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Bayes Estimation for the Reliability and Hazard Rate under the Burr Type X Failure Model

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Abstract

In this paper, we consider a hierarchical Bayes estimation of the parameter, the reliability and hazard rate function based on samples from a Burr type X failure model. Bayes calculations can be implemented by means of the Gibbs sampler and a numerical study is provided.

1. Introduction

The Burr distributions was first introduced and classified twelve types of distributions by Burr(1942). Since the distributions have a wide variety of shapes, these distributions are useful for approximating histograms, particularly when a simple mathematical structure for the fitted cumulative distribution function is required. Other applications include simulation, quantal response, approximation of distributions, and development of nonnormal control charts. A number of standard theoretical distributions are limiting forms of Burr distributions. The Burr distribution was first proposed as a lifetime model by Dubey(1972, 1973). Lewis(1981) noted that Weibull and exponential distributions are special limiting cases of the parameter values of the Burr type XII distribution. She proposed the use of the Burr type XII distribution as a model in accelerated life test data representing times to breakdown of an insulating fluid. Evans and Ragab(1983) obtained Bayes estimators of the two parameters and reliability of the Burr type XII distribution by assuming discrete values on a finite set of points of the prior. Nigam(1988) gave the prediction interval for the k-th order statistic in a future sample from Burr type XII distribution based on a censored sample from the same distribution. Sartawi and Abu-Salih(1991) obtained the prediction interval for the order statistic in the two sample and one sample cases from the Burr type X distribution based on a censored data. AL-Hussaini and Jaheen(1992,1994) developed approximate Bayes estimators of the two parameters, reliability and failure rate functions of the Burr type XII failure model by using the methods

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of Lindley and Tierney and Kadane under type II censored data.

For Gibbs sampler approach, Dey and Lee(1992) considered Bayesian computation for the parameters and the reliability function of 2-parameter exponential distribution, and also constrained parameter and truncated data problem in multivariate life distributions. Tiwari, Yang and Zalkikar(1996) considered Bayesian estimation of the parameters and the reliability function based on type II censored data from a Pareto failure model.

In this paper, we consider the Gibbs sampler approach for the hierarchical Bayes analysis of the Burr type X failure model. In section 2, we give the model and describe the computation methods for Bayes estimation. In section 3, we implement the Burr type X failure model with an illustration from the simulated data.

2. Hierarchical Bayes Model and Gibbs Sampler

We assume that the Burr type X model represent the lifetimes of all item. The Burr type X probability density function(pdf) with parameter θ is given by

$$f(x \mid \theta) = 2\theta x \exp(-x^2)(1-\exp(-x^2))^{\theta-1}, \quad x>0, \theta>0. \tag{2.1}$$

A random sample of n items is drawn from (2.1) and is put on life test. The observed sample consists of the failure times, $x_1, x_2, \cdots, x_n$. The likelihood function of the sample is given as

$$f(\underline{x} \mid \theta) = 2^n \theta^n \Pi_{i=1}^n x_i \exp(-x_i^2)(1-\exp(-x_i^2))^{\theta-1}, \tag{2.2}$$

where $\underline{x} = (x_1, x_2, \cdots, x_n)$. The reliability and hazard rate functions are given by

$$R(t) = 1-(1-\exp(-t^2))^{\theta-1}, \tag{2.3}$$

$$h(t) = \frac{2\theta t \exp(-t^2)(1-\exp(-t^2))^{\theta-1}}{1-(1-\exp(-t^2))^{\theta-1}}. \tag{2.4}$$

For convenience, densities are denoted generically by brackets, so joint, conditional, and marginal forms, for example, appear as $[X, Y]$, $[X \mid Y]$, and $[X]$. Multiplication of densities is denoted by $*$; for example, $[X, Y] = [X \mid Y] * [Y]$.

The general hierarchical Bayes model is as follow:

(1) $[\underline{x} \mid \theta] = 2^n \theta^n \Pi_{i=1}^n x_i \exp(-x_i^2)(1-\exp(-x_i^2))^{\theta-1}$, where $\underline{x} = (x_1, x_2, \cdots, x_n)$.

(2) $[\theta \mid \alpha_1, \beta_1] \sim G(\alpha_1, \beta_1)$, where $G(\alpha_1, \beta_1)$ is gamma density given as

$$f(\theta \mid \alpha_1, \beta_1) = \frac{\beta_1^{\alpha_1}}{\Gamma(\alpha_1)} \theta^{\alpha_1 - 1} \exp(-\beta_1 \theta), \quad \alpha_1 > 0, \beta_1 > 0.$$

(3) We assume that the joint prior is $[\alpha_1, \beta_1] = [\alpha_1] * [\beta_1]$.

$[\alpha_1] \propto \exp(-c\alpha_1)$, where c is a known positive constant and

$[\beta_1] \sim G(\alpha_2, \beta_2)$, where α_2 and β_2 are known positive constants.

In (3), we shall consider several choices of c while doing the data analysis in Section 3.

