셀룰러 오토마타 CA-6의 동적 양상에 관한 연구

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요 약

두가지 상태값 (0, 1)과 서로 다른 네가지 경계조건을 갖는 비선형 셀룰러 오토마타 CA-6의 동적양상을 고정점과 상태 천이 그래프를 생성하는 재귀식의 관점에서 규명하였다. 특히 발견된 재귀식은 다음과 같다:
C(m)=A(m-1)+C(m-1) 그리고 D(m)=B(m-1)+D(m-1)

A Study On The Dynamic Behavior Of Cellular Automata CA-6

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ABSTRACT

The dynamic behavior of nonlinear cellular automata CA-6 with two states (0 and 1) and four different boundary conditions is identified in terms of the fixed point and the recursive formulae generating the state transition graph. The recursive formulae explored are, in particular, as following:
C(m)=A(m-1)+C(m-1) and D(m)=B(m-1)+D(m-1).

1. Introduction

Cellular automata are discrete dynamical systems that generate diverse, complicated behavior[3,8,10,11,13]. First introduced in 1948 by Von Neumann and Ulman[1] as potential models for biological self-reproduction, cellular automata have since been used as mathematical models for many investigations in natural science, combinatorial mathematics and computer science; in particular they represent a natural way of studying the evolution of large physical systems. They also constitute a general paradigm for parallel computation, such as Turing machines do for serial computation[15]. Many scientists, in particular, have been trying to investigate the dynamic behavior of infinite or finite cellular automata with various methods. Aloke tried to characterize cellular automata with matrix algebra and also studied to characterize additive cellular automata based on the depth of state transition graph. Moreover, Voorhees presented an analysis of nearest-neighbor cellular automata based on the seperation, for each automaton rule, of additive and non additive parts. Furthermore, attempt to analyze the dynamic behavior of finite cellular automata by recursive formulae for state transition diagram has been made by Lee.

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However since it is highly difficult to find the recursive formulae for the state transition diagram of even elementary cellular automata clearly, many rules have been still remained unexplored except rules 1,4,5,8,12,19,29,36,72, 76,108,140 and 200. In this paper, it is aimed to identify the dynamic behavior of rule 6 (CA-6) in terms of the general formulae for numbers of fixed points and their patterns, the general formulae for numbers of adjacent nodes which are attracted to nonzero fixed points, maximal cycle length and the recursive formulae for state transition diagram.

In general, cellular automata can be defined as a spatial lattice of sites whose values at each time step t are determined as a transition function of the values of neighboring sites at the previous time step t-1 [2]. This function provides the rule governing the automata's behavior. Specially, consider the class of automata defined on a one dimensional set of sites x_i , each of which assumes any of the values {0,1}. The general form of a rule for such an automaton is then given by

$$x_i^{t+1} = f(x_{i-r}^t, \ldots, x_i^t, \ldots, x_{i+r}^t), \qquad (1.1)$$
$$f:\{0,1\}^{2r+1} \rightarrow \{0,1\}$$

where $r \geq 0$ represents the size of the neighborhood considered by the rule and each site x_i is assigned an initial value x_i^0. Elemetary cellular automata of r=1 are defined by rules of the form:

$$x_i^{t+1} = f(x_{i-1}^t, x_i^t, x_{i+1}^t), \qquad (1.2)$$
$$f:\{0,1\}^3 \rightarrow \{0,1\}$$

A rule is therefore equivalently defined by specifying the value assigned to each of the 2^3 possible 3-tuple configurations of site values: i.e. by specifying the a_i, i=0,....7 such that

$$\begin{array}{cccccccc} 111 & 110 & 101 & 100 & 011 & 010 & 001 & 000 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & a_0 \end{array} \qquad (1.3)$$

Since each $a_i \in \{0,1\}$, there is a total of $2^{2^3}=256$ possible rules[2].

Wofram has defined a labeling scheme according to which a rule is assigned a value

$$\text{rule number} = R = \sum_{i=0}^{7} a_i \cdot 2^i$$

where a_i is the value assigned to the 3-tuple corresponding to the number i in binary representation[3].

CA-6 defined by

$$f(x_{i-1}^t, x_i^t, x_{i+1}^t) = x_i^{t+1} \qquad (1.4)$$
$$= \begin{cases} x_i^t & when\ x_{i-1}^t = x_{i+1}^t = 0 \\ \overline{x_i^t} & when\ x_{i-1}^t = 0, x_{i+1}^t = 1 \\ 0 & else \end{cases}$$

can be rewritten in the form of (1.3) as

$$\begin{array}{cccccccc} 111 & 110 & 101 & 100 & 011 & 010 & 001 & 000 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array} \qquad (1.5)$$

A cellular automaton $CA-R_{a-b}(m)$ with boundary condition a-b (a, b = 0 or 1), cell-size m and rule number R is a dynamical system (X_m, δ_{a-b}^m).

