⊠연구논문

CUSUM of Squares Chart for the Detection of Variance Change in the Process

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Abstract

Traditional statistical process control(SPC) assumes that consecutive observations from a process are independent. In industrial practice, however, observations are often serially correlated. A common approach to building control charts for autocorrelated data is to apply classical SPC to the residuals from a time series model fitted. Unfortunately, one cannot completely escape the effects of autocorrelation by using charts based on residuals of time series model. For the detection of variance change in the process we propose a CUSUM of squares control chart which does not require the model identification. The proposed CUSUM of squares chart and the conventional control charts are compared by a Monte Carlo simulation. It is shown that the CUSUM of squares chart is more effective in the presence of dependency in the processes.

1. Introduction

The problem of testing for changes in the variance of normally distributed data has received less attention, even though it is important in the context of quality control. The standard suggestion is that one should plot CUSUM's of $|X_{t+1}-X_t|$ or $|X_{t+1}-X_t|^2$ if the mean of X_t is unknown and $|X_t-\mu|$ or $|X_t-\mu|^2$ if X_t is known to have mean μ a priori. They require a special development of the theory to half-normal or χ^2 distributed increments. Hawkins(1981) proposed a technique for employing the same cumulative sum procedure used for the mean to control the variance.

An apparent disadvantage of individual observation chart is that one loses the information on the variability among the m samples. Therefore, it is common to try to estimate the process variability by the moving range(MR) [Montgomery, 1991], the mean square of successive differences [Nelson, 1980], the exponentially weighted mean square and the exponentially weighted moving variance [MacGregor and Harris, 1993]. Montgomery and Mastrangelo(1991) also advocated the use of exponentially weighted moving variance type statistics and Bauer and Hackl(1978, 1980) investigated moving sum of squares statistics, again for the case of the omnibus EWMA schemes studied by Domangue and Patch(1991). Ng and Case(1989) investigated some ARL properties for the various exponentially weighted moving averages and moving ranges and Crowder and Hamilton(1992) used exponentially weighting to smooth $\log(s^2)$ obtained from the range statistic in the conventional $\overline{X}$ and R chart.

In this paper centered version of the cumulative sum of squares presented by Brown *et al.*(1975) are used to search for change points systematically at different pieces of the process. A very recent application of this type is the use of cusum of squares to detect changes of variability in the independent process (see Inclán and Tiao(1994) and Ploberger and Krämer(1990) among others). We study the asymptotic properties of the cumulative sum of squares and propose a CUSUM of squares control chart which is useful not only for the independent process but for the dependent processes.

2. Properties of the Cumulative Sum of Squares for Dependent Processes

In this section we show how to design the CUSUM of squares chart for the detection of variance change in the dependent processes. The control limits of the proposed control charts are obtained using the cumulative sum of squares of the observations via the concept of the mixingale and Brownian motion. We consider a process that follows a linear process.

$$X_t = \psi(B)\varepsilon_t = \sum_{k=0}^{\infty} \psi_k \varepsilon_{t-k}, \tag{1}$$

where $\psi(B)=1-\psi_1 B-\psi_2 B^2-\psi_3 B^3-\cdots$, ε_t is independently and identically distributed (i.i.d.) with $E(\varepsilon_t)=0$, $E(\varepsilon_t^2)=\sigma_\varepsilon^2$, and $E|\varepsilon_t|^4<\infty$. It can be readily seen from (1) that ψ_k is geometrically bounded, say, $|\psi_k|\le B\rho^k$ for some $B>0$, $\rho\in(0,1)$.

Let (Ω, F, P) denote a probability space and $\{X_t : t\ge 1\}$ be a sequence of random variables on (Ω, F, P). For weakly dependent data, the standard reference is McLeish (1975). His results includes some maximal inequality for L_2-mixingales, and a strong law of large numbers(SLLN) for L_2-mixingales and α-mixing sequences. His mixingale condition also requires that random variables are L_2-bounded. In this paper, however, maximal inequalities are applied for L_p-mixingales, $p>1$. From now on, $\{F_n : 0\le n\le\infty\}$ denotes the sequence of nondecreasing sub σ-algebras of F. Andrews(1988) defined the L_p-mixingale as follows.

Definition 1 *(Mixingale) The sequence* (X_t, F_t) *is an* L_p*-mixingale if for sequences of finite nonnegative constants* c_t *and* ξ_k *with* $\xi_k\to 0$ *as* $k\to\infty$,

(a) $\|E(X_t|F_{t-k})\|_p\le\xi_k c_t$

(b) $\|X_t-E(X_t|F_{t+k})\|_p\le\xi_{k+1}c_t$

for all $t\ge 1$, $k\ge 0$.

