The Korean Communications
in Statistics Vol. 5, No. 1, 1998
pp. 145-153

Bayes Computations for the Reliability in a Bivariate Exponential Model[1)]

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Abstract

In this paper, a hierarchical Bayesian analysis of a bivariate exponenetial model is discussed using Gibbs sampler. Paramters and reliability estimators are obtained. A numerical study is provided.

1. Introduction

In the problem of life testing and reliability analysis, the exponential distribution plays a central role as useful statistical model. The problem of estimating reliability in the exponential case has been considered in some papers. Tong(1974) derived two expressions for the M.V.U.E. of P(X<Y). Kelly, Kelly and Schucany(1976) derived the M.L.E. and U.M.V.U.E. for P(X<Y). However, in all the previous studies, they have assumed the stochastic independence among the components of system.

But occasionally, independence assumption is not applicable in the practical situation. Naturally, it is more realistic to assume some forms of dependence among the components of the system. This dependence among the components arise from common environmental shocks and stress, or from components depending on common sources of power, and so on. Freund(1961), Marshall and Olkin(1968), Block and Basu et al.(1974) were studied bivariate exponential models. Also, Klein and Basu(1985), Kim and Park(1990) obtained some estimators for the reliability. Cho et al.(1996) obtained some approximate confidence intervals for the reliability.

Let's consider a system which functions only as long as at least one of two identical or very similar components functions, Initially let the two components be independently on test with life distributions that are exponential with parameters λ, denoted $\exp(\lambda)$. Failure of one changes the life distribution of the other to $\exp(\lambda\theta)$, $\theta > 0$, where $\theta = 1$ implies the

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independence of the two components lives. For $\theta > 1$ the workload of the remaining component is increased, thereby decreasing the mean life. In this case, Weier(1981) obtained Bayes estimators of parameters and reliability using conjugate prior.

In this paper, we consider Gibbs sampler approach for the hierarchical Bayes analysis in above bivariate exponential model. We obtain Bayes estimators of parameters and reliability and provide a numerical example.

2. Preliminaries and Notations

Let X and Y denote the lifetimes of the two components. Then the model is most easily derived as a function of $U=\min(X, Y)$ and $W=\max(X, Y)-\min(X, Y)$. (See Weier(1981)). The density of U is then $\exp(2\lambda)$, and by the univariate memoryless property, W is independently distributed as $\exp(\lambda\theta)$. Thus the joint density of U and W is

$$f(u, w)=2\theta\lambda^2\exp(-2\lambda u-\lambda\theta w), \quad u, w>0, \ \lambda, \theta>0.$$

For a random sample size of size k, the likelihood function is

$$L(\lambda, \theta \mid \underline{u}, \underline{w})=2^k\theta^k\lambda^{2k}\exp(-2\lambda\sum_{i=1}^{k} u_i-\lambda\theta\sum_{i=1}^{k} w_i).$$

The MLE's are given as

$$\hat{\lambda}=\frac{k}{2\sum_{i=1}^{k} u_i} \quad \text{and} \quad \hat{\theta}=\frac{2\sum_{i=1}^{k} u_i}{\sum_{i=1}^{k} w_i}.$$

Note that $\sum_{i=1}^{k} U_i$ and $\sum_{i=1}^{k} W_i$ are independent and distributed respectively as gamma $(k, 2\lambda)$ and gamma $(k, \lambda\theta)$.

The joint reliability function is given as:

$$R(u_0, w_0)=P(U>u_0, W>w_0)=\exp(-2u_0\lambda-w_0\lambda\theta).$$

The MLE of $R(u_0, w_0)$ is given as:

$$\hat{R}(u_0, w_0)=\exp(-2u_0\hat{\lambda}-w_0\hat{\lambda}\hat{\theta}).$$

Densities are denoted generically by brackets, so joint, conditional, and marginal forms, for example, appear as $[X, Y]$, $[X \mid Y]$, and $[X]$. Multiplication of densities is denoted by $*$; for example, $[X, Y]=[X \mid Y]*[Y]$.

3. Hierarchical Bayes Structure.

The following hierarchical Bayes model is considered in this paper.

(1) $[\underline{u}, \underline{w} \mid \lambda, \theta] = 2\theta\lambda^2 \exp(-2\lambda u - \lambda\theta w)$.