Here X_m is the set of states and a state transition function δ_{a-b}^m is defined by

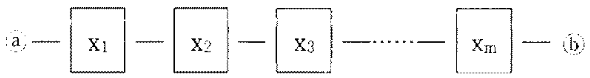
$$\delta_{a-b}^m(x_1 x_2 \cdots x_{m-1} x_m) = f(a x_1 x_2) f(x_1 x_2 x_3) \cdots f(x_{m-1} x_m b) \qquad (1.6)$$

$$\delta_{a-b}^0(\varepsilon) = \varepsilon \text{ (empty string)}$$

where f is a triplet local transition function with rule number R. Its configuration is shown in Fig.1.1.

Now, let us define cellular automata A(m), B(m), C(m) and D(m) for four different boundary conditions as following:

$$A(m) = CA - R_{0-0}(m) = (X_m, \delta^m_{0-0}),$$
$$B(m) = CA - R_{0-1}(m) = (X_m, \delta^m_{0-1}),$$
$$C(m) = CA - R_{1-0}(m) = (X_m, \delta^m_{1-0}),$$
$$D(m) = CA - R_{1-1}(m) = (X_m, \delta^m_{1-1})$$



(Fig. 1.1) A configuration of $CA - R_{a-b}(m)$

Section 2 will analyse fixed points of transition function of CA-6. In section 3, the simple recursive formulae for the transition diagram will be explored. Finally section 4 will make conclusions on the characteristics of CA-6.

2. An analysis of fixed points of CA-6 with boundary conditions

2.1 The characteristics of fixed points in A(m)

A(m) has more than two fixed points but cycles do not exist. Fixed points and number of nodes in the transition graph that are adjacent to the corresponding fixed point is summarized in Table 2.1. The values in each cell indicate fixed points and the values under slash are numbers of adjacent nodes which converge to those fixed points. Table 2.1 tells us that 0^m and $\sum_{i=1}^{\frac{m+1}{2}} 2^{m-(2i-1)}$ are fixed points and the general formular of numbers of fixed points is n+1 whenever cellsize m is both even (2n) and odd (2n 1) as shown in Table 2.2. Moreover, it can be seen from the table that the sequences {1, 1, 2, 4, 6, 9, 14, 21, ⋯} of numbers of adjacent nodes which converge to nonzero fixed points are all the same (see column 3 to 6). The general formular of this sequence is

$$s_n = \frac{k^3 - 3k^2 + 14k + 12}{6},\ k = n-3,\ n \geq 4.$$

<Table 2.1> Fixed points and Numbers of adjacent nodes which converge to nonzero fixed points in A(m)

m \ fixed points	0^m	2^{m-1}	$2^{m-1}+2^{m-3}$	$2^{m-1}+2^{m-3}+2^{m-5}$	$2^{m-1}+2^{m-3}+2^{m-5}+2^{m-}$
1	0 / 1	2^0 / 1	/	/	/
2	0^2 / 2	2^1 / 1	/	/	/
3	0^3 / 3	2^2 / 2	2^2+2^0 / 1	/	/
4	0^4 / 4	2^3 / 4	2^3+2^1 / 1	/	/
5	0^5 / 6	2^4 / 6	2^4+2^2 / 2	$2^4+2^2+2^0$ / 1	/
6	0^6 / 9	2^5 / 9	2^5+2^3 / 4	$2^5+2^3+2^1$ / 1	/
7	0^7 / 13	2^6 / 14	2^6+2^4 / 6	$2^6+2^4+2^2$ / 2	$2^6+2^4+2^2+2^0$ / 1
8	0^8 / 19	2^7 / 21	2^7+2^5 / 9	$2^7+2^5+2^3$ / 4	$2^7+2^5+2^3+2^1$ / 1

〈Table 2.2〉 The number of fixed points in A(m)

m	1	2	3	4	5	6	7	8
numbers of fixed points	2	2	3	3	4	4	5	5

2.2 The characteristics of fixed points in B(m)

B(m) has a single fixed point $\sum_{i=1}^{\frac{m}{2}} 2^{2i-1}$ when cellsize m is even. When cellsize m is odd, B(m) has no fixed points but cycles do exist. The maximal cycle length is 4 for $m \geq 3$.

