(1 - x) + s) y + \lambda^+$. Since for $|y| = 1$,

$$
|(\lambda^+ + \tilde{\mu}(1 - x) + s) y| > |\lambda^- y^2 + \lambda^+|.
$$

Rouche's theorem guarantees that $R(x, y; s)$ as a function of y has exactly one zero in $|y| < 1$, say $y_0(x, s)$ which is given by

$$
y_0(x, s) = \frac{\lambda^+ + \tilde{\mu}(1 - x) + s - \sqrt{(\lambda^+ + \tilde{\mu}(1 - x) + s)^2 - 4\lambda^- \lambda^+}}{2\lambda^-}.
$$

Since for fixed $|x| < 1$ and $s > 0$, $\tilde{G}(x,y;s)$ is analytic in $|y| < 1$, the right hand side of (3.14) vanishes at $y_0(x,s)$ and hence

$$\tilde{G}(x,0;s) = \frac{c\mu}{\lambda^+} y_0(x,s)\tilde{F}(y_0(x,s);s) \tag{3.15}$$

Combining (3.2) and (3.15) yields the following.

THEOREM 1. *In the M/M/c G-queue with FCFS discipline and RCE removal strategy, the LST of sojourn time distribution, jointly with the probability of not being removed, is*

$$W^*(s) = \sum_{n=0}^{c-1} \pi_n F_n^*(s) + \pi_c \frac{c\mu}{\lambda^+} y_0(\rho,s)\tilde{F}(y_0(\rho,s);s), \tag{3.16}$$

where $F_n^*(s)$, $0 \le n \le c-1$ *and* $\tilde{F}(x;s)$ *are given in* (3.10) *and* (3.6) *respectively.*

3.2 RCH with FCFS Discipline

The removal strategy of this subsection is that negative customer chooses randomly one of the customers in service, if any, and removes it at the instant of negative arrival. Thus when the tagged customer is in service and $N = n$, then the tagged customer will be removed by an arrival of negative customer with probability $\frac{1}{\min(n,c-1)+1}$ and not be removed with probability $\frac{\min(n,c-1)}{\min(n,c-1)+1}$. Using the similar arguments in subsection 3.1, we have for $0 \le n \le c-2$

$$\begin{aligned} F_n(t+h) &= \lambda^+ h F_{n+1}(t) + \frac{n}{n+1}(\lambda^- + (n+1)\mu)hF_{n-1}(t) + \mu h \\ &\quad + (1-(\lambda+(n+1)\mu)h)F_n(t) + o(h). \end{aligned} \tag{3.17a}$$

where $F_{-1}(t) = 0$ and for $n \ge c-1$,

$$\begin{aligned} F_n(t+h) &= \lambda^+ h F_{n+1}(t) + \frac{c-1}{c}(\lambda^- + c\mu)hF_{n-1}(t) + \mu h \\ &\quad + (1-(\lambda^+ + \tilde{\mu})h)F_n(t) + o(h). \end{aligned} \tag{3.17b}$$

Dividing both sides of (3.17) by h and then letting $h \to 0$, we have for $0 \le n \le c-2$,

(3.18a)
$$F_n'(t) = \lambda^+ F_{n+1}(t) + \frac{n}{n+1}\mu_{n+1} F_{n-1}(t) + \mu - (\lambda + (n+1)\mu)F_n(t),$$

and for $n \ge c-1$,

$$\text{(3.18b)} \qquad F_n'(t) = \lambda^+ F_{n+1}(t) + \frac{c-1}{c}\tilde{\mu} F_{n-1}(t) + \mu - (\lambda^+ + \tilde{\mu})F_n(t),$$

Taking the LST of the equations (3.18) with $F_k'(0) = \mu, k = 0, 1, \cdots$ we obtain for $0 \le n \le c-2$

$$\text{(3.19a)} \quad (\lambda + (n+1)\mu + s)F_n^*(s) = \lambda^+ F_{n+1}^*(s) + \frac{n}{n-1}\mu_{n+1} F_{n-1}^*(s) + \mu$$

and for $n \ge c-1$,

$$\text{(3.19b)} \qquad (\lambda^+ + \tilde{\mu} + s)F_n^*(s) = \lambda^+ F_{n+1}^*(s) + \frac{c-1}{c}\tilde{\mu} F_{n-1}^*(s) + \mu.$$

Similar calculations to those used to derive (3.6) yield

$$\text{(3.20)} \qquad a(x;s)\tilde{F}(x;s) = -\frac{c-1}{c}\tilde{\mu} x F_{c-2}^*(s) + \lambda^+ F_{c-1}^*(s) - \frac{x}{1-x}\mu,$$

where

$$a(x;s) = \frac{c-1}{c}\tilde{\mu} x^2 - (\lambda^+ + \tilde{\mu} + s)x + \lambda^+.$$

Applying the Rouche's theorem to $a(x;s) = f(x) + g(x)$ with $f(x) = \frac{c-1}{c}\tilde{\mu} x^2 + \lambda^+$ and $g(x) = -(\lambda^+ + \tilde{\mu} + s)x$ yields that for fixed $s > 0$, $a(x;s)$ has exactly one zero, say $x_0(s)$, in the unit disc. $x_0(s)$ is given by

$$\text{(3.21)} \qquad x_0(s) = \frac{\lambda^+ + \tilde{\mu} + s - \sqrt{[\lambda^+ + \tilde{\mu} + s]^2 - 4\frac{c-1}{c}\lambda^+ \tilde{\mu}}}{2\frac{c-1}{c}\tilde{\mu}}.$$

Noting that $a(0;s) = \lambda^+ > 0$ and $a(\rho;s) = -\rho(\frac{\lambda^+}{c} + s) < 0$ we have for $s \geq 0$,

$$0 < x_0(s) < \rho, \;\; s \geq 0. \tag{3.22}$$

By the analyticity of $\tilde{F}(x;s)$ in $|x| < 1$, the right hand side of (3.20) must vanish at $x_0(s)$ given in (3.21), which yields from (3.20) the following relation:

$$\frac{c-1}{c}\tilde{\mu}x_0(s)F^*_{c-2}(s) - \lambda^+ F^*_{c-1}(s) = -\frac{x_0(s)}{1-x_0(s)}\mu, \tag{3.23}$$

