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Fuzzy Maps

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ABSTRACT

We introduce a concept of a 'fuzzy' map between sets by modifying the concept of the extension principle introduced by Dubois and Prade in [1] and by using this we generalize Goguen's and Zadeh's extension principles in [2] and [3].

1. Introduction

We say that a crisp map from a crisp set into another crisp set is fuzzified when it is extended to act on fuzzy sets defined on the crisp sets. Such a principle will be called an extension principle. The extension principle is one of the most basic concepts of fuzzy set theory which can be used to generalize crisp mathematical concepts to fuzzy sets.

In this paper we introduce a concept of a 'fuzzy map' between sets by modifying the concept of the extension principle introduced by Dubois and Prade in [1] and then we show that the concept of fuzzy maps is a generalization of that of crisp maps. Moreover, by using this we make generalizations of the extension principles introduced by Goguen and Zadeh in [2,3].

We will recall basic definitions and notations related to fuzzy set theory. The unit interval will be denoted by I. For a set X, a crisp map $\alpha : X \to I$ is said to be a fuzzy set on X and I^X denotes the set of all fuzzy sets on X. For $\alpha \in I^X$, $S(\alpha)$ denotes $\{x \in X: \alpha(x) > 0\}$. Let $x \in X$ and $t \in I$. Then by a fuzzy point x_t in a set X we mean the fuzzy set in X given below:

$$x_t(y) = \begin{cases} t & \text{if } x = y \\ 0 & otherwise \end{cases}$$

A fuzzy set $\alpha : X \to I$ is a fuzzy point in X if and only if $S(\alpha)$ is a singleton set or $\emptyset$.

Now we will list three types of extension principles:

Dubois and Prade's Extension Principle [1]: For two crisp sets X and Y, a crisp map $f : X \to I^Y$ is said to be a dp-fuzzifying map if there is a crisp map μ: $X \times Y \to$ I such that for all $(x, y) \in X \times Y$, $f(x)(y) = \mu(x, y)$.

Goguen's Extension Principle [2]: For $\alpha \in I^X$ and $\beta \in I^Y$ a crisp map $f : X \to Y$ is said to be a g-fuzzy morphism if $\alpha \leq f^1$ (β).

Zadeh's Extension Principle [3]: Every crisp map f: $X \to Y$ induces two fuzzifing maps $f : I^X \to I^Y$ and f^{-1}: $I^Y \to I^X$ which are defined by

$$f(\alpha)(y) = sup_{y=f(x)} \alpha(x) \text{ for all } \alpha \in I^Y$$

and

$$f^{-1}(\beta)(x) = \beta(f(x)) \text{ for all } \beta \in I^X .$$

2. Fuzzy Maps

In this section we introduce the concepts of fuzzy maps and r-level fuzzy maps, and then investigate some of their properties.

Definition 1.1 Let X and Y be two sets. A dp-fuzzifing map e: $X \to I^Y$ is said to be a fuzzy map if for each $x \in X$, $e(x)$ is a fuzzy point.

Definition 1.2 Let $r \in I$. A fuzzy map e: $X \to I^Y$ is said to be an r-level if for each $x \in X$ there is an $y \in Y$ such that $r \leq e(x)(y)$.

Remark Every fuzzy map is a 0-level fuzzy map and every crisp map is an 1-level fuzzy map.

Notation For an r-level fuzzy map e: $X \to I^Y$, D_e denotes $\{x \in X: S(e(x)) \neq \emptyset\}$ and R_e denotes $\cup \{S(e(x)): x \in X\}$.

Remark If e: $X \to I^Y$ is an r-level fuzzy map and $0<r$, then $D_e=X$.

Proposition 1.3 1) Let e: $X \to I^Y$ be an r-level fuzzy map and let f_e: $D_e \to R_e$ be defined by $f_e(x)=y$ if and only if $y \in S(e(x))$. Then f_e is an onto crisp map.

2) Let f: $X \to Y$ a crisp map and let e_f: $X \to I^Y$ be defined by

$$e_f(x)(y)=\begin{cases}1 & \text{if } f(x)=y\\ 0 & \text{if } f(x)\neq y\end{cases}$$

Then e_f is an 1-level fuzzy map.

3) $f_{e_f}=f$ and $e \leq e_{f_e}=1$ on D_e.

Proof. Straightforward.

Example 1) Let R be the set of real numbers and let e: $R \to I^R$ be defined by

$$e(x)(y)=\begin{cases}(1+(x-10)^{-2})^{-1} & \text{if } y=10 \text{ and } 10<x\\ 0 & \text{otherwise}\end{cases}$$

Then e is a fuzzy map.