2.3 The characteristics of fixed points in C(m)

The behavior of C(m) is quite similar to that of A(m). C(m) has no cycle but fixed points exist. By Table 2.3, fixed points are 0^m and $\sum_{i=1}^{\lfloor \frac{m}{2} \rfloor} 2^{m-2i}$. The number of fixed points is shown in Table 2.4. Furthermore, the sequence of numbers of adjacent nodes which converge to each of nonzero fixed points is the same as that in A(m) given in Table 2.1.

2.4 The characteristics of fixed points in D(m)

Like B(m), D(m) has a single fixed point $\sum_{i=1}^{\frac{m}{2}} 2^{2i-1}$ (m〉1) when cellsize m is odd. When cellsize m is even, D(m) has no fixed points but cycles exist. The maximal cycle length is also 4 for $m \geq 4$.

〈Table 2.4〉 numbers of fixed points in C(m)

m	1	2	3	4	5	6	7	8
numbers of fixed points	1	2	2	3	3	4	4	5

3. The formalism of transition diagrams

The formalization for the transition diagram of cellular automata in an algebraic method is highly difficult because the evolution of cellular automata is unpredictable. In this section, however, one will try to represent the state transition diagrams of CA-6 by simple recursive formulae. To do this, let us introduce two operators ϕ and φ in definition 1 and definition 2 respectively.

Consider X_{m-1} as the set of states of cellular automata with cellsize m-1, i.e. $X_{m-1} = \{x_1 x_2 \cdots x_{m-1} | x_i \in \{0, 1\}\}$ and X_m, X_m^0, X_m^1

〈Table 2.3〉 fixed points and numbers of adjacnet nodes which converge to each of nonzero fixed points

m \ fixed points	0^m	2^{m-2}	$2^{m-2}+2^{m-4}$	$2^{m-2}+2^{m-4}+2^{m-6}$	$2^{m-2}+2^{m-4}+2^{m-6}+2^{m-8}$
1	0 / 2	/	/	/	/
2	0^2 / 3	2^0 / 1	/	/	/
3	0^3 / 5	2^1 / 1	/	/	/
4	0^4 / 8	2^2 / 2	2^2+2^0 / 1	/	/
5	0^5 / 12	2^3 / 4	2^3+2^1 / 1	/	/
6	0^6 / 18	2^4 / 6	2^4+2^2 / 2	$2^4+2^2+2^0$ / 1	/
7	0^7 /	2^5 / 9	2^5+2^3 / 4	$2^5+2^3+2^1$ / 1	/
8	0^8 /	2^6 / 14	2^6+2^4 / 6	$2^6+2^4+2^2$ / 2	$2^6+2^4+2^2+2^0$ / 1

as those with cellsize m, satisfied with

$$X_m = X_m^0 \cup X_m^1 \text{ and } X_m^0 \cap X_m^1 = \emptyset \qquad (3.1)$$

where

$$X_m^0 = \{0x_1x_2\cdots x_{m-1} | x_1x_2\cdots x_{m-1} \in X_{m-1}\}$$
and
$$X_m^1 = \{1x_1x_2\cdots x_{m-1} | x_1x_2\cdots x_{m-1} \in X_{m-1}\}$$

Definition 1

A transition function φ_m is defined as

$$\varphi_m: X_m^0 \rightarrow X_m,\ \varphi_m(X_m^0) \subset \delta_{a-b}^m(X_m)$$

such that $\varphi_m(0x_1x_2\cdots x_{m-1}) = \delta_{a-b}^{m-1}(x_1x_2\cdots x_{m-1})$ for all $0x_1x_2\cdots x_{m-1} \in X_m^0$ and $x_1x_2\cdots x_{m-1} \in X_{m-1}$ where the transition function δ_{a-b}^{m-1} is that defined in (1.6).

Definition 2

A transition function ψ_m is defined as $\psi_m: X_m^1 \rightarrow X_m$, $\psi_m(X_m^1) \subset \delta_{a-b}^m(X_m)$ such that $\psi_m(1x_1x_2\cdots x_{m-1}) = 0\delta_{a-b}^{m-1}(x_1x_2\cdots x_{m-1})$ for all $1x_1x_2\cdots x_{m-1} \in X_m^1$ and $x_1x_2\cdots x_{m-1} \in X_{m-1}$.

In the following theorem, we want to show that C(m) in the rule number 6 is partitioned by the diagraphs derived by A(m-1) and C(m-1). Thus for convience, we consider φ_m and ψ_m as

$$\varphi_m(0x_1x_2\cdots x_{m-1}) = 0\delta_{0-0}^{m-1}(x_1x_2\cdots x_{m-1})$$

and $\psi_m(1x_1x_2\cdots x_{m-1}) = 0\delta_{1-0}^{m-1}(x_1x_2\cdots x_{m-1})$

Theorem 1

In the rule number 6, dynamic systems (X_m^0, φ_m) and (X_m^1, ψ_m) are a partition of (X_m, δ_{1-0}^m) which is a dynamic system C(m).