Combining (3.19a) and (3.23), we have the linear system for $F_0^*(s), \cdots, F^*_{c-1}(s)$

(3.24)

$$\begin{pmatrix} b_0 & c_0 & & & & \\ a_1 & b_1 & c_1 & & \boldsymbol{O} & \\ & a_2 & b_2 & c_2 & & \\ & & \ddots & \ddots & \ddots & \\ & \boldsymbol{O} & & a_{c-2} & b_{c-2} & c_{c-2} \\ & & & & a_{c-1} & b_{c-1} \end{pmatrix} \begin{pmatrix} F_0^*(s) \\ F_1^*(s) \\ F_2^*(s) \\ \vdots \\ F^*_{c-2}(s) \\ F^*_{c-1}(s) \end{pmatrix} = -\mu \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \\ \frac{x_0(s)}{1-x_0(s)} \end{pmatrix},$$

where

$$a_k = \begin{cases} \frac{k}{k+1}\mu_{k+1}, & 1 \leq k \leq c-2 \\ \frac{c-1}{c}\tilde{\mu}x_0(s), & k = c-1 \end{cases}$$

$$b_k = \begin{cases} -(\lambda + (k+1)\mu + s), & 0 \leq k \leq c-2 \\ -\lambda^+, & k = c-1 \end{cases}$$

$$c_k = \lambda^+, \;\; 0 \leq k \leq c-2.$$

Noting that $|b_k| > |a_k| + |c_k|$, $0 \leq c-2$ with $a_0 = 0$ and that from (3.22),

$$|b_{c-1}| - |a_{c-1}| = \tilde{\mu}(\rho - \frac{c-1}{c}x_0(s)) > 0,$$

shows the linear system (3.24) has unique solution for each $s \geq 0$.

Using the similar arguments to those used to derive $G_{n,k}(t)$, $G^*_{n,k}(s)$ and $\tilde{G}(x,y;s)$ in subsection 3.1, we have the following:

$$G'_{n,k}(t) = \lambda^+ G_{n,k+1}(t) + \tilde{\mu} G_{n-1,k}(t) - (\lambda^+ + \tilde{\mu}) G_{n,k}(t),\ n \geq c,\ k \geq 0,$$
$$(\lambda^+ + \tilde{\mu} + s) G^*_{n,k}(s) = \lambda^+ G^*_{n,k+1}(s) + \tilde{\mu} G^*_{n-1,k}(s),\ n \geq c,\ k \geq 0$$

and

$$R(x,y;s)\tilde{G}(x,y;s) = \lambda^+ \tilde{G}(x,0;s) - \tilde{\mu} y \tilde{F}(y;s), \tag{3.25}$$

where $R(x,y;s) = -[(\lambda^+ + \tilde{\mu} + s) - \tilde{\mu}x]y + \lambda^+$. From the analyticity of $\tilde{G}(x,y;s)$ in $|y| < 1$, the right hand side of (3.25) is zero at $y_0(x,s)$ which is given by

$$y_0(x,s) = \frac{\lambda^+}{s + \lambda^+ + \tilde{\mu} - \tilde{\mu}x} \tag{3.26}$$

and hence

$$\tilde{G}(x,0;s) = \frac{y_0(x,s)}{\rho} \tilde{F}(y_0(x,s);s). \tag{3.27}$$

Combining (3.2) and (3.27) with $\pi_c = \rho\pi_{c-1}$ yields the following.

THEOREM 2. *In the $M/M/c$ G-queue with FCFS discipline and RCH removal strategy, the LST of sojourn time distribution, jointly with the probability of not being removed, is*

$$W^*(s) = \sum_{n=0}^{c-1} \pi_n F^*_n(s) + \pi_{c-1} y_0(\rho,s) \tilde{F}(y_0(\rho,s);s), \tag{3.28}$$

where $F^*_n(s)$, $0 \leq n \leq c-1$ and $\tilde{F}(x;s)$ are given in (3.24) and (3.20) respectively and $y_0(\rho,s) = \frac{\lambda^+}{\lambda^- + c\mu + s}$.

4. Preemptive LCFS Dicipline

Under the preemptive LCFS discipline, when a tagged customer finds n customers upon his arrival, if $n \geq c$, then he chooses one customer in service randomly and let the customer chosen go to the head of the queue and starts his service and if $n \leq c-1$, then no customers in service go to the queue. Hence we get the LST $W^*(s)$ of the distribution function $W(t)$ of the sojourn time W,

$$\begin{aligned} W^*(s) &= \sum_{n=0}^{c-2} \pi_n F_n^*(s) + \pi_{c-1} \sum_{n=c-1}^{\infty} \rho^{n-c+1} F_n^*(s) \\ &= \sum_{n=0}^{c-2} \pi_n F_n^*(s) + \pi_{c-1} \tilde{F}(\rho; s). \end{aligned} \tag{4.1}$$

4.1 RCE with LCFS Disciplne

The removal strategy we consider in this subsection is the same as that in the subsection 3.1. Using the similar arguments to those used to have $G_{n,k}(t)$, $G_{n,k}^*(s)$ and $\tilde{G}(x,y;s)$ in subsection 3.1 and taking $G_{n,-1}(t) = 0$, we have for $n \geq c$, $k \geq 0$,

$$\begin{aligned} G_{n,k}(t+h) &= \lambda^+ h G_{n+1,k}(t) + \lambda^- h G_{n,k-1}(t) + c\mu h G_{n-1,k}(t) \\ &\quad + (1 - (\lambda^+ + \tilde{\mu})h) G_{n,k}(t) + o(h), \\ G'_{n,k}(t) &= \lambda^+ G_{n+1,k}(t) + \lambda^- G_{n,k-1}(t) + c\mu G_{n-1,k}(t) - (\lambda^+ + \tilde{\mu}) G_{n,k}(t) \\ (\lambda^+ + \tilde{\mu} + s) G_{n,k}^*(s) &= \lambda^+ G_{n+1,k}^*(s) + \lambda^- G_{n,k-1}^*(s) + c\mu G_{n-1,k}^*(s) \end{aligned}$$

(4.2)

$$R(x,y;s)\tilde{G}(x,y;s) = \lambda^+ \tilde{G}(0,y;s) - c\mu x \tilde{F}(y;s),$$

where $R(x,y;s) = c\mu x^2 + (\lambda^- y - (\lambda^+ + \tilde{\mu} + s))x - \lambda^+$.