The Gibbs sampler is an iterative Monte Carlo integration method, developed formally by Geman and Geman(1984) in the context of image restoration. In Statistical framework Tanner and Wong(1987) used essentially this algorithm in their substitution sampling approach. Gelfand and Smith(1990) developed the Gibbs sampler for fairly general parametric settings. To summarize the method briefly, suppose we have a collection of p r.v.'s $U_1, \cdots, U_p$ whose full conditional distributions, denoted generally by $[U_s \mid U_r, r \neq s]$, $s = 1, \cdots, p$ are available for sampling. Under mild conditions, these full conditional distributions uniquely determine the full joint distribution $[U_1, \cdots, U_p]$ and hence all the marginal distributions $[U_s], s = 1, \cdots, p$. The Gibbs sampler generates from the conditional distributions as follows: Given an arbitrary starting set of values $U_1^{(0)}, \cdots, U_p^{(0)}$, we draw $U_1^{(1)}$ from $[U_1 \mid U_2^{(0)}, \cdots, U_p^{(0)}]$, $U_2^{(1)}$ from $[U_2 \mid U_1^{(1)}, U_3^{(0)}, \cdots, U_p^{(0)}]$, and so on up to $U_p^{(1)}$ from $[U_p \mid U_1^{(1)}, \cdots, U_{p-1}^{(1)}]$ to complete one iteration of the scheme. After t such iterations we arrive at a joint sample $(U_1^{(t)}, \cdots, U_p^{(t)})$ from $[U_1, \cdots, U_p]$. Under mild conditions, Geman and Geman(1984) showed that $(U_1^{(t)}, \cdots, U_p^{(t)}) \to^d (U_1, \cdots, U_p) \sim [U_1, \cdots, U_p]$ as $t \to \infty$. Hence for sufficiently large t, $U_s^{(t)}$ can be regarded as a sample from $[U_s]$. Parallel replications m times yields m i.i.d. p-tuples: $(U_{1j}^{(t)}, \cdots, U_{pj}^{(t)})$, $j = 1, \cdots, m$. For any function T of $U_1, \cdots, U_p$ whose expectation exists,

$$\frac{1}{m} \sum_{i=1}^{m} T(U_1^{(t)}, \cdots, U_p^{(t)}) \to \mathrm{E}[T(U_1, \cdots, U_p)] \text{ as } m \to \infty$$

almost surely. The distribution of $(U_1, \cdots, U_p)$ can be approximated by the empirical distribution of $(U_{1j}^{(t)}, \cdots, U_{pj}^{(t)})$, $j = 1, \cdots, m$. Similarly the marginal of U_s can be approximated by the empirical distribution of $U_{sj}^{(t)}$, $j = 1, \cdots, m$. If $[U_s \mid U_r, r \neq s]$ can be computed, then

$$\widehat{[U_s]} = \frac{1}{m} \sum_{j=1}^{m} [U_s \mid U_{rj}^{(t)}, r \neq s]. \tag{2.5}$$

For any $T(U_1, \cdots, U_p)$, let $T_j^{(t)} \equiv T(U_{1j}^{(t)}, \cdots, U_{pj}^{(t)})$, $j=1,\cdots,m$, the empirical distribution of $T_1^{(t)}, \cdots, T_m^{(t)}$ provides an estimate of $[T(U_1, \cdots, U_p)]$.

In implementing Gibbs sampler, we follow the recommendation of Gelman and Rubin(1992) and run $m(\geq 2)$ parallel chains, each for $2d$ iterations with starting points drawn from an overdispersed distribution. But to diminish the effects of the starting distributions, the first d iterations of each chain are discarded. After d iterations, all the subsequence iterates are retained for finding the desired posterior distributions, posterior mean and variance, as well as for monitoring the convergence of the Gibbs sampler. The convergence monitoring is discussed in detail in Section 3.

To implement the Gibbs sampler, we need to calculate the full conditional distributions. From the model, the joint posterior density of θ, α_1 and β_1 is

$$\begin{aligned}[\theta, \alpha_1, \beta_1 \mid \underline{x}] &\propto [\underline{x} \mid \theta] * [\theta \mid \alpha_1, \beta_1] * [(\alpha_1] * [\beta_1] \\ &\propto \exp[(\theta-1) \sum_{i=1}^{n} \log(1-\exp(-x_i^2)) - \beta_1\theta - \beta_2\beta_1 - c\alpha_1] \\ &\quad \frac{1}{\Gamma(\alpha_1)} \theta^{(n+\alpha_1-1)} \beta_1^{(\alpha_1+\alpha_2-1)}. \end{aligned} \tag{2.6}$$

From (2.6), the full conditional distributions are given by

(1) $[\theta \mid \alpha_1, \beta_1, \underline{x}] \propto \exp[-\theta(\beta_1 - \sum_{i=1}^{n} \log(1-\exp(-x_i^2)))] \theta^{(n+\alpha_1-1)}$,

that is, $$[\theta \mid \alpha_1, \beta_1, \underline{x}] \sim G[n+\alpha_1, \beta_1 - \sum_{i=1}^{n} \log(1-\exp(-x_i^2))]. \tag{2.7}$$