2) Let X be a crisp set and $r \in I$. Let 1_X^r: $X \to I^X$ be defined by

$$1_X^r(x)(y)=\begin{cases}r & \text{if } x=y\\ 0 & \text{otherwise}\end{cases}$$

Then 1_X^r is an r-level fuzzy map, which is called the r-level fuzzy identity map on X.

Definition 1.4 An r-level fuzzy map e: $X \to I^Y$ is said to be *equipotent* if there is an r-level fuzzy map e: $Y \to I^X$ such that $e(x)(y)=e'(y)(x)$ for each $(x, y) \in X \times Y$. In this case we say that X and Y are *r-equipotent* with respect to e and we write $e'=e^{-1}$ and $X \cong_r Y$.

Proposition 1.5 If e: $X \to I^Y$ is an r-level equipotent fuzzy map, then f_e: $D_e \to R_e$ is a crisp 1-1 map and hence D_e and R_e are equipotent.

Proof. Suppose $f_e(x)=f_e(x')$. Then there is a unique $y \in Y$ with $e(x)(y) \wedge e(x')(y) \geq r>0$. Since e is r-level equipotent, $e^{-1}(y)(x)=e^{-1}(y)(x')$ and hence $x=x'$.

Proposition 1.6 Let e: $X \to I^Y$ and e': $Y \to I^Z$ *be r-level* fuzzy maps. For each $(x, z) \in X \times Z$, let $(e' \circ e)(x)(z)=\bigvee_{y \in Y}(e(x)(y) \wedge e'(y)(z))$. Then $e' \circ e$: $X \to I^Z$ is an r-level fuzzy map.

Proof. Suppose $(e' \circ e)(x)(z) \wedge (e' \circ e)(x)(z') \geq r>0$. Then there is a unique $y \in Y$ with $e'(y)(z) \wedge e'(y)(z') \geq r>0$ and so $z=z'$.

Proposition 1.7 1) If f: $X \to Y$ and g: $Y \to Z$ are crisp maps. Then $e_{g \circ f}=e_g \circ e_g$.

2) If e: $X \to I^Y$ and e': $Y \to I^Z$ are fuzzy maps, then $f_{e' \circ e}=f_{e'} \circ f_e$.

Proof. 1) Let $(x,z) \in X \times Z$. Then $e_{g \circ f}(x)(z)=1$ if and only if $(g \circ f)(x)=z$ if and only if there is a unique $y \in Y$ such that $f(x)=y$ and $g(y)=z$ if and only if there is a unique $y \in Y$ such that $e_f(x)(y) \wedge e_g(y)(z)=1$ if and only if $(e_g \circ e_f)(x)(z)=1$. Thus $e_{g \circ f}= e_g \circ e_f$.

2) Let $x \in X$ and $z \in Z$. $f_{e' \circ e}(x)=z$ if and only if there is a unique $y \in Y$ such that $e(x)(y) \wedge e'(y)(z)>0$ if and only if there is a unique $y \in Y$ such that $f_e(x)=y$ and $f_e(y)=z$ if and only if $(f_{e'} \circ f_e)(x)=z$. Thus $f_{e \circ e}=f_{e' \circ fe}$.

Proposition 1.8 Let e: $X \to I^Y$, e': $Y \to I^Z$ and e'': $Z \to I^W$ be r-level fuzzy maps. Then $(e'' \circ e') \circ e=e'' \circ (e' \circ e)$.

Proof. Straightforward.

Proposition 1.9 1) For a crisp set X, 1_X^r: $X \to I^X$ is an r-level equipotent fuzzy map.

2) Let e: $X \to I^Y$ and e': $Y \to I^Z$ be r-level equipotent fuzzy maps. Then $e' \circ e$: $X \to I^Z$ is an r-level equipotent fuzzy map.

Proof. Straightforward.

Theorem 1.10 For any $r \in I$, $\cong_r$ is an equivalence relation on the class of all sets.

Proof. It is immediate from Proposition 1.9.

3. Fuzzy Morphisms

In this section we introduce the concepts of fuzzy morphisms and then investigate some of their properties.

For a fuzzy map e: $X \to I^Y$ and $\alpha \in I^X$, $S_\alpha(e(x))$ denotes $\{y \in Y \mid e(x)(y) \geq \alpha(x)\}$.

Definition 2.1 Let α and β be fuzzy sets in X and

Y, respectively. Then a fuzzy map e: $X \to I^Y$ is said to be:

1) α-stable if for each $x \in X$, $S_\alpha(e(x)) \neq \emptyset$.