From the analyticity of $\tilde{G}(x,y;s)$ in $|x| < 1$, the right hand side of (4.2) is zero at $z_0(y,s)$ which is the unique zero of $R(x,y;s)$ as a function of x in the x-unit disc and is given by

(4.3)

$$z_0(y,s) = \frac{(s + \lambda^+ + \tilde{\mu} - \lambda^- y) - \sqrt{(s + \lambda^+ + \tilde{\mu} - \lambda^- y)^2 - 4\lambda^+ c\mu}}{2c\mu}.$$

Sustituting $z_0(y,s)$ into (4.2) yields the following:

$$\lambda^+ \tilde{G}(0,y;s) = c\mu z_0(y,s)\tilde{F}(y;s). \tag{4.4}$$

Since

$$R(0,0;s)R(1,0;s) = (\lambda^+)(-s-\lambda) < 0$$
$$R(0,\rho;s)R(\rho,\rho;s) = \lambda^+(-s\rho) < 0,$$

we have from the intermediate value theorem that for $s > 0$

$$0 < z_0(0,s) < 1, \qquad 0 < z_0(\rho,s) < \rho.$$

It is clear that $z_0(\rho,0) = \rho$ from $R(\rho,\rho;0) = 0$.

Now we derive the $F_n^*(s)$. When tagged customer is in service and $N = n$, the possible events that can be occurred during $[0,h]$ for infinitesimal $h > 0$ and the resultants are as follows:

(i) an arrival of positive customer with probability $\lambda^+ h + o(h)$ and $N_h = n+1$. If $n \le c-2$, then the tagged customer is not affected by the new customer. If $n \ge c-1$, then either the tagged customer is chosen and joins the head of queue with probability $\frac{1}{c}$ in this case $A = c$ and $B = n - c + 1$ or the tagged customer continues his service with probability $\frac{c-1}{c}$.

(ii) an arrival of negative customer with probability $\lambda^- h + o(h)$ and $N_h = n-1$. If $n \le c-1$, then either the tagged customer is removed with probability $\frac{1}{n+1}$ and $W_h = \infty$ or the tagged customer is not removed with probability $\frac{n}{n+1}$. If $n \ge c$, then the customer at the end of the queue is removed.

(iii) service completion of the tagged customer with probability $\mu h + o(h)$ and $W_h = 0$

(iv) service completion of a customer in service other than the tagged customer with probability $\min(c-1,n)\mu h + o(h)$ and $N_h = n-1$

(v) no changes of the number of customers in the system with probability $1 - (\lambda + \min(c,n+1)\mu)h + o(h)$.

Using the Markov property and the law of total probability, we have the following equations for transitions: for $0 \le n \le c-2$

$$\begin{aligned} F_n(t+h) &= \lambda^+ h F_{n+1}(t) + \frac{n}{n+1}(\lambda^- + (n+1)\mu) h F_{n-1}(t) + \mu h \\ &\quad + (1 - (\lambda + (n+1)\mu)h) F_n(t) + o(h), \end{aligned} \tag{4.5a}$$

where $F_{-1}(t) = 0$ and for $n = c-1$,

$$\begin{aligned} F_{c-1}(t+h) &= \lambda^+ h \big[\frac{1}{c} G_{c,0}(t) + \frac{c-1}{c} F_c(t)\big] \\ &\quad + \frac{c-1}{c}(\lambda^- + c\mu) h F_{c-2}(t) + \mu h \\ &\quad + (1 - (\lambda^+ + \tilde{\mu})h) F_{c-1}(t) + o(h), \end{aligned} \tag{4.5b}$$

and for $n \ge c$,

$$\begin{aligned} F_n(t+h) &= \lambda^+ h \big[\frac{1}{c} G_{c,n-c+1}(t) + \frac{c-1}{c} F_{n+1}(t)\big] \\ &\quad + (\lambda^- + (c-1)\mu) h F_{n-1}(t) \\ &\quad + \mu h + (1 - (\lambda^+ + \tilde{\mu})h) F_n(t) + o(h). \end{aligned} \tag{4.5c}$$

Dividing both sides of (4.5) by h and then letting $h \to 0$, we have for $0 \le n \le c-2$,

$$F_n'(t) = \lambda^+ F_{n+1}(t) + \frac{n}{n+1}\mu_{n+1} F_{n-1}(t) + \mu - (\lambda + (n+1)\mu) F_n(t), \tag{4.6a}$$

and for $n = c-1$,

$$\begin{aligned} F_{c-1}'(t) &= \lambda^+ \big[\frac{1}{c} G_{c,0}(t) + \frac{c-1}{c} F_c(t)\big] \\ &\quad + \frac{c-1}{c}\tilde{\mu} F_{c-2}(t) + \mu - (\lambda^+ + \tilde{\mu}) F_{c-1}(t), \end{aligned} \tag{4.6b}$$

and for $n \ge c$,

$$\begin{aligned} F_n'(t) &= \lambda^+ \big[\frac{1}{c} G_{c,n-c+1}(t) + \frac{c-1}{c} F_{n+1}(t)\big] + \mu_{c-1} F_{n-1}(t) + \mu \\ &\quad - (\lambda^+ + \tilde{\mu}) F_n(t). \end{aligned} \tag{4.6c}$$