We will denote it as

$$C(m) = A(m-1) + C(m-1).$$

Proof.

By eq.(3.1),

$$X_m = X_m^0 \cup X_m^1 \text{ and}$$
$$X_m^0 \cap X_m^1 = \emptyset.$$

Hence it will suffice to prove the following relations:

For each $(0x_1x_2\cdots x_{m-1}) \in X_m^0$,

$$\delta_{1-0}^m(0x_1x_2\cdots x_{m-1}) = \varphi_m(0x_1x_2\cdots x_{m-1}) \quad (3.2)$$

and for each $(1x_1x_2\cdots x_{m-1}) \in X_m^1$

$$\delta_{1-0}^m(1x_1x_2\cdots x_{m-1}) = \psi_m(1x_1x_2\cdots x_{m-1}) \quad (3.3)$$

Let us prove this by mathematical induction.

(i) When m is 1, X_m is {0, 1} where X_m^0 is {0} and X_m^1 is {1}. Thus

$$\delta_{1-0}^1(0) = 0 \qquad (3.4)$$

since $\delta_{1-0}^1(0) = f(100) = 0$

and

$$\delta_{1-0}^1(1) = 0 \qquad (3.5)$$

since $\delta_{1-0}^1(1) = f(110) = 0$.

On the other hand, X_{m-1} is empty string.

By definition 1,

$$\varphi_1(0) = \varphi_1(0\varepsilon) = 0\delta_{0-0}^0(\varepsilon) = 0\varepsilon = 0 \qquad (3.6)$$

By definition 2,

$$\psi_1(1) = \psi_1(1\varepsilon) = 0\delta_{1-0}^0(\varepsilon) = 0\varepsilon = 0 \qquad (3.7)$$

By eq.(3.4) and eq.(3.6),

$$\delta_{1-0}^1(0) = \varphi_1(0)$$

and

by eq.(3.5) and eq.(3.7),

$$\delta_{1-0}^1(1) = \psi_1(1).$$

Hence it holds when the cellsize m is 1.

(ii) Assume that it is true when the cellsize is m, then we need to show that it holds when the cellsize is m+1.

$\delta^{m+1}_{1-0}(0x_1x_2\cdots x_m)$

$=f(10x_1)\cdot f(0x_1x_2)\cdots f(x_{m-1}x_m0)$

$=0\cdot f(0x_1x_2)\cdot f(x_1x_2x_3)\cdots f(x_{m-1}x_m0)$

$=0\cdot \delta^{m}_{0-0}(x_1x_2\cdots x_m)=\varphi_{m+1}(0x_1x_2\cdots x_m)$

by eq.(1.5), eq.(1.6) and eq.(3.2).

Also

$\delta^{m+1}_{1-0}(1x_1x_2\cdots x_m)$

$=f(11x_1)\cdot f(1x_1x_2)\cdot\cdots\cdot f(x_{m-1}x_m0)$

$=0\cdot f(1x_1x_2)\cdot f(x_1x_2x_3)\cdot\cdots\cdot f(x_{m-1}x_m0)$

$=0\,\delta^{m}_{1-0}(x_1x_2\cdots x_m)=\psi_{m+1}(x_1x_2\cdots x_m)$

by eq.(1.5), eq.(1.6) and eq.(3.3).

These prove our theorem. □

Let us consider φ_m and ψ_m as

$$\varphi_m(0x_1x_2\cdots x_{m-1})=0\delta^{m-1}_{0-1}(x_1x_2\cdots x_{m-1})$$

and

$$\psi_m(1x_1x_2\cdots x_{m-1})=0\delta^{m-1}_{1-1}(x_1x_2\cdots x_{m-1})$$

.

Then we have theorem 2. The proof is word for word the same as proof of theorem 1, so will not be reproduced.

Theorem 2

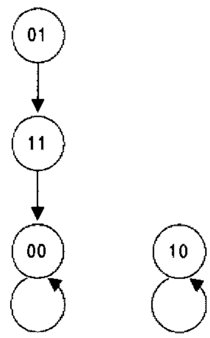
In the rule number 6, dynamic systems (X^0_m, φ_m) and (X^1_m, ψ_m) are a partition of (X_m, δ^m_{1-1}) which is a dynamic system D(m). In other word, D(m) in the rule 6 is partitioned by the diagraphs derived by B(m-1) and D(m-1).