Definition 2 *(Size)* $\{\xi_k\}$ *is of size* $-p$ *if there exist a positive sequence* $\{L(k)\}$ *such that*

(a) $\sum_k (kL(k))^{-1}<\infty$,

(b) $L(k)-L(k-1)=O(k^{-1}L(k))$,

(c) $L(k)$ *is eventually nondecreasing*

(d) $\xi_k=O((n^{1/2}L(k))^{-2p})$.

In this section, we consider the centered random variable $Z_t=X_t^2-\sigma_X^2$, where $\{X_t\}$ follow the linear process in (1) and $\sigma_X^2=E(X_t^2)$. In the followings we suggest a control scheme based upon the cumulative sum of Z_t. The approach presented here uses the cumulative sum of squares to search for change points systematically. The idea is similar to that of Kim(1996), where the cumulative sum of squares is used in GARCH models.

Theorem 1 *If* $\{X_t\}$ *follows the linear process in (1),* $\{(X_t^2-\sigma_X^2, F_t)\}$ *is an* L_p *-mixingale of size* $-1/2$.

Proof. In the definition of the mixingale, the condition (b) is trivially satisfied because Z_t is F_t-measurable. For the condition (a),

$$\begin{aligned}
\|E(Z_t|F_{t-m})\|_p &= \|E(X_t^2|F_{t-m})-\sigma_X^2\|_p \\
&= \|E(\sum_{k=0}^{\infty}\psi_k^2\varepsilon_{t-k}^2|F_{t-m})+E(\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\psi_k\psi_l\varepsilon_{t-k}\varepsilon_{t-l}|F_{t-m})-\sigma_X^2\|_p \\
&= \|\sum_{k=m}^{\infty}\sum_{l=m}^{\infty}\psi_k\psi_l\varepsilon_{t-k}\varepsilon_{t-l}+\sigma_\varepsilon^2\sum_{k=m}^{\infty}\psi_k^2\|_p \\
&\le K\sum_{k=m}^{\infty}\sum_{l=m}^{\infty}|\psi_k||\psi_l|\|\varepsilon_{t-k}\varepsilon_{t-l}\|_p \\
&\le \tilde{K}(\sum_{k=m}^{\infty}|\psi_k|)^2 \\
&= \tilde{K}\xi_m.
\end{aligned}$$

Therefore, $\{X_t^2-\sigma_X^2\}$ is mixingale and ξ_m is of size -1/2 with $L(m)=m^{1/2}$.

Theorem 2 *If* $\{(Z_t, F_t)\}$ *is an* L_p*-mixingale with* ξ_m *of size* $-1/2$, *then*

$$\sigma_Z^{-1}T^{-1/2}\sum_{t=1}^{[Tr]}Z_t \rightarrow W(r), \tag{2}$$

where $\sigma_Z^2=E(Z_t^2)+2\sum_{k=1}^{\infty}E(Z_0Z_k)$ *and* W *is the standard Brownian motion.*

Proof. The proof is similar to that given by Theorem 2.6 of McLeish(1975). Hence it is omitted.

3. CUSUM of Squares Control Chart

In this section, we present how to obtain the empirical control limits of the centered CUSUM of squares chart for the process in (1). To design a CUSUM of squares chart for the detection of variance change, the following three steps are proposed. Let $S_k^{(2)}=\sum_{t=1}^{k}Z_t$, where $Z_t=X_t^2-\sigma_X^2$.

Step 1. *Estimate* σ_Z^2 *using the spectral based estimator*

$$\hat{\sigma}_Z^{\ 2} = \hat{\gamma}_0 + 2\sum_{k=1}^{M} W_n(k)\ \hat{\gamma}_k,$$

where $\hat{\gamma}_k = \sum_{t=1}^{n-k}(Z_t - \overline{Z})(Z_{t-k} - \overline{Z}) / \sum_{t=1}^{k}(Z_t - \overline{Z})^2$ *and estimate* ρ_1 *by* $\widehat{\rho_1} = \widehat{\gamma_1}/\widehat{\gamma_0}$.

Since the variance of $S_k^{(2)}$, σ_Z^2, and the lag-one autocorrelation ρ_1 are unknown we estimate them using the in-control data.

Step 2. *Set up the control limits using* $\hat{\sigma}_Z$

$$\begin{aligned} UCL &= \sigma_0 + C \cdot K\,\hat{\sigma}_Z, \\ LCL &= \sigma_0 - C \cdot K\,\hat{\sigma}_Z, \end{aligned} \tag{3}$$

where σ_0 *is a nominal level for the process, UCL is the upper control limit, LCL is the lower control limit,* K *is the control limit constant which ensures the in-control ARL, and* C *is a correction constant.*

$$C = 1 - \rho_1^2/5.$$

The control limit constant K is required to implement the CUSUM of squares. This constant is most conveniently determined by calculating the ARL for the test procedure. We will show how to select the control limit constant K in Section 4. We also consider the correction constant C to adjust for the autocorrelated process from experience since the ARL of CUSUM of squares chart depends on ρ_1.