Taking the LST of the equations (4.6) and noting that $F_n'(0) = \mu, n = 0, 1, \cdots$ we obtain for $0 \le n \le c-2$

$$\text{(4.7a)} \quad (s + \lambda + (n+1)\mu)F_n^*(s) = \lambda^+ F_{n+1}^*(s) + \frac{n}{n+1}\mu_{n+1}F_{n-1}^*(s) + \mu$$

and for $n = c-1$,
(4.7b)

$$(s + \lambda^+ + \tilde{\mu})F_{c-1}^*(s) = \frac{\lambda^+}{c}G_{c,0}^*(s) + \frac{c-1}{c}\lambda^+ F_c^*(s) + \frac{c-1}{c}\tilde{\mu}F_{c-2}^*(s) + \mu$$

and for $n \ge c$,
(4.7c)

$$(\lambda^+ + \tilde{\mu} + s)F_n^*(s) = \frac{\lambda^+}{c}G_{c,n-c+1}^*(s) + \frac{c-1}{c}\lambda^+ F_{n+1}^*(s) + \mu_{c-1}F_{n-1}^*(s) + \mu.$$

Solving for $\tilde{F}(x;s)$ by multiplying x^{n-c+1} to (4.7b) and (4.7c) and summing over $n, \ge n \ge c-1$ and using (4.4), we have

$$\text{(4.8)} \quad a(x;s)\tilde{F}(x;s) = -\frac{c-1}{c}\tilde{\mu}xF_{c-2}^*(s) + \frac{c-1}{c}\lambda^+ F_{c-1}^*(s) - \frac{x}{1-x}\mu,$$

where

$$a(x;s) = \mu_{c-1}x^2 - (\lambda^+ + \tilde{\mu} + s)x + \mu z_0(x,s) + \frac{c-1}{c}\lambda^+.$$

Note that for $|x| = 1$,

$$\begin{aligned} |\mu_{c-1}x^2 + \mu z_0(x,s) + \frac{c-1}{c}\lambda^+| &\le \mu_{c-1}x^2 + \mu|z_0(x,s)| + \frac{c-1}{c}\lambda^+ \\ &< \tilde{\mu} + \frac{c-1}{c}\lambda^+ \\ &< |-(\lambda^+ + \tilde{\mu} + s)x|. \end{aligned}$$

By the Rouche's theorem, there exists exactly one zero of $a(x;s)$ in $|x| < 1$, say $x_0(s)$.

Using the facts $0 < z_0(0,s) < 1$ and $0 < z_0(\rho, s) < \rho$, we have

$$a(0;s) = \mu z_0(0,s) + \frac{c-1}{c}\lambda^+ > 0$$

and

$$\frac{a(\rho;s)}{\rho} = -(s+\lambda+c\mu) + \frac{\lambda^+(\lambda^- + (c-1)\mu)}{\lambda^- + c\mu} + \frac{z_0(\rho,s)}{\rho}\mu + \frac{c-1}{c}\frac{\lambda^+}{\rho}$$
$$< -(s+\lambda+c\mu) + \lambda^+ - \rho\mu + \mu + \frac{c-1}{c}\lambda^- + (c-1)\mu$$
$$< -(\frac{\lambda^-}{c} + s) < 0$$

and hence

$$0 < x_0(s) < \rho.$$

Since $\tilde{F}(x;s)$ is analytic in $|x| < 1$ for each fixed $s > 0$, the right hand side of (4.8) vanishes at $x = x_0(s)$ and hence we have

$$(4.9) \qquad \frac{c-1}{c}(c\mu + \lambda^-)z_0(s)F^*_{c-2} - \frac{c-1}{c}\lambda^+ F^*_{c-1}(s) = -\frac{x_0(s)}{1-x_0(s)}\mu.$$

Combining (4.7a) and (4.9) we have the tridiagonal system for $F^*_n(s)$, $0 \le n \le c-1$;

(4.10)

$$\begin{pmatrix} b_0 & c_0 & & & & \\ a_1 & b_1 & c_1 & & \boldsymbol{O} & \\ & a_2 & b_2 & c_2 & & \\ & & \ddots & \ddots & \ddots & \\ & \boldsymbol{O} & & a_{c-2} & b_{c-2} & c_{c-2} \\ & & & & a_{c-1} & b_{c-1} \end{pmatrix} \begin{pmatrix} F^*_0(s) \\ F^*_1(s) \\ F^*_2(s) \\ \vdots \\ F^*_{c-2}(s) \\ F^*_{c-1}(s) \end{pmatrix} = -\mu \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \\ \frac{x_0(s)}{1-x_0(s)} \end{pmatrix},$$

where

$$a_k = \begin{cases} \frac{k}{k+1}((k+1)\mu + \lambda^-), & 1 \le k \le c-2 \\ \frac{(c-1)}{c}(\lambda^- + c\mu)x_0(s), & k = c-1 \end{cases}$$
$$b_k = \begin{cases} -(s+\lambda+(k+1)\mu), & 0 \le k \le c-2 \\ -\frac{c-1}{c}\lambda^+, & k = c-1 \end{cases}$$
$$c_k = \lambda^+, \ 0 \le k \le c-2.$$

It is clear that $|b_0| > |c_0|$ and $|b_k| > |a_k| + |c_k|$, $1 \le k \le c-2$. We also have from $x_0(s) < \rho$ that

$$|b_{c-1}| - |a_{c-1}| = \frac{c-1}{c}(c\mu + \lambda^-)(\rho - \ x_0(s)|) > 0.$$

Thus the linear system (4.10) has a unique solution for each $s \ge 0$.

THEOREM 3. *In the $M/M/c$ G-queue with LCFS discipline and RCE removal strategy, the LST of sojourn time distribution, jointly with the probability of not being removed, is*

$$W^*(s) = \sum_{n=0}^{c-2} \pi_n F_n^*(s) + \pi_{c-1} \sum_{n=c-1}^{\infty} \rho^{n-c+1} F_n^*(s)$$

$$= \sum_{n=0}^{c-2} \pi_n F_n^*(s) + \pi_{c-1} \tilde{F}(\rho; s), \tag{4.11}$$

where $F_n^*(s)$, $0 \le n \le c-1$ and $\tilde{F}(x; s)$ are given in (4.10) and (4.8) respectively.