(2) Prior distributions

$[\lambda \mid \beta_1] \sim$ Gamma(α_1, β_1), where α_1 is known positive constant.

$[\lambda\theta \mid \lambda, \beta_2] \sim$ Gamma(α_2, β_2), where α_2 is known positive constant.

Then the joint prior is

$$[\lambda, \theta \mid \beta_1, \beta_2] = [\lambda \mid \beta_1] * [\lambda\theta \mid \lambda, \beta_2]$$

$$= \frac{1}{\Gamma(\alpha_1)\beta_1^{\alpha_1}} \lambda^{\alpha_1 - 1} \exp(-\lambda/\beta_1) \cdot \frac{1}{\Gamma(\alpha_2)\beta_2^{\alpha_2}} (\lambda\theta)^{\alpha_2 - 1} \exp(-\lambda\theta/\beta_2),$$

where a Gamma (α, β) variable, say z has pdf

$$\frac{1}{\Gamma(\alpha)\beta^{\alpha}} z^{\alpha-1} \exp(-z/\beta).$$

(3) Hyper prior distributions

$[\beta_1 \mid c_1, d_1] \sim IG(c_1, d_1)$ and $[\beta_2 \mid c_2, d_2] \sim IG(c_2, d_2)$,

where c_1, d_1, c_2, d_2 are known positive constants and $IG(c, d)$ variable, say t has pdf

$$g(t) = \frac{1}{\Gamma(c) d^c t^{c+1}} \exp(-1/dt).$$

The joint distribution of $[\lambda, \theta, \beta_1, \beta_2, \underline{u}, \underline{w}]$ is given as

$$[\lambda, \theta, \beta_1, \beta_2, \underline{u}, \underline{w}]$$

$$\propto [\underline{u}, \underline{w} \mid \lambda, \theta] * [\lambda \mid \beta_1] * [\lambda\theta \mid \lambda, \beta_2] * [\beta_1 \mid c_1, d_1] * [\beta_2 \mid c_2, d_2]$$

$$\propto \theta^{k+\alpha_2-1} \lambda^{2k+\alpha_1+\alpha_2-2} \beta_1^{-\alpha_1-c_1-1} \beta_2^{-\alpha_2-c_2-1}$$

$$\cdot \exp\left[-2\lambda \sum_{i=1}^{k} u_i - \lambda\theta \sum_{i=1}^{k} w_i - \lambda/\beta_1 - \lambda\theta/\beta_2 - 1/d_1\beta_1 - 1/d_2\beta_2\right].$$

From the joint distribution of $[\lambda, \theta, \beta_1, \beta_2, \underline{u}, \underline{w}]$, the full conditional distributions are given as:

(1) $[\lambda \mid \underline{u}, \underline{w}, \theta, \beta_1, \beta_2]$

$$\propto \lambda^{2k+\alpha_1+\alpha_2-2} \cdot \exp\left[-\lambda\left(2\sum u_i + \theta\sum w_i + \frac{1}{\beta_1} + \frac{\theta}{\beta_2}\right)\right], \tag{3.5}$$

that is, $[\lambda \mid \cdot\,] \sim \text{Gamma}\left(2k + \alpha_1 + \alpha_2 - 1, \left(2\sum u_i + \theta\sum w_i + \frac{1}{\beta_1} + \frac{\theta}{\beta_2}\right)^{-1}\right)$.

(2) $$[\theta \mid \underline{u}, \underline{w}, \lambda, \beta_1, \beta_2] \propto \theta^{k+\alpha_2-1} \cdot \exp\left(-\theta\left(\lambda\sum w_i + \frac{\lambda}{\beta_2}\right)\right), \tag{3.6}$$

that is, $[\theta \mid \cdot\,] \sim \text{Gamma}\left(k + \alpha_2,\ (\lambda\sum w_i + \frac{\lambda}{\beta_2})^{-1}\right)$.