Step 3. *Calculate the cumulative sum of* Z_t *after the* k*-th observation,*

$$k^{-1/2}S_k^{(2)} = k^{-1/2}\sum_{t=1}^{k} Z_t, \tag{4}$$

and plot $(k, k^{-1/2}S_k^{(2)})$ *against* k.

Any point which cross the control limits is declared as a signal which requires an action. Control charts based on the CUSUM of squares are easy to implement for dependent processes and can be interpreted according to traditional guidelines for uncorrelated data. The main advantage of CUSUM type control chart as compared to the usual R-chart is that it does not require the subgrouping of observations. Thus it enables one to take advantage of new information immediately upon its arrival. This also ensures a better statistical performance as compared to other types of R-chart. However, the chart based on (4) is more difficult to design since the sequence (4) is autocorrelated. Therefore, a design that ignores this fact will inevitably lead to a high rate of false alarms. This can be handled by selecting the appropriate value of K.

4. Simulation Study

In this section we show how to select the control limit constant K and compare the performance of the proposed CUSUM of squares chart with other control charts, Hawkins'(1981) CUSUM, MR and Crowder and Hamilton's(1992) EWMA when the underlying processes follow AR(1) and MA(1).

The usual performance criterion is the ARL. The ARL can be calculated via a Monte Carlo simulation, once the distributional properties of the observations are specified. In order to compare the performance of the charts more meaningfully for different autoregressive and moving average parameter values, σ_X or σ_Z multipliers of the control charts(K) are manipulated so that the ARL's, when there is no shift in the variance, are the same for all four charts. The chart with lowest ARL when a change in the variance occurred is considered superior since the ARL when the process variance deviates from the nominal value, i.e., out-of-control, should be as short as possible subject to a specified ARL when the process is in-control. However, the autocorrelation in the process data degrades the ARL performance.

4.1 Design of CUSUM of Squares Chart

The design of control chart procedure is usually based on the ARL of the scheme, although other factors should be taken into consideration. The design strategy is to choose the control limit constant K which for a given in-control ARL, minimizes the out-of-control ARL in the presence of the change in the process variance. The contour nomogram is a convenient device for the design of control chart, since the method of construction is relatively simple and straightforward. The ARL surfaces can be easily visualized to get an intuitive insight when designing a control chart. Hawkins(1981) provided tables and nomograms for the scaled cumulative sum absolute procedure. But he assumed that the underlying process is independently and identically normally distributed.

For dependent processes, computation of the ARL's of the control chart is analytically intractable and they are thus determined via simulation. For the simulation, it is assumed that only one observation is available at each time period and that all parameters are known exactly. Time series of sample size 2100 are generated according to the models

$$\text{AR}(1): \quad X_t = \phi_1 X_{t-1} + \varepsilon_t,$$

$$\text{MA}(1): \quad X_t = \varepsilon_t - \theta_1 \varepsilon_{t-1}.$$

The first 100 observations are discarded to reduce the effect of initialization. The white noise are drawn from normal distribution with mean zero and unit variance. In the above models, $(X_t^2 - \sigma_X^2)$ satisfy the L_p-mixingale condition and ε_t is i.i.d. $N(0, \sigma_\varepsilon^2)$. Thus variances are

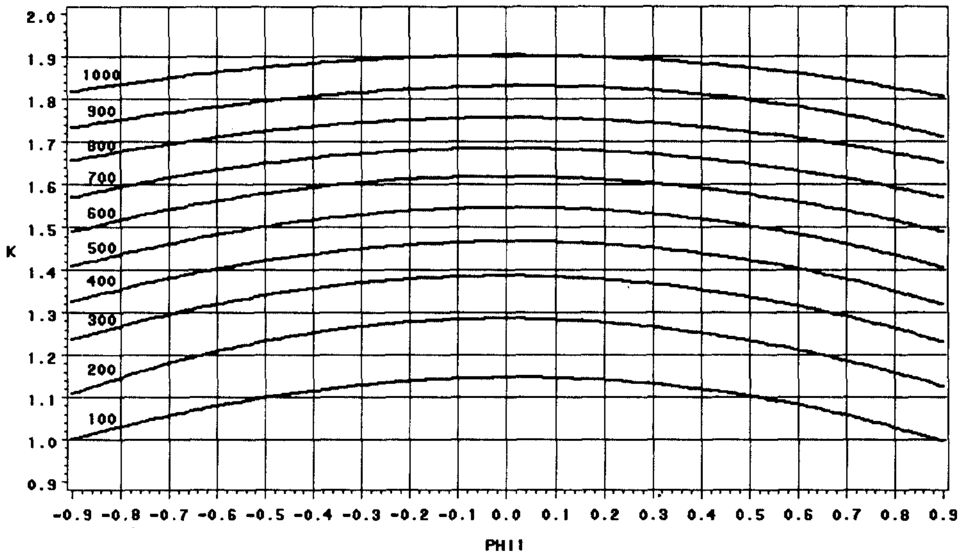
$$\text{Var}(X_t^2 - \sigma_X^2) = 2\frac{(1+\phi_1^2)}{(1-\phi_1^2)^3}\sigma_\varepsilon^4,$$