We will denote it as

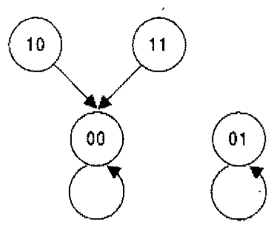
$$D(m)=B(m-1)+D(m-1).$$

For example, let us show that C(3)=A(2)+C(2). Transition diagrams of A(2) and C(2) are graphs with two connected components as shown in Fig.3.1, Fig.3.2 and Fig.3.3.

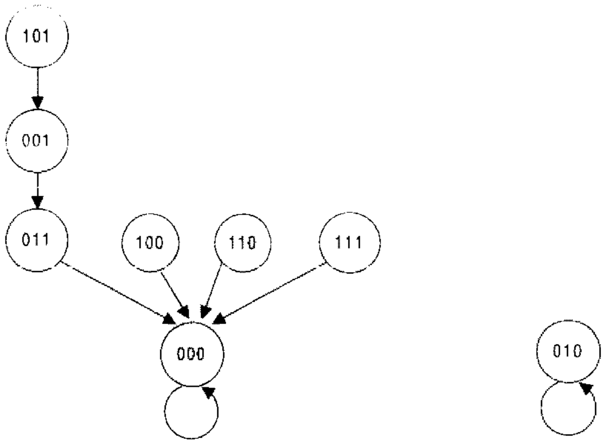
Graph G_1 which is created by prefixing 0 to each site string of A(2) as shown in Fig.3.1 becomes a subgraph of C(3) as shown in Fig.3.3. Strings $1x_1x_2$ created by prefixing 1 to each site string x_1x_2 of C(2) are transitted to the strings created by prefixing 0 to $\delta_{1-0}(x_1x_2)$ such that $\psi(1x_1x_2)=0\delta_{1-0}(x_1x_2)$ where δ_{1-0} is the transition function of C(2) as shown in Fig.3.2. Let us denote this graph as G_2. Then G_2 is also a subgraph of C(3). It can be noticed that the sets of {G_1, G_2} are a partition of C(3).



(Fig. 3.1) A(2)



(Fig.3.2) C(2)



(Fig. 3.3) C(3)

4. Conclusion

In this paper, the dynamic behavior of nonlinear cellular automata CA-6 with two states 0 and 1 and four different boundary conditions has been identified in terms of fixed points, maximal cycle length and the recursive formulae for state transition diagrams. It is to say that the dynamic behavior of A(m) is similar to C(m) and also the dynamic behavior of B(m) is similar to that of D(m). A(m) and C(m) have fixed points but cycles do not exist. On the other hand, B(m) and D(m) have cycles with no fixed points or have just single fixed point with no cycles. 0^m and $\sum_{i=1}^{\lfloor \frac{m+1}{2} \rfloor} 2^{m-(2i-1)}$ are fixed points of A(m) and the general formula of numbers of fixed points is n+1 whenever cellsize m is both even (2n) and odd (2n-1). Moreover the general expression of sequences of numbers of adjacent nodes which converge to nonzero fixed points is

$$s_n = \frac{k^3 - 3k^2 + 14k + 12}{6}, k = n-3, n \geq 4.$$

Fixed points of C(m) are 0^m and $\sum_{i=1}^{\lfloor \frac{m}{2} \rfloor} 2^{m-2i}$. The sequence of numbers of adjacent nodes which converge to each of nonzero fixed points is the same as that in A(m). B(m) has single fixed point $\sum_{i=1}^{\frac{m}{2}} 2^{2i-1}$ only when cellsize m is even. When cellsize m is odd, B(m) has no fixed points but cycles whose maximal cycle length is 4 ($m \geq 3$) do exist. Like B(m), D(m) has a single fixed point $\sum_{i=1}^{\frac{m}{2}} 2^{2i-1}$ (m>1) when cellsize m is odd. When cellsize m is even, D(m) has no fixed points but cycles exist. The maximal cycle length is 4 for $m \geq 4$. The recursive formulae for state transition diagrams are C(m)=A(m-1)+C(m-1) and D(m)=B(m-1)+D(m-1). This recursive formulae can make to generate the state stransition diagram automatically. However many rules of one dimensional cellular automata have been still remained unexplored. Thus we will try to find the recursive formula of those rules continuously. Studies on the formalization of finite cellular automata which have more than 3-neighbor, more than two states and more than two dimension remain as further works.

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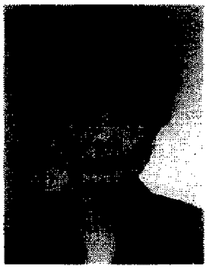

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