(3) $[\beta_1 \mid \underline{u}, \underline{w}, \lambda, \theta, \beta_2] \propto \beta_1^{-(\alpha_1+c_1+1)} \cdot \exp\left(-\frac{1}{\beta_1}\left(\lambda+\frac{1}{d_1}\right)\right),$ (3.7)

that is, $[\beta_1 \mid \cdot\,] \sim IG\left(\alpha_1+c_1, \left(\lambda+\frac{1}{d_1}\right)^{-1}\right).$

(4) $[\beta_2 \mid \underline{u}, \underline{w}, \lambda, \theta, \beta_1] \propto \beta_2^{-(\alpha_2+c_2+1)} \cdot \exp\left(-\frac{1}{\beta_2}\left(\lambda\theta+\frac{1}{d_2}\right)\right),$ (3.8)

that is, $[\beta_2 \mid \cdot\,] \sim IG\left(\alpha_2+c_2, \left(\lambda\theta+\frac{1}{d_2}\right)^{-1}\right).$

In this section, we use Gibbs sampling originally introduced in Geman and Geman(1984), and more recently popularized by Gelfand and Smith(1990). Also Gelman and Rubin(1992) introduced iterative simulation using multiple sequences. In this section, we use Gelman and Rubin's method as follows.

First, independently simulate $m \geq 2$ sequences, each of length $2n$, with starting points drawn from an over-dispersed distribution. To diminish the effect of the starting distribution, discard the first n iteration of each sequence, and focus attention on the last n. For an arbitrary starting set of values $U_1^{(0)}, U_2^{(0)}, \cdots, U_p^{(0)}$, we draw $U_1^{(1)} \sim [U_1 \mid U_2^{(0)}, U_3^{(0)}, \cdots, U_p^{(0)}]$, $U_2^{(1)} \sim [U_2 \mid U_1^{(1)}, U_3^{(0)}, \cdots, U_p^{(0)}]$, $\cdots$, $U_p^{(1)} \sim [U_p \mid U_1^{(1)}, U_2^{(1)}, \cdots, U_{p-1}^{(1)}]$. Thus each variable is visited in the natural order and a cycle in this scheme requires p random variate generations. After $2n$ such iterations, one arrives at $(U_1^{(2n)}, U_2^{(2n)}, \cdots, U_p^{(2n)})$. Under mild conditions(Geman and Geman, 1984), $(U_1^{(n)}, \cdots, U_p^{(n)}) \xrightarrow{d} (U_1, \cdots, U_p)$, as $n \to \infty$. Gibbs sampling through m replications of the aforementioned last $n-$iterations generates mn i.i.d. $p-$tuples $(U_{1j}^{(l)}, \cdots, U_{pj}^{(l)})$ $(j=1,2,\cdots,m,\ l=n+1,\cdots,2n)$; $U_1, \cdots, U_p$ could possibly be vectors in the above scheme.

To obtain a pdf estimate(of any posterior) we use a Rao-Blackwell argument:

$$\widehat{[U_s]} \simeq (mn)^{-1} \sum_{j=1}^{m} \sum_{l=n+1}^{2n} [U_s \mid U_{rj}^{(l)}, r \neq s].$$

3.1 Parameter estimation

To estimate the posterior moments, we use Rao-Blackwellized estimates as in Gelfand and Smith(1991). Using above step (1), the posterior mean and the posterior variance for λ are approximated by

$E[\lambda \mid \underline{u}, \underline{w}] = E[E(\lambda \mid \theta, \beta_1, \beta_2, \underline{u}, \underline{w}) \mid \underline{u}, \underline{w}]$

$$= E\left[(2k+\alpha_1+\alpha_2-1)\cdot(2\sum_{i=1}^{k} u_i+\theta\sum_{i=1}^{k} w_i+1/\beta_1+\theta/\beta_2)^{-1} \mid \underline{u},\underline{w}\right].$$

$$\simeq (mn)^{-1}\sum_{j=1}^{m}\sum_{l=n+1}^{2n}\left[(2k+\alpha_1+\alpha_2-1)\cdot(2\sum_{i=1}^{k} u_i+\theta_j^{(l)}\sum_{i=1}^{k} w_i+1/\ \beta_{1j}{}^{(l)}+\theta_j^{(l)}/\ \beta_{2j}{}^{(l)})^{-1}\right].$$