$$\text{Var}(X_t^2 - \sigma_X^2) = 2(1+4\theta_1^2+\theta_1^4)\sigma_\varepsilon^4,$$

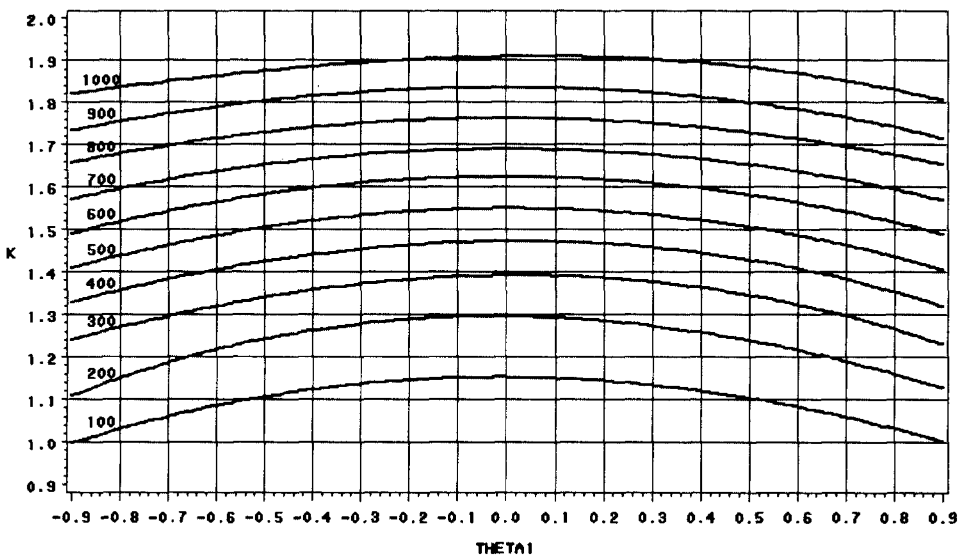
respectively.

The run length is measured until the first out-of-control condition is signaled. The process is repeated 5000 times in order to obtain the ARL. Based on the simulation results we construct the new nomograms for the CUSUM of squares charts. In <Figure 1> and <Figure 2>, optimal K's are given for in-control ARL's ranging from 100 to 1000.

To construct the CUSUM of squares chart we have to choose K which ensures the predetermined in-control ARL. Crowder and Hamilton(1992) suggested an ARL of 200 when the process is in-control. For this in-control ARL value, choice of K depends on the degree of autocorrelation. From <Figure 1> and <Figure 2>, we see that K is ranging from 1.15 to 1.25 when the process is moderately or highly autocorrelated($0.4 \le |\phi_1|, |\theta_1| \le 0.8$) and is ranging from 1.26 to 1.3 when the process is slightly autocorrelated ($0.1 \le |\phi_1|, |\theta_1| \le 0.3$) or independent.



< Figure 1 > Nomogram for the Choice of Optimal K for CUSUM of Squares Charts of AR(1) Processes



< Figure 2 > Nomogram for the Choice of Optimal K for CUSUM of Squares Charts of MA(1) Processes

In general, optimal choice of K can be chosen according to the following steps. First, choose the smallest acceptable ARL when there is no change in the process variance. This corresponds to fixing the false alarm rate. Then find the control limit constant K which satisfies the in-control ARL value considering the degree of autocorrelation obtained from the in-control data.

4.2 Power Comparisons

We perform a simulation study to compare the performance of the proposed CUSUM of squares control chart with MR, EWMA and conventional CUSUM charts which have been used to monitor the process variance. The effects of serial correlations represented by ϕ's and θ's are discussed on the basis of the simulated ARL's.