4.2 RCH with LCFS Disciplne

Here, if an arriving negative customer finds any customers in service, he chooses one among the customers in service randomly and then removes it. Following the the similar procedures to those used in subsection 4.1, we have for $n \ge c$, $k \ge 0$,

$$G_{n,k}(t+h) = \lambda^+ h G_{n+1,k}(t) + (\lambda^- + c\mu) h G_{n-1,k}(t) + (1 - (\lambda^+ + \tilde{\mu})h) G_{n,k}(t) + o(h)$$

and

$$G'_{n,k}(t) + (\lambda^+ + \tilde{\mu}) G_{n,k}(t) = \lambda^+ G_{n+1,k}(t) + \tilde{\mu} G_{n-1,k}(t)$$

and the LST $G^*_{n,k}(s)$ of $G_{n,k}(t)$ have the following recursive relation,

$$(s + \lambda^+ + \tilde{\mu}) G^*_{n,k}(s) = \lambda^+ G^*_{n+1,k}(s) + \tilde{\mu} G^*_{n-1,k}(s). \tag{4.12}$$

Multiplying $x^{n-c} y^k$ to the both sides of (4.12) and summing over $n \ge c$ and $k \ge 0$ and then using (2.2), we obtain after simplication

$$R(x; s) \tilde{G}(x, y; s) = \lambda^+ \tilde{G}(0, y; s) - \tilde{\mu} x \tilde{F}(y; s), \tag{4.13}$$

where $R(x; s) = \tilde{\mu} x^2 - (s + \lambda^+ + \tilde{\mu}) x + \lambda^+$.

Using the Rouche's theorem, it is easy to show that for each fixed $s > 0$, $R(x; s)$ has exactly one zero, say $z_0(s)$, in $|x| < 1$ and $z_0(s)$ is given by

$$z_0(s) = \frac{(s + \lambda^+ + \tilde{\mu}) - \sqrt{(s + \lambda^+ + \tilde{\mu})^2 - 4\lambda^+ \tilde{\mu}}}{2\tilde{\mu}}. \tag{4.14}$$

It is easy to see that $0 < z_0(s) < \rho$ for $s > 0$ and $z_0(0) = \rho$ by using the intermediate value theorem.

From the analyticity of $\tilde{G}(x, y; s)$ in $|x| < 1$, the right hand side of (4.13) vanishes at $z_0(s)$ and hence we have

$$\tilde{G}(0, y; s) = \frac{z_0(s)}{\rho} \tilde{F}(y; s). \tag{4.15}$$

Again using the similar arguments in subsection 4.1, we have the following equations for transitions: for $0 \le n \le c - 2$

$$\begin{aligned} F_n(t + h) &= \lambda^+ h F_{n+1}(t) + \frac{n}{n+1}(\lambda^- + (n + 1)\mu) h F_{n-1}(t) \\ &\quad + \mu h + (1 - (\lambda + (n + 1)\mu)h) F_n(t) + o(h), \end{aligned} \tag{4.16a}$$

where $F_{-1}(t) = 0$ and for $n \ge c - 1$,

$$\begin{aligned} F_n(t + h) = \lambda^+ h \Big(\frac{1}{c} G_{c,n-c+1}(t) &+ \frac{c-1}{c} F_{n+1}(t)\Big) \\ &+ \frac{c-1}{c}(\lambda^- + c\mu) h F_{n-1}(t) \\ &+ \mu h + (1 - (\lambda^+ + \tilde{\mu})h) F_n(t) + o(h). \end{aligned} \tag{4.16b}$$

Dividing both sides of (4.16) by h and then letting $h \to 0$, we have for $0 \le n \le c - 2$

$$F_n'(t) = \lambda^+ F_{n+1}(t) + \frac{n}{n+1} \mu_{n+1} F_{n-1}(t) + \mu - (\lambda + (n+1)\mu) F_n(t) \tag{4.17a}$$

where $F_{-1}(t) = 0$ and for $n \ge c - 1$,

$$\begin{aligned} F_n'(t) &= \lambda^+ \left(\frac{1}{c} G_{c,n-c+1}(t) + \frac{c-1}{c} F_{n+1}(t)\right) + \frac{c-1}{c} \tilde{\mu} F_{n-1}(t) + \mu \\ &\quad - (\lambda^+ + \tilde{\mu}) F_n(t). \end{aligned} \tag{4.17b}$$

Taking the LST of the equations (4.17) and noting that $F_n'(0) = \mu, n = 0, 1, \cdots$ we obtain for $0 \le n \le c-2$

$$(4.18a)\quad (s+\lambda+(n+1)\mu)F_n^*(s) = \lambda^+ F_{n+1}^*(s) + \frac{n}{n+1}\mu_{n+1}F_{n-1}^*(s) + \mu$$

and for $n \ge c-1$,

$$(4.18b)\quad \begin{aligned}(\lambda^+ + \tilde{\mu} + s)F_n^*(s) &= \frac{\lambda^+}{c}G_{c,n-c+1}^*(s) \\ &+ \frac{c-1}{c}\lambda^+ F_{n+1}^*(s) + \frac{c-1}{c}\tilde{\mu}F_{n-1}^*(s) + \mu.\end{aligned}$$

We have from (4.18b) and (4.15) that

$$(4.19)\quad a(x;s)\tilde{F}(x;s) = -\tilde{\mu}xF_{c-2}^*(s) + \lambda^+ F_{c-1}^*(s) - \frac{x}{1-x}\frac{c-1}{c}\mu,$$

where

$$a(x;s) = \tilde{\mu}x^2 - \frac{c}{c-1}(\lambda^+ + \tilde{\mu} + s)x + \frac{1}{c-1}\tilde{\mu}z_0(s)x + \lambda^+.$$

Applying the Rouche's theorem to $a(x;s) = f(x) + g(x)$ with $f(x) = \tilde{\mu}x^2 + \frac{1}{c-1}\tilde{\mu}z_0(s)x + \lambda^+$ and $g(x) = -\frac{c}{c-1}(\lambda^+ + \tilde{\mu} + s)x$, it is easy to see that there exists exactly one zero of $a(x;s)$ in $|x| < 1$, say $x_0(s)$. It can be easily seen that $0 < x_0(s) < \rho$ by using the intermideate value theorem. Since $\tilde{F}(x;s)$ is analytic in $|x| < 1$ for each fixed $s > 0$, the right hand side of (4.19) vanishes at $x = x_0(s)$ and hence we have

$$(4.20)\quad \frac{c-1}{c}(\lambda^- + c\mu)x_0(s)F_{c-2}^* - \frac{c-1}{c}\lambda^+ F_{c-1}^*(s) = -\mu\frac{x_0(s)}{1-x_0(s)}.$$