$$Var[\lambda \mid \underline{u},\underline{w}] = E[\,Var(\lambda \mid \underline{u},\underline{w},\theta,\beta_1,\beta_2) \mid \underline{u},\underline{w}] + Var[\,E(\lambda \mid \underline{u},\underline{w},\theta,\beta_1,\beta_2) \mid \underline{u},\underline{w}]$$
$$= E\left[(2k+\alpha_1+\alpha_2-1)\cdot(2\sum_{i=1}^{k} u_i+\theta\sum_{i=1}^{k} w_i+1/\beta_1+\theta/\beta_2)^{-2} \mid \underline{u},\underline{w}\right]$$
$$+ Var\left[(2k+\alpha_1+\alpha_2-1)\cdot(2\sum_{i=1}^{k} u_i+\theta\sum_{i=1}^{k} w_i+1/\beta_1+\theta/\beta_2)^{-1} \mid \underline{u},\underline{w}\right].$$
$$\simeq (mn)^{-1}\sum_{j=1}^{m}\sum_{l=n+1}^{2n}\left[(2k+\alpha_1+\alpha_2-1)\cdot(2\sum_{i=1}^{k} u_i+\theta_j^{(l)}\sum_{i=1}^{k} w_i+1/\ \beta_{1j}{}^{(l)}+\theta_j^{(l)}/\ \beta_{2j}{}^{(l)})^{-2}\right]$$
$$+ (mn)^{-1}\sum_{j=1}^{m}\sum_{l=n+1}^{2n}\left[(2k+\alpha_1+\alpha_2-1)\cdot(2\sum_{i=1}^{k} u_i+\theta_j^{(l)}\sum_{i=1}^{k} w_i+1/\ \beta_{1j}{}^{(l)}+\theta_j^{(l)}/\ \beta_{2j}{}^{(l)})^{-1}\right]^2$$
$$- \left[(mn)^{-1}\sum_{j=1}^{m}\sum_{l=n+1}^{2n} (2k+\alpha_1+\alpha_2-1)\cdot(2\sum_{i=1}^{k} u_i+\theta_j^{(l)}\sum_{i=1}^{k} w_i+1/\ \beta_{1j}{}^{(l)}+\theta_j^{(l)}/\ \beta_{2j}{}^{(l)})^{-1}\right]^2.$$

By similar method, the posterior mean and the posterior variance for θ are given as

$$E[\theta \mid \underline{u},\underline{w}] = E\left[(k+\alpha_2)\cdot(\lambda\sum_{i=1}^{k} w_i+\lambda/\beta_2)^{-1} \mid \underline{u},\underline{w}\right]$$
$$\simeq (mn)^{-1}\sum_{j=1}^{m}\sum_{l=n+1}^{2n}\left[(k+\alpha_2)\cdot(\lambda_j^{(l)}\sum_{i=1}^{k} w_i+\lambda_j^{(l)}/\ \beta_{2j}{}^{(l)})^{-1}\right].$$

$$Var[\theta \mid \underline{u},\underline{w}]$$
$$= E\left[(k+\alpha_2)\cdot(\lambda\sum_{i=1}^{k} w_i+\lambda/\beta_2)^{-2} \mid \underline{u},\underline{w}\right] + Var\left[(n+\alpha_2)\cdot(\lambda\sum_{i=1}^{n} w_i+\lambda/\beta_2)^{-1} \mid \underline{u},\underline{w}\right].$$
$$\simeq (mn)^{-1}\sum_{j=1}^{m}\sum_{l=n+1}^{2n}\left[(k+\alpha_2)\cdot(\lambda_j^{(l)}\sum_{i=1}^{k} w_i+\lambda_j^{(l)}/\ \beta_{2j}{}^{(l)})^{-2}\right]$$
$$+ (mn)^{-1}\sum_{j=1}^{m}\sum_{l=n+1}^{2n}\left[(k+\alpha_2)\cdot(\lambda_j^{(l)}\sum_{i=1}^{k} w_i+\lambda_j^{(l)}/\ \beta_{2j}{}^{(l)})^{-1}\right]^2$$
$$- \left[(mn)^{-1}\sum_{j=1}^{m}\sum_{l=n+1}^{2n} (k+\alpha_2)\cdot(\lambda_j^{(l)}\sum_{i=1}^{k} w_i+\lambda_j^{(l)}/\ \beta_{2j}{}^{(l)})^{-1}\right]^2.$$

3.2 Reliability estimation

To estimate the posterior distribution of R, it is necessary to find the full conditional distribution of R.