For MR chart, Crowder(1987) studied the ARL of combined control chart for individuals and MR charts, where the moving range is defined as $MR_t = |X_t - X_{t-1}|$. Crowder produced ARL's for various settings of the control limits and shifts in the process mean and standard deviation. In general, his work showed that the ARL of combined procedure will generally be much less than the ARL of standard Shewhart control chart when the process is in-control, if we use the conventional 3σ control limits on the charts. In general, results closer to the Shewhart in-control ARL are obtained if we compute the upper control limit on the MR chart from

$$UCL = D\overline{MR},$$

where the constant D should be chosen such that $4 \le D \le 5$. Choosing the control limit constant $D=3.3$ yields an in-control ARL of approximately 125.

For the conventional CUSUM chart, Hawkins(1981) proposed a technique which is based on CUSUM of

$$Y_t = \frac{|X_t/\sigma|^{1/2} - E(|X_t/\sigma|^{1/2})}{\sqrt{Var(|X_t/\sigma|^{1/2})}}$$

Choosing the control limit constant $K=1.1791$ yields an in-control ARL of approximately 126.

For EWMA chart, Crowder and Hamilton(1992) proposed one-sided EWMA chart as follows,

$$S_t = \max\left((1-\lambda)S_{t-1} + \lambda \log(\sigma^2_{X,t}),\ \log(\sigma^2_Y) \right),$$

where $S_0 = \log(\sigma^2_Y)$ and $\sigma^2_{X,t}$ values are successive sample variances. The upper control limit for one-sided EWMA chart is

$$UCL = K\sigma_S.$$

Choosing combination of the design parameter, $(\lambda, K) = (0.16, 1.45)$ yields an in-control ARL of 134.68.

The in-control ARL's of four charts for various of ϕ_1's and θ_1's are given in <Table 1> and <Table 2>. <Table 1> and <Table 2> are obtained from the simulation of the AR(1) model with $-1 < \phi_1 < 1$ and the MA(1) model with $-1 < \theta_1 < 1$. From <Table 1> and <Table 2> we see that ARL's of four charts are not much different from the theoretical ARL's when the process is independent, $\phi_1 = 0$ and $\theta_1 = 0$. The simulated ARL of Hawkins' CUSUM chart shows that the ARL gets smaller as $|\phi_1|$ gets larger and the pattern is symmetric against $\phi_1 = 0$ or $\theta_1 = 0$ because the variance σ^2_Z is estimated under the independence assumption. EWMA chart shows the different pattern from Hawkins' CUSUM chart. For the positively autocorrelated case, $\phi_1 > 0$ ($\theta_1 < 0$), the ARL of EWMA chart gets larger as ϕ_1(θ_1) increases (decreases) from 0. For the negatively autocorrelated case it decreases for $-0.5 < \phi_1 < 0$ and for all θ_1's considered and then increases for $\phi_1 \le -0.5$. MR chart shows the opposite pattern to EWMA chart. The ARL of MR chart continuously decreases(increases) as ϕ_1(θ_1) increases(decreases) from -0.75(0.75) to 0.75(-0.75). These patterns indicate that three charts are affected by the dependency in the process. In contrast, the ARL of the CUSUM of squares chart is relatively flat regardless of the magnitudes of $|\phi_1|$ and $|\theta_1|$ although it slightly increases as $|\phi_1|$ gets larger. Hence the proposed control chart is robust to the dependency than the other control charts.

Relative performances of four control charts, Hawkins' CUSUM, EWMA, MR, and the CUSUM of squares in the presence of variance change are compared by a Monte Carlo simulation. To evaluate the out-of-control ARL's the values of step change ranged from 0 to 5 standard deviations of the process are introduced at time 0 and the number of samples until the point fall outside control limits are counted. For each chart, the above procedure is repeated 5000 times in order to obtain the ARL's.

< Table 1 > ARL of the Control Chart when the Process is In-Control AR(1) Process

ϕ_1	Hawkins' CUSUM chart	CUSUM of squares chart	EWMA chart	MR chart
0.75	13.25 (0.65)	185.25 (5.82)	365.94 (9.29)	119.16 (1.66)
0.60	27.32 (1.94)	159.05 (5.56)	257.01 (8.14)	122.41 (1.69)
0.45	54.27 (3.33)	141.93 (5.45)	217.47 (7.66)	121.12 (1.69)
0.30	90.25 (4.67)	137.60 (5.39)	190.07 (7.30)	126.46 (1.75)
0.15	110.70 (5.34)	123.38 (5.15)	162.92 (6.82)	124.69 (1.77)
0.00	120.07 (5.65)	115.38 (5.01)	143.40 (6.47)	128.47 (1.82)
-0.15	113.99 (5.46)	133.44 (5.54)	121.56 (5.90)	133.99 (1.87)
-0.30	87.92 (4.66)	130.70 (5.21)	127.62 (6.08)	145.02 (2.03)
-0.45	53.24 (3.19)	140.12 (5.30)	127.60 (6.00)	156.90 (2.19)
-0.60	27.75 (1.88)	164.75 (5.74)	140.77 (6.27)	178.03 (2.51)
-0.75	13.48 (0.82)	183.04 (5.74)	159.52 (6.63)	213.46 (2.96)