Combining (4.18a) and (4.20) we have the following linear system for $F_n^*(s)$, $0 \le n \le c-1$;
(4.21)

$$\begin{pmatrix} b_0 & c_0 & & & & \\ a_1 & b_1 & c_1 & & O & \\ & a_2 & b_2 & c_2 & & \\ & & \ddots & \ddots & \ddots & \\ & O & & a_{c-2} & b_{c-2} & c_{c-2} \\ & & & & a_{c-1} & b_{c-1} \end{pmatrix} \begin{pmatrix} F_0^*(s) \\ F_1^*(s) \\ F_2^*(s) \\ \vdots \\ F_{c-2}^*(s) \\ F_{c-1}^*(s) \end{pmatrix} = -\mu \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \\ \frac{x_0(s)}{1-x_0(s)} \end{pmatrix},$$

where

$$a_k = \begin{cases} \frac{k}{k+1}\mu_{k+1}, & 1 \le k \le c-2 \\ x_0(s)(\frac{c-1}{c})\tilde{\mu}, & k = c-1 \end{cases}$$
$$b_k = \begin{cases} -(\lambda^+ + \mu_{k+1} + s), & 0 \le k \le c-2 \\ -(\frac{c-1}{c})\lambda^+, & k = c-1 \end{cases}$$
$$c_k = \lambda^+, \ 0 \le k \le c-2.$$

It is clear that the linear system (4.21) has a unique solution for each $s \ge 0$ from that $|b_0| > |c_0|$ and $|b_k| > |a_k| + |c_k|$, $1 \le k \le c-2$ and $|b_{c-1}| > |a_{c-1}|$ which is equivalent to $\rho > |x_0(s)|$.

THEOREM 4. *In the $M/M/c$ G-queue with LCFS discipline and RCH removal strategy, the LST of sojourn time distribution, jointly with the probability of not being removed, is*

$$W^*(s) = \sum_{n=0}^{c-2} \pi_n F_n^*(s) + \pi_{c-1}\tilde{F}(\rho; s). \tag{4.22}$$

where $F_n^*(s)$, $0 \le n \le c-1$ and $\tilde{F}(x; s)$ are given in (4.21) and (4.19) respectively.

5. RCR with PS Discipline

Here the servers work according to the processor sharing discipline. If there are fewer customers than servers in the system, each customer is served by a single server. Otherwise total service capacity is shared equally among all customers in the system. We assume each server works at a constant total rate 1. Let X be the service demand of a tagged customer and N the number of customers in the system other than tagged customer. We define

$$F_n(t; x) = P(W \le t | X = x, N = n)$$

with corresponding LST $F_n^*(s; x) = \int_0^\infty e^{-st} F(dt; x)$ and

$$W_n(s; \omega) = \int_0^\infty e^{-\omega x} F_n^*(s; x)\, dx, \ \omega > 0$$

and the generating function

$$W(s;\omega,y) = \sum_{n=c-1}^{\infty} y^{n-c+1} W_n(s;\omega), \ |y| < 1.$$

Then the LST of the sojourn time distribution, conditioned on the service requirement being x, is given by

$$W^*(s|x) = \sum_{n=0}^{c-2} \pi_n F_n^*(s;x) + \pi_{c-1} \sum_{n=c-1}^{\infty} \rho^{n-c+1} F_n^*(s;x) \tag{5.1}$$

and that of unconditioned distribution

$$\begin{aligned} W^*(s) &= \int_0^\infty W^*(s|x)\mu e^{-\mu x}\,dx \\ &= \mu \sum_{n=0}^{c-2} \pi_n W_n(s;\mu) + \pi_{c-1}\mu W(s;\mu,\rho). \end{aligned} \tag{5.2}$$

If there are fewer customers than servers, that is, $N = n \le c-1$, then the tagged customer's service demand is reduced with rate 1 and for $n \ge c$ the tagged customer is served with rate $\frac{c}{n+1}$. Hence analogous to the previous sections, we obtain for $0 \le n \le c-1$

$$\begin{aligned} &\frac{F_n(t+h;x) - F_n(t;x)}{h} \\ &= \lambda^+ F_{n+1}(t;x-h) + \left(\frac{n}{n+1}\lambda^- + n\mu\right)F_{n-1}(t;x-h) \\ &\quad - \frac{F_n(t;x) - F_n(t;x-h)}{h} - (\lambda + n\mu)F_n(t;x-h) + \frac{o(h)}{h} \end{aligned} \tag{5.3a}$$

and for $n \ge c$, letting $h_n = \frac{c}{n+1}h$

$$\begin{aligned} &\frac{F_n(t+h;x) - F_n(t;x)}{h} \\ &= \lambda^+ F_{n+1}(t;x-h_n) + \frac{n}{n+1}(\lambda^- + c\mu)F_{n-1}(t;x-h_n) \\ &\quad - \frac{F_n(t;x) - F_n(t;x-h_n)}{h} - (\lambda + \frac{n}{n+1}c\mu)F_n(t;x-h_n) + \frac{o(h)}{h}. \end{aligned} \tag{5.3b}$$

Letting $h \to 0$ in (5.3) and rearranging, we have for $0 \le n \le c-1$,

$$\text{(5.4a)} \quad \frac{\partial F_n}{\partial t} = -\frac{\partial F_n}{\partial x} - (\lambda + n\mu)F_n + \lambda^+ F_{n+1} + \frac{n}{n+1}(\lambda^- + (n+1)\mu)F_{n-1}$$

and for $n \ge c$,

$$\text{(5.4b)} \quad \frac{\partial F_n}{\partial t} = -\frac{c}{n+1}\frac{\partial F_n}{\partial x} - (\lambda + \frac{n}{n+1}c\mu)F_n + \lambda^+ F_{n+1} + \frac{n}{n+1}\tilde{\mu}F_{n-1}.$$