With Gibbs sequences from the full conditional distributions, we can obtain

$R_j^{(l)} = \exp(-2u_0\lambda_j^{(l)} - w_0\lambda_j^{(l)}\theta_j^{(l)})$, $j=1,2,\cdots,m$ and $l=n+1,n+2,\cdots,2n$. Note that the $R_j^{(l)}$ can be regarded as samples from unknown posterior distribution of R because of continuity of R.

From the Gibbs sampler procedure, if $\{R_1^{n+1},\cdots,R_m^{2n}\}$ is a sample from posterior distribution of R, Bayes estimator of the joint reliability function is approximated by

$$\hat{R} \simeq (mn)^{-1}\sum_{j=1}^{m}\sum_{l=n+1}^{2n}[\exp(-2u_0\lambda_j^{(l)} - w_0\lambda_j^{(l)}\theta_j^{(l)})].$$

Also the 90% credibility interval[6] (equal tails) is :

$$(R_{[0.05m]}\ ,\ R_{[0.95m]}),$$

where $[0.05m]$ and $[0.95m]$ are the $0.05m$th and $0.95m$th order statistics.

4. A Numerical Example.

In implementing the Gibbs sampler, one should be able to draw samples from the full conditional densities given in (3.5)-(3.8) by using IMSL software.

In our simulated data, we take sample size $k=30$ and $\theta=1.5$ and $\lambda=1$. Table 1 gives the generated data. For the hyperparameters we take $a_1=a_2=5$. To diffuse in hyper prior stage, we take $c_1=c_2=d_1=d_2=10^{-5}$. In Gibbs samper, we use 5 sequences and 3000 iterations for each seqnece. We can easily compute the posterior means for parameters λ, θ and standard errors for those estimators. The posterior means(standard errors) of λ, θ are $\hat{\lambda}=0.9671(0.1650)$ and $\hat{\theta}=1.6748(0.4204)$, respectively. We then obtain the estimated posterior pdf's of $[\lambda \mid \underline{u}, \underline{w}]$, $[\theta \mid \underline{u}, \underline{w}]$ as shown figures 1-2. Each figures are almost peaked at the true value of λ and θ, respectively.

Table 2 contains the true values of R, $\hat{R}$, posterior mean, the residuals, and the 90% credibility intervals for R for several choices of mission time u_0, w_0. Also Figure 3 indicates graph of estimated pdf of $[R \mid \underline{u}, \underline{w}]$ for $R=0.5018$, 0.7022 and 0.9032, respectively. The figure is almost peaked at the true values of R.

Table 1. Generated data

6) A Bayes credibility interval is analogous to a classical s-interval(See Dey and Lee(1992).

i	(u_i, w_i)	i	(u_i, w_i)
1	(0.2776, 0.0767)	16	(0.4376, 0.2977)
2	(0.8861, 0.6789)	17	(0.0877, 1.2083)
3	(0.3274, 1.3390)	18	(0.9190, 0.1517)
4	(0.1425, 0.5384)	19	(0.2799, 0.0827)
5	(0.7552, 1.0107)	20	(0.4084, 0.6302)
6	(0.7456, 1.0712)	21	(0.5223, 0.5101)
7	(3.2087, 2.0753)	22	(0.1117, 1.0661)
8	(0.4030, 0.5419)	23	(0.5609, 1.5151)
9	(0.6692, 0.3919)	24	(0.6153, 0.2474)
10	(0.6290, 1.2050)	25	(1.2334, 0.2357)
11	(0.0244, 0.9264)	26	(1.3615, 0.1798)
12	(0.6578, 0.2004)	27	(0.1221, 1.9795)
13	(0.0268, 1.8200)	28	(0.1923, 0.0253)
14	(0.1831, 1.3504)	29	(0.4803, 1.4495)
15	(0.3376, 0.0370)	30	(0.5440, 0.0627)

Table 2. Posterior means and 90% credibility intervals of R

$u_0 = w_0$	R	$\hat{R}$	$R-\hat{R}$	credibility interval
0.655	0.1010	0.1044	-0.0034	(0.0622 , 0.1567)
0.456	0.2022	0.2057	-0.0035	(0.1447 , 0.2751)
0.343	0.3010	0.3034	-0.0024	(0.2336 , 0.3788)
0.261	0.4011	0.4027	-0.0016	(0.3307 , 0.4778)
0.197	0.5018	0.5027	-0.0009	(0.4338 , 0.5726)
0.145	0.6019	0.6024	-0.0005	(0.5408 , 0.6634)
0.101	0.7022	0.7022	0.0000	(0.6517 , 0.7514)
0.062	0.8049	0.8048	0.0001	(0.7689 , 0.8391)
0.003	0.9032	0.9002	0.0030	(0.8806 , 0.9186)