* ( ) : standard error of ARL

< Table 2 > ARL of the Control Chart when the Process is In-Control MA(1) Process

θ_1	Hawkins' CUSUM chart	CUSUM of squares chart	EWMA chart	MR chart
0.75	52.86 (3.34)	120.19 (4.86)	110.29 (5.67)	145.56 (2.01)
0.60	59.00 (3.50)	121.59 (4.87)	124.22 (6.02)	143.97 (1.97)
0.45	78.10 (4.25)	127.44 (5.16)	122.63 (5.96)	139.31 (1.89)
0.30	91.55 (4.75)	130.90 (5.30)	123.85 (5.96)	136.85 (1.87)
0.15	111.97 (5.40)	123.58 (5.19)	132.02 (6.19)	132.70 (1.85)
0.00	112.65 (5.37)	120.43 (5.11)	138.46 (6.31)	126.35 (1.73)
-0.15	107.07 (5.25)	124.34 (5.20)	159.78 (6.72)	124.53 (1.74)
-0.30	92.99 (4.82)	121.22 (5.02)	173.07 (6.94)	120.85 (1.69)
-0.45	74.04 (4.13)	134.82 (5.36)	190.22 (7.33)	122.76 (1.71)
-0.60	60.22 (3.59)	123.71 (5.14)	198.91 (7.41)	124.28 (1.73)
-0.75	53.81 (3.31)	124.60 (5.11)	199.74 (7.29)	128.33 (1.76)

* ( ) : standard error of ARL

Simulation results are summarized in <Table 3(a)> for $\phi_1 = 0$, <Table 3(b)> for $\phi_1 = 0.6$, and <Table 3(c)> for $\phi_1 = -0.6$ when the underlying process is AR(1). We choose the control limit constant $K = 1.1791$ for the independent case, $K = 1.1193$ for the positively autocorrelated case, and $K = 1.1266$ for the negatively autocorrelated case for the CUSUM of squares chart using <Figure 1>. The same constant $K = 1.1791$ for the Hawkins' CUSUM chart, $\lambda = 0.16$ and $K = 1.45$ for

EWMA chart and $K=3.3$ for MR chart are used in the comparison since with these control limits the ARL associated with no change in the variance is approximately equal to 130. <Table 3> compares the ARL of four charts in the presence of change Δ in the process variance.

For the independent case, <Table 3(a)>, it is observed that the CUSUM of squares chart performs equally well and even better than EWMA chart. CUSUM and MR charts are very slow in detecting even the small change and the ARL's are not much different for all the values of the changes Δ considered. <Table 3(b)>, the positively autocorrelated case, clearly shows that the CUSUM of squares chart outperforms the other charts. The small ARL of CUSUM chart for $\Delta=1.0$ indicates that it gives the false alarm too often when there is no change in the process variance. Since the ARL's of EWMA are about 2 times of the ARL's of the CUSUM of squares chart for all the Δ's considered, it is not comparable. However, we can see that it is affected by the dependency in the process. On the other hand the ARL of CUSUM of squares chart is not much different from the independent case which indicates that the CUSUM of squares chart is robust to the dependency in the process. Although the ARL's of MR chart for positively autocorrelated case is not much different from the independent case the larger values of the ARL's for $\Delta>1.0$ indicate that it cannot detect the shift in the process as quickly as the CUSUM of squares chart.

For the negatively autocorrelated case, <Table 3(c)>, the performances of CUSUM and MR charts are similar to the positively autocorrelated case. The ARL of CUSUM for $\Delta=1.0$ is again too small compared to the independent case and is not much different from the ARL's for other values of Δ's considered. The ARL of MR chart is larger than that of the independent case and is relatively flat for $\Delta>1.0$. EWMA chart can detect small change in the process but is slower than the CUSUM of squares chart for larger changes.

In summary, for the positively and negatively autocorrelated processes, Hawkins' CUSUM and MR charts are not useful. EWMA chart performs better than those two charts but is more affected by the dependency in the process than the CUSUM of squares chart. Since the CUSUM of squares chart is very simple to construct and does not require the independence assumption it is a useful tool when the process is autocorrelated.