2) an (α, β)-fuzzy morphism (or simply fuzzy morphism) if $y \in S_\alpha(e(x))$ implies $\alpha(x) \leq \beta(y)$.

Proposition 2.2 Let e: $X \to I^Y$ be an equipotent fuzzy morphism and e^{-1}: $Y \to I^X$ a fuzzy morphism. If $\alpha(x) \vee \beta(y) \leq e(x)(y)$ then $\alpha(x) = \beta(y)$.

Proof. Straightforward.

Proposition 2.3 Let f: $X \to Y$ be a map and let α and β fuzzy sets in X and Y respectively. Then f is a g-fuzzy morphism if and only if e_f: $X \to I^Y$ is an α-stable fuzzy morphism.

Proof. ($\Rightarrow$) Take any $x \in X$. Then there is a unique $y \in Y$ with $e_f(x)(y)=1$ and hence $\alpha(x) \leq e_f(x)(y)$. Thus e_f: $X \to I^Y$ is α-stable. Suppose that $\alpha(x) \leq e_f(x)(y)$. If $\alpha(x)=0$ then $\alpha(x) \leq \beta(y)$. If $\alpha(x)>0$, then $e_f(x)(y)=1$ and hence $f(x)=y$. Since f is a g-fuzzy morphism, $\alpha(x) \leq f^1(\beta)(x)=\beta(f(x))=\beta(y)$. So $\alpha(x) \leq \beta(y)$. Thus e_f is a fuzzy morphism.

($\Rightarrow$) Take any $x \in X$. Then there is a $y \in Y$ with $f(x)=y$. Then $e_f(x)(y)=1$ and so $\alpha(x) \leq \beta(y)$. Hence $\alpha(x) \leq f^1(\beta)(x)$.

Proposition 2.4 Let e: $X \to I^Y$ and e': $Y \to I^Z$ be fuzzy maps and let α, β and γ be fuzzy sets on X, Y and Z, respectively. Then one has the following:

1) Suppose $y \in S_\alpha(e(x))$ implies $z \in S_\beta(e'(y))$ for some $z \in Z$. Then if e and e' are fuzzy morphisms, then $e' \circ e$: $X \to I^Z$ is a fuzzy morphism.

2) If e is an α-stable fuzzy morphism and e' is a β-stable fuzzy morphism, then $e' \circ e$ is a α-stable fuzzy morphism.

Proof. 1) Suppose $\alpha(x) \leq (e' \circ e)(x)(z)$. If $\alpha(x)=0$ then $\alpha(x) \leq \gamma(z)$. If $\alpha(x)>0$, then there is a unique $y \in Y$ such that $0<\alpha(x) \leq e(x)(y) \wedge e'(y)(z)$. Since e is a fuzzy morphism, $\alpha(x) \leq \beta(y)$. Since $\alpha(x) \leq e(x)(y)$, by the assumption, there is $z' \in Z$ $\beta(y) \leq e'(y)(z')$. Since e' is a fuzzy morphism and $0<e'(y)(z) \wedge e'(y)(z')$, $z=z'$ and $\beta(y) \leq \gamma(z)$. Thus $\alpha(x) \leq \gamma(z)$.

2) Take any $x \in X$. If $\alpha(x)=0$ then $S_\alpha((e' \circ e)(x)) \neq \emptyset$. Suppose $\alpha(x)>0$. Since e is α-stable fuzzy morphism, there is $y \in Y$ with $\alpha(x) \leq e(x)(y)$ and $\alpha(x) \leq \beta(y)$. Since e' is β-stable, there is $z \in Z$ with $\beta(y) \leq e'(y)(z)$. Thus $\alpha(x) \leq (e' \circ e)(x)(z)$ and so $S_\alpha((e' \circ e)(x)) \neq \emptyset$. Therefore $e' \circ e$ is an α-stable fuzzy map. Suppose $\alpha(x) \leq (e' \circ e)(x)(z)$. If $\alpha(x)=0$ then $\alpha(x) \leq \gamma(z)$. If $\alpha(x)>0$, then there is a unique $y \in Y$ such that $0<\alpha(x) \leq e(x)(y) \wedge e'(y)(z)$. Since e is a fuzzy morphism, $\alpha(x) \leq \beta(y)$. Since e' is a β-stable fuzzy map, $0<\beta(y) \leq e'(y)(z)$. Since e' is a fuzzy morphism, $\beta(y) \leq \gamma(z)$. Thus $\alpha(x) \leq \gamma(z)$.