Taking the LST of $F_n(t;x)$ with respect to t and multiplying $(n+1)$ to the both sides of (5.3), since $F_n'(0;x) = 0$, $n \ge 0, x > 0$, we have for $0 \le n \le c-1$

$$\text{(5.5a)} \quad \begin{aligned} (n+1)\frac{\partial F_n^*(s;x)}{\partial x} = &-(n+1)(s+\lambda+n\mu)F_n^*(s;x) \\ &+ (n+1)\lambda^+ F_{n+1}^*(s;x) + n(\lambda^- + n\mu)F_{n-1}^*(s;x) \end{aligned}$$

and for $n \ge c$

$$\text{(5.5b)} \quad \begin{aligned} c\frac{\partial F_n^*(s;x)}{\partial x} = &-[(n+1)(s+\lambda) + nc\mu]F_n^*(s;x) \\ &+ (n+1)\lambda^+ F_{n+1}^*(s;x) + n\tilde{\mu}F_{n-1}^*(s;x). \end{aligned}$$

Noting that $F_n^*(s;0) = 1$, $n \ge 0$, we get that

$$\int_0^\infty \frac{\partial F_n^*(s;x)}{\partial x} e^{-\omega x}\, dx = -1 + \omega W_n(s;\omega)$$

and hence that for $0 \le n \le c-1$

$$\text{(5.6a)} \quad \begin{aligned} &n(\lambda^- + (n+1)\mu)W_{n-1} - (n+1)(s+\omega+\lambda+n\mu) \\ &\qquad W_n + (n+1)\lambda^+ W_{n+1} = -(n+1), \end{aligned}$$

and for $n \ge c$

$$\text{(5.6b)} \quad n\tilde{\mu}W_{n-1} - (c\omega + (n+1)(s+\lambda) + nc\mu)W_n + (n+1)\lambda^+ W_{n+1} = -c.$$

Multipling (5.6a) for $n = c-1$ and (5.6b) by y^{n-c+1} and summing over n from $n = c-1$ to ∞ yield, after simplication,

$$(5.7) \qquad -Q(y)\frac{\partial W(s;\omega,y)}{\partial y} = P(y)W(s;\omega,y) + g(s;\omega,y),$$

where

$$Q(y) = \tilde{\mu}y^2 - (s+\lambda^+ + \tilde{\mu})y + \lambda^+$$

$$(5.8) \qquad P(y) = c\tilde{\mu}y - c(\omega + s + \lambda^+ + \mu_{c-1}) + \frac{(c-1)\lambda^+}{y}$$

$$g(y) = \frac{c}{1-y} + (c-1)\tilde{\mu}W_{c-2}(s;\omega) - \frac{(c-1)\lambda^+}{y}W_{c-1}(s;\omega)$$

and the boundary condition is given by

$$(5.9) \qquad W(s;\omega,0) = W_{c-1}(s;\omega).$$

Now we solve the linear differential equation (5.7) with the boundary condition (5.9) by using the integrating factor. The solutions y_1 and y_2 of $Q(y) = 0$ with $0 < y_1 < 1 < y_2$ are given by

$$y_1 = \frac{1}{2\tilde{\mu}}\left[(\lambda + s + c\mu) - \sqrt{D}\right], \quad y_2 = \frac{1}{2\tilde{\mu}}\left[(\lambda - s + c\mu) + \sqrt{D}\right],$$

where

$$D = (\lambda + s + c\mu)^2 - 4\lambda^+\tilde{\mu}.$$

The integrating factor $M(y)$ of (5.7) is given by

$$(5.10) \qquad \begin{aligned} M(y) &= \exp\left[\int \frac{P(y)}{Q(y)}\,dy\right] \\ &= y^{c-1}(y-y_1)^{A_1}(y-y_2)^{A_2}, \end{aligned}$$

where

$$A_1 = A_1(s,\omega) = \frac{1}{2\sqrt{D}}(s + \lambda + c(2\omega - \mu) + \sqrt{D})$$

$$A_2 = A_2(s,\omega) = -\frac{1}{2\sqrt{D}}(s + \lambda + c(2\omega - \mu) - \sqrt{D}).$$

Taking branch cuts $\{u+iv : v = 0, y_1 < u < y_2\}$ for $y < y_1$ and $\{u+iv : v=0, u<y_1 \text{ or } u>y_2\}$ for $y_1 < y < y_2$, $M(y)$ can be written by

$$(5.11) \qquad M(y) = \begin{cases} -y^{c-1}|y-y_1|^{A_1}|y-y_2|^{A_2}, & \text{if } 0 \le y \le y_1 \\ e^{iA_2\pi}y^{c-1}|y-y_1|^{A_1}|y-y_2|^{A_2}, & \text{if } y_1 \le y < 1. \end{cases}$$

Thus we get

(5.12)

$$-\frac{M(y)}{Q(y)} = \begin{cases} \frac{1}{\tilde{\mu}}y^{c-1}|y-y_1|^{A_1-1}|y-y_2|^{A_2-1}, & \text{if } 0 \le y \le y_1 \\ e^{iA_2\pi}\frac{1}{\tilde{\mu}}y^{c-1}|y-y_1|^{A_1-1}|y-y_2|^{A_2-1}, & \text{if } y_1 \le y < 1. \end{cases}$$

For the notational convenience, we let for $0 < y < 1$,

$$\begin{aligned} U(y) &= y^{c-1}|y-y_1|^{A_1}|y-y_2|^{A_2}, \\ V_1(y) &= \frac{y^{c-1}}{1-y}|y-y_1|^{A_1-1}|y-y_2|^{A_2-1}, \\ V_2(y) &= y^{c-1}|y-y_1|^{A_1-1}|y-y_2|^{A_2-1}, \\ V_3(y) &= y^{c-2}|y-y_1|^{A_1-1}|y-y_2|^{A_2-1}, \end{aligned}$$

and define

$$J_i(y) = \int_{y_1}^{y} V_i(y)\,dy, \quad y_1 < y < 1, \; i = 1,2,3,$$

$$K_i(y) = \begin{cases} \int_0^y V_i(y)\,dy, & \text{if } 0 < y \le y_1 \\ \int_0^{y_1} V_i(y)\,dy + e^{iA_2\pi}J_i(y), & \text{if } y_1 < y < 1, \end{cases}, \quad i = 1,2,3.$$

