The Korean Communications
in Statistics Vol. 5, No. 1, 1998
pp. 231-238

A Study on the Efficiency of a Two Stage Shrinkage Testimator for the Mean of an Exponential Distribution[1]

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Abstract

A two stage shrinkage testimator for the mean of an exponential distribution is considered with the assumption that an initial estimate of the mean is available. Mean squared error(MSE) of testimator and its relative efficiency (to usual single sample mean) are briefly reviewed. It is shown that relative efficiency depends only on the ratio of true mean value and its initial estimate.

1. Introduction

Testimator is a kind of adaptive estimators (Hogg(1974), Katti(1962), and Waikar and Katti(1971)) based on a preliminary test on an available initial estimate. Waikar, Schuurmann and Raghunathan(1984) developed a testimator for the mean of a normal distribution. Later, Adke, Waikar and Schuurmann(1987) extended their results to the testimation of the mean of an exponential distribution, and provided a table containing the relative efficiency of testimator for the various values of true mean when the initial estimate is fixed to 1. Hence, if we are able to show that relative efficiency depends only on the ratio of true mean value and its initial estimate, then the relative efficiency results given in their table can be applied to other values of initial estimate, and it greatly simplifies the computation of relative efficiency. This paper focuses on that point.

Two stage shrinkage testimation of the mean of an exponential distribution, based on Adke, Waikar and Schuurmann(1987), is a main topic of this paper. Two stage shrinkage testimator of the mean, its MSE and relative efficiency are briefly reviewed in section 2. The same basic notation given in Adke, Waikar and Schuurmann is used in this paper with some modifications as necessary. Invariance property of relative efficiency to the ratio of true mean

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value and its initial estimate is proved in section 3. An example including simulation results is provided in section 4 to clarify what is meant in section 3. Some concluding remarks are given in the final section.

2. Two stage shrinkage testimator of the mean, its MSE and relative efficiency: Exponential distribution

Let X be a random variable following an exponential distribution with mean θ, and suppose that an initial estimate θ_0 of θ is available. Adke, Waikar and Schuurmann(1987) proposed a two stage shrinkage testimator of θ based on θ_0 which is defined as follows:

Step 1: Obtain n_1 first stage samples($X_{1i},\ i = 1, 2, \cdots, n_1$), and test

$$H_0:\ \theta = \theta_0 \text{ vs. } H_1:\ \theta \neq \theta_0$$

at level α using the first stage sample mean $\overline{X}_1$.

Step 2: If H_0 is accepted, testimator is defined as

$$\hat{\theta} = w\,\overline{X}_1 + (1 - w)\,\theta_0 \qquad 0 < w < 1.$$

If H_0 is rejected, obtain n_2 second stage samples($X_{2i},\ i = 1, 2, \cdots, n_2$) and define testimator as the combined sample mean, that is

$$\hat{\theta} = (n_1\,\overline{X}_1 + n_2\,\overline{X}_2)/(n_1 + n_2).$$

The UMPU test for testing above hypothesis is given by:

$$\text{Reject } H_0 \text{ if } Z_1 = \sum_{i=1}^{n_1} X_{1i} < k_1 \quad \text{or if} \quad Z_1 > k_2,$$

where k_1 and k_2 are chosen to satisfy

$$\begin{aligned} 1 - \alpha &= \gamma(n_1, k_2/\theta_0) - \gamma(n_1, k_1/\theta_0) \\ &= \gamma(n_1 + 1, k_2/\theta_0) - \gamma(n_1 + 1, k_1/\theta_0) \end{aligned} \tag{2.1}$$

where $\gamma(a, x) = \frac{1}{\Gamma(a)} \int_0^x y^{a-1} e^{-y} dy$ represents a usual incomplete gamma function. They defined the weighing factor w as $|Z_1 - n_1\theta_0|/(k_2 - k_1)$ so that a higher weight is given to θ_0 when $\overline{X}_1$ is closer to θ_0.

To compare the property of testimator with that of usual single sample mean, they derived MSE of testimator and it is rewritten here for later use.

$$\mathrm{MSE}(\hat{\theta}\,|\,\theta) = \mathrm{E}(\hat{\theta}^2) - 2\theta \cdot \{ b_0(\theta) + \theta \cdot b_1(\theta) + \theta^2 \cdot b_2(\theta)\} + \theta^2, \tag{2.2}$$

where $\mathrm{E}(\hat{\theta}^2) = \theta^2 + \theta^2/(n_1+n_2) + \sum_{j=0}^{4} c_j^* \cdot \gamma(n_1+j, k_1/\theta, k_2/\theta)$

$$+ \sum_{j=0}^{2} d_j^* \cdot \gamma(n_1+j,\ k_1/\theta,\ n_1\theta_0/\theta,\ k_2/\theta),$$

and where $\gamma(n,a,b) = \gamma(n,b) - \gamma(n,a)$ and $\gamma(n,a,b,c) = \gamma(n,c) - 2\gamma(n,b) + \gamma(n,a)$.
The remaining notation are defined as follows:

$b_0(\theta) = \theta_0\,\gamma(n_1,\ k_1/\theta,\ k_2/\theta) + [n_1\theta_0^2/(k_2-k_1)] \cdot \gamma(n_1,\ k_1/\theta,\ n_1\theta_0/\theta,\ k_2/\theta)$,

$b_1(\theta) = 1 - [n_2/(n_1+n_2)] \cdot \gamma(n_1, k_1/\theta, k_2/\theta) - [n_1/(n_1+n_2)] \cdot \gamma(n_1+1, k_1/\theta, k_2/\theta)$
$\qquad - [2n_1\theta_0/(k_2-k_1)] \cdot \gamma(n_1+1,\ k_1/\theta,\ n_1\theta_0/\theta,\ k_2/\theta)$,

$b_2(\theta) = [(n_1+1)/(k_2-k_1)] \cdot \gamma(n_1+2,\ k_1/\theta,\ n_1\theta_0/\theta,\ k_2/\theta)$,

$$c_0 = \frac{(n_1+n_2)^2\theta_0^2 - n_2\theta^2(n_2+1)}{(n_1+n_2)^2} + \frac{(n_1\theta_0)^4}{n_1^2(k_2-k_1)^2},\quad c_1 = -\frac{4(n_1\theta_0)^3}{n_1^2(k_2-k_1)^2} - \frac{2n_2\theta}{(n_1+n_2)^2},$$

$$c_2 = \frac{6\theta_0^2}{(k_2-k_1)^2} - \frac{1}{(n_1+n_2)^2},\quad c_3 = -\frac{4\theta_0}{n_1(k_2-k_1)^2},\quad c_4 = \frac{1}{n_1^2(k_2-k_1)^2},$$

$$d_0 = \frac{2n_1\theta_0^3}{(k_2-k_1)},\quad d_1 = -\frac{4\theta_0^2}{(k_2-k_1)},\quad d_2 = \frac{2\theta_0}{n_1(k_2-k_1)},$$

$c_j^* = n_1(n_1+1)\cdots(n_1+j-1)\theta^j c_j,\quad j = 0,1,2,3,4,$ and

$d_j^* = n_1(n_1+1)\cdots(n_1+j-1)\theta^j d_j,\quad j = 0,1,2.$

Therefore, its relative efficiency to usual single sample mean is defined by

$$eff_{\bar{X},\hat{\theta}}(\theta) = \frac{\theta^2/n^*}{MSE(\hat{\theta}\,|\,\theta)}, \tag{2.3}$$

where $n^* = n_1 + n_2 \cdot [1 - (\gamma(n_1, k_2/\theta) - \gamma(n_1, k_1/\theta))]$ and $MSE(\hat{\theta}\,|\,\theta)$ is given in (2.2).

3. Invariance of relative efficiency to the ratio $\frac{\theta}{\theta_0}$

If the relative efficiency defined in (2.3) is invariant to the ratio $\frac{\theta}{\theta_0}$, it will greatly simplify the computation of relative efficiency. In this section, it is shown that relative efficiency depends only on the ratio $\frac{\theta}{\theta_0}$.

Suppose that initial estimates θ_{10} of mean θ_1, and θ_{20} of mean θ_2 are available. To prove the invariance property mentioned above, it is sufficient to show $eff_{\overline{X},\,\widehat{\theta_1}}(\theta_1) = eff_{\overline{X},\,\widehat{\theta_2}}(\theta_2)$ if $\frac{\theta_1}{\theta_{10}} = \frac{\theta_2}{\theta_{20}}$, where $\widehat{\theta_1}$ ($\widehat{\theta_2}$) is a testimator of θ_1 (θ_2).

When θ_1 (θ_2) is true mean, define k_{11} (k_{21}) and k_{12} (k_{22}) as two constants that take the role of k_1 and k_2 in (2.1) with θ_0 replaced by θ_{10} (θ_{20}). Also define $b_j(\theta_1)$ and $b_j(\theta_2)$, $j=0,1,2$, as $b_j(\theta)$ with θ replaced by θ_1 and θ_2, and θ_0 replaced by θ_{10} and θ_{20}, respectively. The notation $c_{ij}^*(i=1,2;\ j=0,1,\cdots,4)$ and $d_{ij}^*(i=1,2;\ j=0,1,2)$ are defined similarly. Using above definitions, some lemmas useful in proving the invariance property are given in the following.