Proposition 2.5 Let e: $X \to I^Y$ be a fuzzy map and let α and β be fuzzy sets in X and Y, respectively. Then one has the following:

1) Let $e^{-1}(\beta)$ be defined by $e^{-1}(\beta)(x)=\vee\{\beta(y) \wedge e(x)(y) | y \in Y\}$. Then $e^{-1}(\beta)$ is a fuzzy set on X.

2) Let $e(\alpha)$ be defined by $e(\alpha)(y)=\bigvee_{y \in S_\alpha(e(x))} \alpha(x)$. Then $e(\alpha)$ is a fuzzy set on Y.

Proof. Straightforward.

Proposition 2.6 Let e: $X \to I^Y$ be a fuzzy map and suppose that $0<t \leq e(x)(y)$. Then one has the following:

1) $e(x_t)=y_t$.

2) For each $z \in X$, $e^{-1}(y_t)(z)=t \wedge e(z)(y)$.

3) If e is equipotent, then $e^{-1}(y_t)=x_t$.

Proof. 1) If $x \neq u$ then $x_t(u)=0$ and so $e(x_t)(y)=\bigvee_{y \in S_t(e(u))} x_t(u)=x_t(x)=t$, for $0<t \leq e(x)(y)$. Suppose $z \neq y$. Then $e(x)(z)=0$. If $x_t(u) \leq e(u)(z)$ then $x \neq u$ and so $x_t(u)=0$. Thus $e(x_t)(z)=\bigvee_{z \in S_t(e(u))}=0$. Hence $e(x_t)=y_t$.

2) Take any $z \in X$. Then $e^{-1}(y_t)(z)=y_t(y) \wedge e(z)(y)=t \wedge e(z)(y)$, because $y_t(u)=0$ for every $u \in Y$-$\{y\}$.

3) Since e is equipotent and $0<t \leq e(x)(y)$, $e(z)(y)=0$ for every $z \in X$-$\{x\}$. Thus, by 2), $e^{-1}(y_t)=x_t$.

Proposition 2.7 Let e: $X \to I^Y$ be a fuzzy map. Then one has the following:

1) $e(\alpha)=0$ if and only if $\alpha=0$, where $S(\alpha) \subseteq \{x$: $S_\alpha(e(x)) \neq \emptyset\}$.

2) If $\alpha_1 \leq \alpha_2$ then $e(\alpha_1) \leq e(\alpha_2)$, where $S_{\alpha_1}(e(x)) \subseteq S_{\alpha_2}(e(x))$ for all $x \in X$.

3) $e(\bigvee_{j \in J} \alpha_j) \leq \bigvee_{j \in J} e(\alpha_j)$.

4) $e(\bigwedge_{j \in J} \alpha_j) \leq \bigwedge_{j \in J} e(\alpha_j)$.

5) If $\beta_1 \leq \beta_2$ then $e^{-1}(\beta_1) \leq e^{-1}(\beta_2)$.

6) $e^{-1}(\bigvee_{j \in J} \beta_j)=\bigvee_{j \in J} e^{-1}(\beta_j)$.

7) $e^{-1}(\wedge_{j\in J} \beta_j)=\wedge_{j\in J} e^{-1}(\beta_j)$.

8) $\alpha\le e^{-1}(e(\alpha))$, where $S(\alpha)\subseteq\{x: S_\alpha(e(x))\neq\emptyset\}$.

9) $e(e^{-1}(\beta))\le\beta$.

10) If for each $j\in J$, e is an α_j-stable fuzzy map, then $e(\vee_{j\in J} \alpha_j)=\vee_{j\in J} e(\alpha_j)$.

Proof. 1)~7) are obvious.

8) Suppose $0<\alpha(x)$. Then $\alpha(x)\le e(x)(y)$ for some $y\in Y$. Then $e^{-1}(e(\alpha))(x)=e(\alpha)(y)\wedge e(x)(y)=e(x)(y)\wedge(\vee\{\alpha(a)|y\in S(e(a))\}=\vee\{e(x)(y)\wedge\alpha(a)|e(a)(y)>0\}\ge e(x)(y)\wedge\alpha(x)=\alpha(x)$.

9) Note that if $y\in S(e(x))$ then $e^{-1}(\beta)(x)=\beta(y)\wedge e(x)(y)$. Take any $y\in Y$. Then $e(e^{-1}(\beta))(y)=\vee\{e^{-1}(\beta)(x) \mid y\in S(e(x))\}=\{e(x)(y)\wedge\beta(y) \mid y\in S(e(x))\}\le\beta(y)$. Hence $e(e^{-1}(\beta))\le\beta$.