< Table 3 > Comparison of Chart Performance : AR(1) Process

(a) Random Process : $\phi_1 = 0.0$

Δ	Hawkins' CUSUM chart $K=1.1791$	CUSUM of squares chart $K=1.1791$	EWMA chart $\lambda=0.16$ $K=1.45$	MR chart $K=3.3$
1.00	121.53 (5.65)	122.88 (5.16)	139.43 (6.37)	133.68 (1.84)
1.50	109.35 (5.28)	3.16 (0.07)	3.84 (0.05)	127.54 (1.76)
2.00	115.50 (5.47)	1.56 (0.02)	2.83 (0.02)	127.92 (1.76)
2.50	120.42 (5.61)	1.26 (0.01)	2.56 (0.02)	129.14 (1.79)
3.00	111.68 (5.37)	1.13 (0.01)	2.43 (0.01)	127.69 (1.75)
3.50	116.91 (5.52)	1.08 (0.00)	2.34 (0.01)	124.43 (1.69)
4.00	120.78 (5.65)	1.07 (0.00)	2.28 (0.01)	129.52 (1.79)
4.50	106.08 (5.21)	1.04 (0.00)	2.22 (0.01)	126.93 (1.75)
5.00	116.33 (5.50)	1.03 (0.00)	2.19 (0.01)	129.35 (1.78)

* ( ) : standard error of ARL

(b) Positively Autocorrelated Process : $\phi_1 = 0.6$

Δ	Hawkins' CUSUM chart $K=1.1193$	CUSUM of squares chart $K=1.1193$	EWMA chart $\lambda=0.16$ $K=1.45$	MR chart $K=3.3$
1.00	21.44 (1.56)	114.10 (4.45)	258.86 (8.16)	119.67 (1.69)
1.50	19.47 (1.42)	7.05 (0.16)	6.49 (0.10)	125.33 (1.72)
2.00	20.95 (1.57)	2.05 (0.04)	3.95 (0.04)	118.60 (1.62)
2.50	21.40 (1.49)	1.42 (0.02)	3.12 (0.03)	121.49 (1.71)
3.00	18.83 (1.32)	1.19 (0.01)	2.83 (0.02)	120.06 (1.66)
3.50	17.26 (1.18)	1.13 (0.01)	2.63 (0.02)	118.29 (1.61)
4.00	19.17 (1.33)	1.09 (0.01)	2.50 (0.01)	119.01 (1.64)
4.50	19.69 (1.35)	1.06 (0.00)	2.42 (0.01)	120.36 (1.72)
5.00	20.44 (1.47)	1.05 (0.00)	2.36 (0.01)	116.94 (1.59)

* ( ) : standard error of ARL

< Table 3 > (continued)

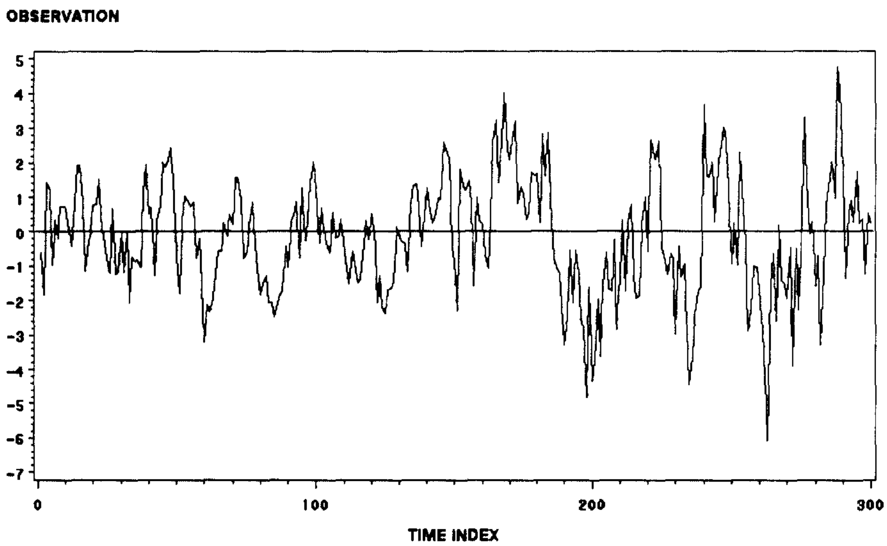
(c) Negatively Autocorrelated Process : $\phi_1 = -0.6$