Lemma 1. With k_{ij}'s $(i,j=1,2)$ defined in the above, we have

$$k_{12} = (\theta_{10}/\theta_{20})\cdot k_{22} \quad \text{and} \quad k_{11} = (\theta_{10}/\theta_{20})\cdot k_{21}.$$

Proof. From (2.1), it follows

$$\begin{aligned} 1-\alpha &= \gamma(n_1, k_{12}/\theta_{10}) - \gamma(n_1, k_{11}/\theta_{10}) \\ &= \gamma(n_1, k_{22}/\theta_{20}) - \gamma(n_1, k_{21}/\theta_{20}), \end{aligned}$$

and

$$\begin{aligned} 1-\alpha &= \gamma(n_1+1, k_{12}/\theta_{10}) - \gamma(n_1+1, k_{11}/\theta_{10}) \\ &= \gamma(n_1+1, k_{22}/\theta_{20}) - \gamma(n_1+1, k_{21}/\theta_{20}) \end{aligned}$$

for all α. Hence the results follow. □

Lemma 2. With k_{ij}'s $(i,j=1,2)$ defined in the above, if $\frac{\theta_1}{\theta_{10}} = \frac{\theta_2}{\theta_{20}}$, then

$$\gamma(n_1+j,\ k_{11}/\theta_1,\ k_{12}/\theta_1) = \gamma(n_1+j,\ k_{21}/\theta_2,\ k_{22}/\theta_2), \qquad j=0,1,2,3,4,$$

and

$\gamma(n_1+j,\ k_{11}/\theta_1,\ n_1\theta_{10}/\theta_1,\ k_{12}/\theta_1) = \gamma(n_1+j,\ k_{21}/\theta_2,\ n_1\theta_{20}/\theta_2,\ k_{22}/\theta_2),\quad j=0,1,2,3.$

Proof. It is evident using Lemma 1 and the given condition $\theta_1 = (\theta_{10}/\theta_{20}) \cdot \theta_2$. □

Lemma 3. With k_{ij}'s ($i, j = 1,2$) defined in the above, if $\dfrac{\theta_1}{\theta_{10}} = \dfrac{\theta_2}{\theta_{20}}$, then it follows:

i) $E(\widehat{\theta}_1^{\,2}) = (\theta_{10}/\theta_{20})^2 \cdot E(\widehat{\theta}_2^{\,2})$, and

ii) $E(\widehat{\theta}_1) = (\theta_{10}/\theta_{20}) \cdot E(\widehat{\theta}_2)$.

Proof.

i) It can be shown that $c_{10} = (\theta_{10}/\theta_{20})^2 \cdot c_{20}$ using Lemma 1 and the given condition, which results in $c_{10}^* = (\theta_{10}/\theta_{20})^2 \cdot c_{20}^*$. Similarly we have

$$c_{1j}^* = (\theta_{10}/\theta_{20})^2 \cdot c_{2j}^*,\quad j=1,2,3,4,\ \text{and}\ d_{1j}^* = (\theta_{10}/\theta_{20})^2 \cdot d_{2j}^*,\quad j=0,1,2.$$

Now, the result follows by applying Lemma 2 and the given condition.

ii) Again by applying Lemma 1, Lemma 2 and the given condition, it follows that

$$b_0(\theta_1) = (\theta_{10}/\theta_{20}) \cdot b_0(\theta_2),\ b_1(\theta_1) = b_1(\theta_2)\ \text{and}\ b_2(\theta_1) = (\theta_{20}/\theta_{10}) \cdot b_2(\theta_2).$$

Hence, the result follows immediately. □

Finally, the invariance property of relative efficiency of testimator to $\dfrac{\theta}{\theta_0}$ is stated in the following theorem.

Theorem 1. If $\dfrac{\theta_1}{\theta_{10}} = \dfrac{\theta_2}{\theta_{20}}$, then we have $eff_{\overline{X},\widehat{\theta_1}}(\theta_1) = eff_{\overline{X},\widehat{\theta_2}}(\theta_2)$.

Proof. From the results of Lemma 3, it can be shown that

$$\mathrm{MSE}(\widehat{\theta}_1 \mid \theta_1) = (\theta_{10}/\theta_{20})^2 \cdot \mathrm{MSE}(\widehat{\theta}_2 \mid \theta_2).$$

Therefore,

$$\begin{aligned} eff_{\overline{X},\widehat{\theta_1}}(\theta_1) &= \frac{\theta_1^2/n^*}{\mathrm{MSE}(\widehat{\theta}_1 \mid \theta_1)} \\ &= \frac{(\theta_{10}/\theta_{20})^2 \cdot \theta_2^2/n^*}{(\theta_{10}/\theta_{20})^2 \cdot \mathrm{MSE}(\widehat{\theta}_2 \mid \theta_2)} \\ &= eff_{\overline{X},\widehat{\theta_2}}(\theta_2), \end{aligned}$$

which completes the proof. □

4. An Example

An example is given in this section to illustrate what is meant by Theorem 1. Simulation is used since it is difficult to illustrate by real data. The following values of parameter, initial estimate and sample sizes are used in simulation.