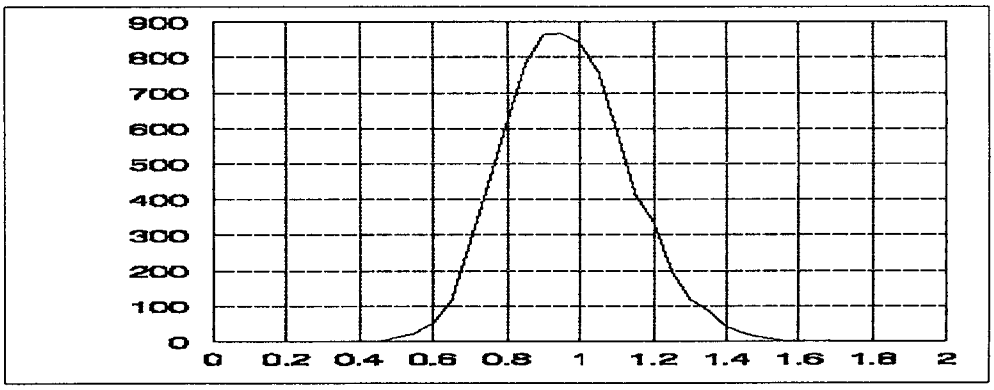

Figure 1. Estimated pdf of λ

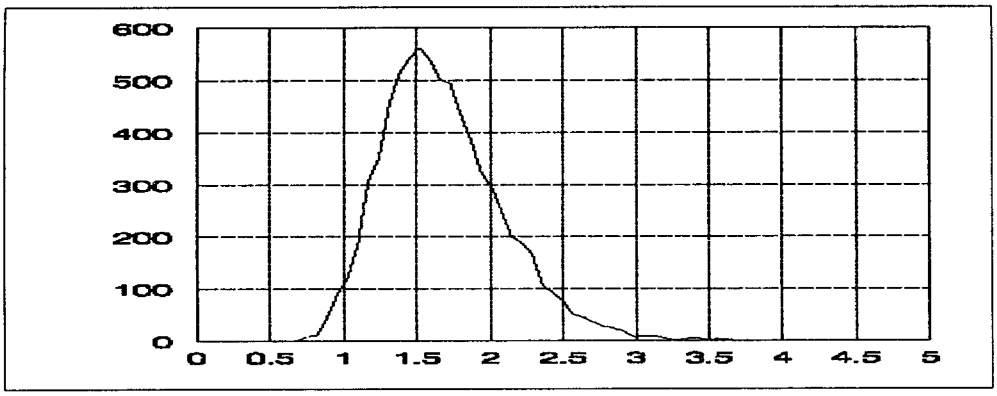

Figure 2. Estimated pdf of $\hat{\theta}$

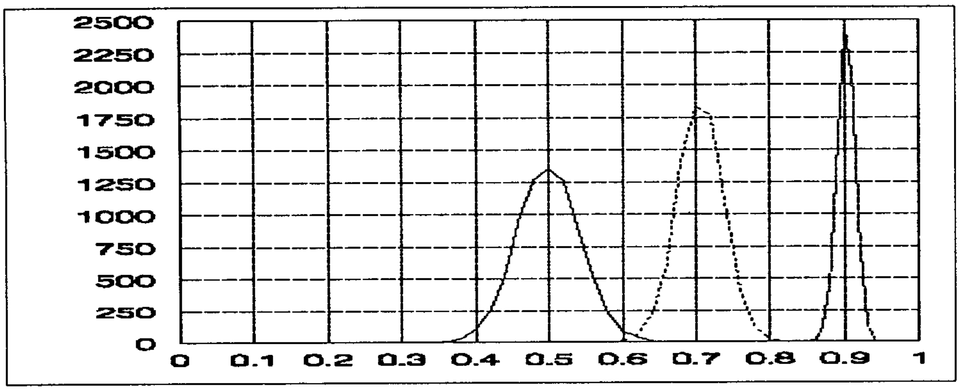

Figure 3. Estimated pdf of $\hat{R}$

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