Δ	Hawkins' CUSUM chart $K=1.1266$	CUSUM of squares chart $K=1.1266$	EWMA chart $\lambda=0.16$ $K=1.45$	MR chart $K=3.3$
1.00	18.60 (1.23)	123.68 (4.72)	135.35 (6.14)	177.46 (2.40)
1.50	21.25 (1.42)	7.25 (0.16)	3.97 (0.07)	174.59 (2.35)
2.00	19.43 (1.26)	2.15 (0.05)	2.86 (0.03)	176.38 (2.40)
2.50	17.25 (1.00)	1.41 (0.02)	2.53 (0.02)	173.44 (2.44)
3.00	23.39 (1.64)	1.21 (0.01)	2.37 (0.01)	173.09 (2.39)
3.50	24.10 (1.76)	1.13 (0.01)	2.30 (0.01)	173.82 (2.41)
4.00	23.90 (1.66)	1.09 (0.01)	2.23 (0.01)	176.14 (2.37)
4.50	22.87 (1.61)	1.07 (0.00)	2.22 (0.01)	171.43 (2.38)
5.00	22.41 (1.54)	1.05 (0.00)	2.17 (0.01)	171.90 (2.36)

* ( ) : standard error of ARL

4.3 Example

We simulate a time series of 300 observations from the AR(1) process with $\phi_1 = 0.6$, assuming single readings taken every time point. The simulated process undergoes a step change with $\Delta = 1.5$ in the middle of the series at time $t = 150$. <Figure 3> is the plot of the series. From a visual examination alone, we see that the series, in terms of variance, is obviously out-of-control with evidence of positively autocorrelated and variance shifted behavior.

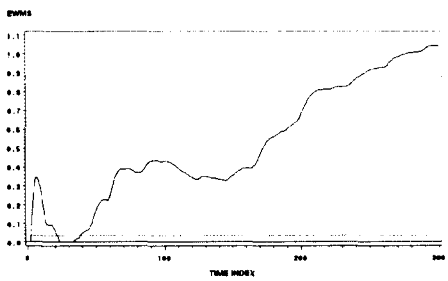


< Figure 3 > Plot of the Simulated Series

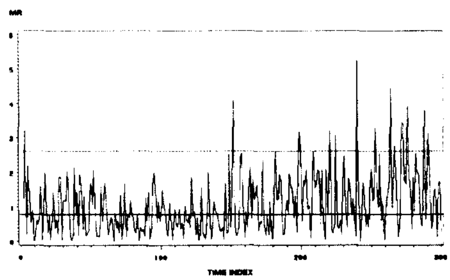
<Figure 4> to <Figure 7> are EWMA, MR, Hawkins' CUSUM and the CUSUM of squares chart, respectively. <Figure 4> is EWMA chart with control parameters $\lambda = 0.16$ and $K = 1.45$. We see in <Figure 4> that these limits are tighter than real limits and that it has a very poor performance. <Figure 6> is Hawkins' CUSUM chart with control limits ± 3.4044, which is obtained by $K \times \hat{\sigma}_Y$ ($K = 1.1791$ and $\hat{\sigma}_Y = 2.8873$). Both charts give the false alarm too often, i.e., too many points fall outside the control limits. <Figure 5> is MR chart with control limits $UCL = 2.6161$, which is obtained by $D \times \overline{MR}$ ($D = 3.3$ and $\overline{MR} = 0.7928$) and $LCL = 0$. These figures clearly show that the performances of EWMA and MR charts for the dependent processes are very poor as we explained

in Section 2. <Figure 7> is the CUSUM of squares chart with control limits ± 3.2829 which is calculated using the estimated parameter $\hat{\phi}_1 = 0.65$. <Figure 7> clearly indicates that CUSUM is out-of-control after observation 172.

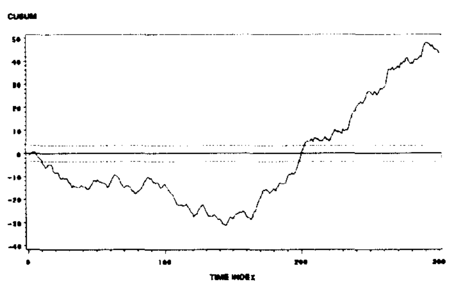
As a conclusion it is shown that all but the CUSUM of squares chart have too many out-of-control points. Hence, traditional means of monitoring quality for the detection of variance change in dependent processes may have yielded misleading results. The CUSUM of squares chart which is designed and implemented without relying on the independence assumption and model identification can detect the change in the process rather quickly without false alarm.



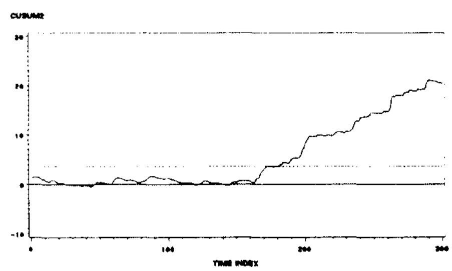
< Figure 4 > EWMA Chart for the Simulated Series



< Figure 5 > MR Chart for the Simulated Series



< Figure 6 > Hawkins' CUSUM Chart for the Simulated Series



< Figure 7 > CUSUM of Squares Chart of the Simulated Series

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