Set I: $n_1 = 10$, $n_2 = 10 \sim 50\ (10)$, $\alpha = 0.05$, $\theta/\theta_0 = 1.00,\ 1.20,\ 1.40$,

Set II: $n_1 = 20$, $n_2 = 10 \sim 50\ (10)$, $\alpha = 0.01$, $\theta/\theta_0 = 1.00,\ 1.20,\ 1.40$,

Set III: $n_1 = 25$, $n_2 = 10 \sim 50\ (10)$, $\alpha = 0.10$, $\theta/\theta_0 = 1.00,\ 0.90,\ 0.80$.

For each combination of n_1, n_2, θ, θ_0 and α values given in each set, simulation was performed 100,000 times. Usual single sample mean, and testimator value were calculated for each simulated data set and the empirical relative efficiency was obtained from them. IMSL subroutine RNEXP was used in generating exponential variates.

Empirical relative efficiency of testimator is summarized in Table 1. In simulation, initial estimate (θ_0) was fixed to 2.0 and true mean value (θ) was changed according to the ratio θ/θ_0 values given in each set. In Table 2, theoretical relative efficiency of testimator obtained using (2.3) is simply reproduced from Adke, Waikar and Schuurmann(1987) for comparison purpose. Initial estimate value was fixed to 1.0 in this table. Note that the ratio θ/θ_0 values given in both tables are the same although θ_0 values are different.

Comparison of both tables reveals that they are almost the same for every combination given in Set I, II and III. Therefore, it is confirmed empirically what is stated in Theorem 1 from this simulation result(In fact, they are exactly the same if theoretical calculation is made using (2.3) in Table 1 for each combination).

5. Conclusion

The relative efficiency of a two stage shrinkage testimator of the mean of an exponential distribution is considered with the assumption that an initial estimate of the mean is available. Two stage shrinkage testimation procedure is briefly reviewed and it is proved that relative efficiency to usual single sample mean is invariant to the ratio of the true mean and its initial estimate. As a result of this proof, the computation of relative efficiency is greatly simplified. That is, since the relative efficiency depends only on the ratio θ/θ_0, it is possible to find the relative efficiency for any combination of θ and θ_0 values if we have relative efficiency calculation result for the simplest one(for example when $\theta_0 = 1.0$, as is given in a table of Adke, Waikar and Schuurmann, p.1829-1831).

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Table 1. (Simulation results) Relative efficiency of testimator w.r.t. $\overline{X}$ when $\theta_0 = 2$

	$n_1 = 10$, $\alpha = 0.05$, $k_1 = 9.95780$, $k_2 = 35.22680$			$n_1 = 20$, $\alpha = 0.01$, $k_1 = 21.09372$, $k_2 = 67.79314$			$n_1 = 25$, $\alpha = 0.10$, $k_1 = 35.26020$, $k_2 = 68.36680$		
n_2 \ θ	2.00	2.40	2.80	2.00	2.40	2.80	2.00	1.80	1.60
10	5.60439	2.63196	1.32675	7.76312	1.91571	0.87462	3.05554	1.96471	0.85192
20	6.96481	2.91874	1.30357	8.82881	2.09009	0.89166	4.00971	2.21175	0.83525
30	7.47947	2.95118	1.21372	9.34586	2.15282	0.87034	4.69950	2.34381	0.79839
40	7.74713	2.91716	1.11746	9.81253	2.17610	0.83581	5.36551	2.42038	0.75898
50	7.71956	2.80543	1.02262	10.03330	2.19478	0.80119	5.84194	2.41773	0.72151

Table 2. Theoretical relative efficiency of testimator w.r.t. $\overline{X}$ when $\theta_0 = 1$

	$n_1 = 10$, $\alpha = 0.05$, $k_1 = 4.97890$, $k_2 = 17.61340$			$n_1 = 20$, $\alpha = 0.01$, $k_1 = 10.54686$, $k_2 = 33.89657$			$n_1 = 25$, $\alpha = 0.10$, $k_1 = 17.63010$, $k_2 = 34.18340$		
n_2 \ θ	1.00	1.20	1.40	1.00	1.20	1.40 .	1.00	0.90	0.80
10	5.58081	2.61506	1.31660	7.76734	1.92096	0.87745	3.04473	1.96358	0.85385
20	6.93081	2.93425	1.29682	8.82327	2.08520	0.88954	3.99187	2.21345	0.83363
30	7.49003	2.97211	1.20894	9.41610	2.15539	0.87039	4.76338	2.34596	0.79814
40	7.67698	2.90744	1.11334	9.76593	2.18008	0.83974	5.36587	2.40593	0.75840
50	7.68186	2.80515	1.02466	9.97998	2.18097	0.80544	5.82292	2.42125	0.71880