10) Let $\beta=\vee_{j\in J}\alpha_j$. Then, by the assumption, if $a_j(x)>0$ then $S_{\alpha_j}(e(x))=S_\beta(e(x))$ for each $j\in J$. Thus, by 2), $e(\alpha_j)\le e(\beta)$ for each $j\in J$ and hence $\vee_{j\in J} e(\alpha_j)\le e(\vee_{j\in J}\alpha_j)$. Hence, by 3), $e(\vee_{j\in J}\alpha_j)=\vee_{j\in J} e(\alpha_j)$.

Theorem 2.8 Let X be a set and β_j a fuzzy set in a set X_j ($j\in J$). Suppose $(e_j: X\to I^{X_j})_{j\in J}$ is a source of fuzzy maps. Then there is a unique fuzzy set α in X satisfying the following:

1) For each $j\in J$, $e_j: X\to I^{X_j}$ is a α-stable fuzzy morphism.

2) Let Z be a set and γ a fuzzy set in Z. If $e: Z\to I^X$ is a γ-stable fuzzy map such that for each $j\in J$, $e_j\circ e$ is a fuzzy morphism then e is a fuzzy morphism.

Proof. Let $\alpha=\wedge\{e_j^{-1}(\beta_j)|j\in J\}$. Then for each $x\in X$,

$$\alpha(x)=\wedge\{\beta_j(x_j)\wedge e_j(x)(x_j)|\,j\in J\}.$$

Take any $x\in X$. If $\alpha(x)=0$ then for each $j\in J$, $S_\alpha(e_j(x))\neq\emptyset$. Suppose $\alpha(x)>0$. Then for each $j\in J$, $0<\alpha(x)\le e_j^{-1}(\beta_j)(x)$. Hence there is $x_j\in X_j$ with $0<\alpha(x)\le\beta_j(x_j)\wedge e_j(x)(x_j)$ and so $0<\alpha(x)\le e_j(x)(x_j)$. Thus for each $j\in J$, $S_\alpha(e_j(x))\neq\emptyset$. Clearly each e_j is a fuzzy morphism. Suppose $\gamma(z)\le e(x)(z)$. If $\gamma(z)=0$ then $\gamma(z)\le\alpha(x)$. Suppose $\gamma(z)>0$. Since for each $j\in J$, $e_j\circ e$ is a γ-stable fuzzy map, there is a $x_j\in X_j$ such that $\gamma(z)\le(e_j\circ e)(z)(x_j)$ and so $\gamma(z)\le e_j(x)(x_j)$. Since $e_j\circ e$ is a fuzzy morphism, $\gamma(z)\le\beta_j(x_j)$. Thus $\gamma(z)\le e_j(x)(x_j)\wedge\beta_j(x_j)=e_j^{-1}(\beta_j)(x)$ and hence $\gamma(z)\le\alpha(x)$. Thus e is a fuzzy morphism. Clearly such an α is unique.

Theorem 2.9 Let X be a set and α_j a fuzzy set in a set X_j ($j\in J$). Suppose $(e_j: X_j\to I^X)_{j\in J}$ is a sink of α_j-stable fuzzy maps. Then there is a unique fuzzy set α in X satisfying the following:

1) For each $j\in J$, $e_j: X_j\to I^X$ is a fuzzy morphism.

2) Let Z be a set and γ a fuzzy set in Z. If $e: X\to I^Z$ is a α-stable fuzzy map such that for each $j\in J$, $e\circ e_j: X_j\to I^Z$ is a fuzzy morphism, then e is a fuzzy morphism.

Proof. Let $\alpha=\vee\{e_j(\alpha_j)|j\in J\}$. Then for each $x\in X$,

$$\alpha(x)=\vee\{\alpha_j(x_j)|x\in S_{\alpha_j}(e_j(x_j))\}|\,j\in J\}.$$

Then $\alpha_j(x_j)\le e_j(x_j)(x)$ implies $\alpha_j(x_j)\le\alpha(x)$ and hence each e_j is a fuzzy morphism. Suppose $\alpha(x)\le e(x)(z)$. Then $\vee\{\alpha_j(x_j)|x\in S_{\alpha_j}(e_j(x_j))\}\le e(x)(z)$ for all $j\in J$. Hence if $x\in S_{\alpha_j}(e_j(x_j))$, then $\alpha_j(x_j)\le e(x)(z)$ and so

$$\alpha_j(x_j)\le e(x)(z)\wedge e_j(x_j)(x)\le(e\circ e_j)(x_j)(z).$$

Since $e\circ e_j: X_j\to I^Z$ is a fuzzy morphism, $\alpha_j(x_j)\le\gamma(z)$ and hence $\alpha(x)\le\gamma(z)$. Clearly such an α is unique.

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