Note that since $A_1 > 0$ for all $s \ge 0, \omega \ge 0$ and $y_2 > 1$, we have $A_1 - 1 > -1$ and for $0 < y < 1$, the integrals $J_i(y)$ and $K_i(y)$ are well-defined. The solution of (5.7) with (5.8) and (5.9) is given by

$$\begin{aligned} M(y)W(s;\omega,y) &= \int_0^y \Big(-\frac{M(x)}{Q(x)}\Big)g(x)\,dx \\ &= \frac{c}{\tilde{\mu}}K_1(y) + (c-1)W_{c-2}(s;\omega)K_2(y) - (c-1)\rho W_{c-1}(s;\omega)K_3(y). \end{aligned} \qquad (5.13)$$

Since $W(s,\omega,y)$ is analytic in $|y| < 1$ and $M(y_1) = 0$, we have

(5.14)
$$\frac{c}{\tilde{\mu}}K_1(y_1) + (c-1)W_{c-2}(s;\omega)K_2(y_1) - (c-1)\rho K_3(y_1)W_{c-1}(s;\omega) = 0.$$

Combining (5.13) and (5.14), we have for $0 < y < y_1$,

(5.15a)
$$W(s;\omega,y) = -\frac{1}{U(y)}\Big[\frac{c}{\tilde{\mu}}K_1(y) + (c-1)W_{c-2}(s;\omega)K_2(y) - (c-1)\rho W_{c-1}(s;\omega)K_3(y)\Big]$$

and for $y_1 < y < 1$

(5.15b)
$$W(s;\omega,y) = \frac{1}{U(y)}\Big[\frac{c}{\tilde{\mu}}J_1(y) + (c-1)W_{c-2}(s;\omega)J_2(y) - (c-1)\rho W_{c-1}(s;\omega)J_3(y)\Big]$$

and for $y = y_1$

(5.15c)
$$W(s;\omega,y_1) = \frac{1}{A_1|y_1-y_2|}\Big[\frac{c}{\tilde{\mu}}\frac{1}{1-y_1} + (c-1)W_{c-2}(s;\omega) - (c-1)\rho W_{c-1}(s;\omega)\frac{1}{y_1}\Big].$$

Combining (5.6a) for $0 \le n \le c-2$ and (5.14) yields the following linear system of equations

(5.16)
$$\begin{pmatrix} b_0 & c_0 & & & & \\ a_1 & b_1 & c_1 & & \boldsymbol{O} & \\ & a_2 & b_2 & c_2 & & \\ & & \ddots & \ddots & \ddots & \\ & \boldsymbol{O} & & a_{c-2} & b_{c-2} & c_{c-2} \\ & & & & a_{c-1} & b_{c-1} \end{pmatrix} \begin{pmatrix} W_0(s;\omega) \\ W_1(s;\omega) \\ W_2(s;\omega) \\ \vdots \\ W_{c-2}(s;\omega) \\ W_{c-1}(s;\omega) \end{pmatrix} = \begin{pmatrix} d_0 \\ d_1 \\ d_2 \\ \vdots \\ d_{c-2} \\ d_{c-1} \end{pmatrix},$$

where

$$a_k = \begin{cases} k(\lambda^- + (k+1)\mu), & 1 \le k \le c-2 \\ (c-1)K_2(y_1), & k = c-1 \end{cases}$$
$$b_k = \begin{cases} -(k+1)(s+\omega+\lambda+k\mu), & 0 \le k \le c-2 \\ -(c-1)\rho K_3(y_1), & k = c-1 \end{cases}$$
$$c_k = (k+1)\lambda^+, \ 0 \le k \le c-2.$$
$$d_k = \begin{cases} -(k+1), & 1 \le k \le c-2 \\ -\frac{c}{\mu}K_1(y_1), & k = c-1. \end{cases}$$

To show that (5.14) has unique solution, we show that the matrix in (5.14) is strictly diagonally dominant. We define

$$K(y) = \rho|K_3(y)| - |K_2(y)|.$$

Since $0 < y_1 < 1 < y_2$ and $y_1 y_2 = \rho > y_1$, we have for $0 < y < y_1$,

$$K'(y) = |y - y_1|^{A_1 - 1}|y - y_2|^{A_2 - 1} y^{c-2}(\rho - y) > 0$$

and $K(0) = 0$. Thus the coefficient matrix in (5.16) is strictly diagonally dominant and hence (5.16) has a unique solution for each $s \ge 0$ and $\omega \ge 0$.

THEOREM 5. *In the M/M/c G-queue with PS discipline and RCR removal strategy, $W(s,\omega)$ is given by*

$$W(s,\omega) = \sum_{n=0}^{c-2} \pi_n W_n(s;\omega) + \pi_{c-1} W(s,\omega,\rho), \tag{5.17}$$

where $W_n(s,\omega)$, $0 \le n \le c-2$ and $W(s,\omega,\rho)$ are given in (5.16) and (5.15) respectively. The LST of sojourn time distribution, jointly with the probability of not being removed is

$$W^*(s) = \mu \sum_{n=0}^{c-2} \pi_n W_n(s;\mu) + \pi_{c-1}\mu W(s,\mu,\rho). \tag{5.18}$$

Since $\lim_{s\to 0} y_1 = \rho$, *we get after routine calculation,*

$$W(0;\mu,\rho) = \lim_{s\to 0} W(s,\mu,y_1)$$

$$= \frac{c}{\tilde{\mu}(1-\rho)} + (c-1)[W_{c-2}(0;\mu) - W_{c-1}(0;\mu)] \tag{5.19}$$

and the probability that the tagged customer is not removed is given by

$$W^*(0) = \mu \sum_{n=0}^{c-2} \pi_n W_n(0;\mu) + \pi_{c-1}\mu W(0,\mu,\rho).$$

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