(2) $[\beta_1 \mid \theta, \alpha_1, \underline{x}] \propto \exp[-\beta_1(\theta+\beta_2)] \beta_1^{(\alpha_1+\alpha_2-1)}$,

that is, $$[\beta_1 \mid \theta, \alpha_1, \underline{x}] \sim G[\alpha_1+\alpha_2, \theta+\beta_2], \tag{2.8}$$

(3) $$[\alpha_1 \mid \theta, \beta_1, \underline{x}] \propto \exp(-c\alpha_1) \frac{1}{\Gamma(\alpha_1)} \beta_1^{\alpha_1} \theta^{\alpha_1}. \tag{2.9}$$

To implement the Gibbs sampler, We should be able to draw samples from the conditional densities given in (2.7)-(2.9). Simulation from the conditional densities (2.7) and (2.8) which are both gamma densities can be done by standard methods. However, in order to simulate from the posterior density(2.9), one approach is to use the adaptive rejection sampling algorithm by Gilks and Wild(1992). Fortunately, the use of the adaptive rejection sampling algorithm becomes simple for us because of the following result of lemma.

Lemma $[\alpha_1 \mid \theta, \beta_1, \underline{x}]$ is a log-concave function of α_1.

Proof. Consider

$$[\alpha_1 \mid \theta, \beta_1, \underline{x}] \propto \exp(-c\alpha_1) \frac{1}{\Gamma(\alpha_1)} \beta_1^{\alpha_1} \theta^{\alpha_1},$$

then

$$\log[\alpha_1 \mid \theta, \beta_1, \underset{\sim}{x}] = a + \alpha_1 \log(\beta_1 \theta) - \log\Gamma(\alpha_1) - c\alpha_1,$$

where a is the norming constant. Hence

$$\frac{\partial \log[\alpha_1 \mid \theta, \beta_1, \underset{\sim}{x}]}{\partial \alpha_1} = \log(\beta_1\theta) - c - \frac{d}{d\alpha_1}\log\left(\frac{\Gamma(\alpha_1+1)}{\alpha_1}\right)$$

$$= \log(\beta_1\theta) - c + \frac{1}{\alpha_1} - \frac{\int_0^\infty \log e^{-z} z^{\alpha_1} dz}{\int_0^\infty e^{-z} z^{\alpha_1} dz}.$$

Therefore

$$\frac{\partial^2 \log[\alpha_1 \mid \theta, \beta_1, \underset{\sim}{x}]}{\partial \alpha_1^2} = -\frac{1}{\alpha_1^2} - \left[\frac{\int_0^\infty (\log z)^2 e^{-z} z^{\alpha_1} dz}{\int_0^\infty e^{-z} z^{\alpha_1} dz} - \left(\frac{\int_0^\infty \log z e^{-z} z^{\alpha_1} dz}{\int_0^\infty e^{-z} z^{\alpha_1} dz}\right)^2\right]$$

$$= -\frac{1}{\alpha_1^2} - \mathrm{Var}(\log z) < 0, \quad \text{since } z \sim G(1, \alpha_1+1).$$

So all required random variated generation is straightforward.

Using the Gibbs sampler, the posterior distribution of θ given $\underset{\sim}{x}$ is approximated by

$$[\theta \mid \underset{\sim}{x}] \approx (md)^{-1} \sum_{k=1}^{m} \sum_{l=d+1}^{2d} [\theta \mid \alpha_1 = \alpha_{1kl}, \beta_1 = \beta_{1kl}, \underset{\sim}{x}]. \tag{2.10}$$

Also following Gelfand and Smith(1991), Rao-Blackwellized estimates of posterior mean and variance of the θ are given by

$$\mathrm{E}(\theta \mid \underset{\sim}{x}) \approx (md)^{-1} \sum_{k=1}^{m} \sum_{l=d+1}^{2d} \frac{n + \alpha_{1kl}}{\beta_{1kl} - \sum_{i=1}^{n} \log(1 - \exp(-x_i^2))} \tag{2.11}$$

and

$$\begin{aligned} \mathrm{Var}(\theta \mid \underset{\sim}{x}) &\approx (md)^{-1} \sum_{k=1}^{m} \sum_{l=d+1}^{2d} \frac{n + \alpha_{1kl}}{(\beta_{1kl} - \sum_{i=1}^{n} \log(1 - \exp(-x_i^2)))^2} \\ &+ (md)^{-1} \sum_{k=1}^{m} \sum_{l=d+1}^{2d} \frac{(n + \alpha_{1kl})^2}{(\beta_{1kl} - \sum_{i=1}^{n} \log(1 - \exp(-x_i^2)))^2} \\ &- [(md)^{-1} \sum_{k=1}^{m} \sum_{l=d+1}^{2d} \frac{n + \alpha_{1kl}}{\beta_{1kl} - \sum_{i=1}^{n} \log(1 - \exp(-x_i^2))}]^2. \end{aligned} \tag{2.12}$$

3. A Numerical Example

In this Section, an illustrative example is represented by simulated data. In our simulation data, we take $n=20$ and $\theta=3.0$ and generate the observations x_j from the Burr type X failure model with parameter θ. Using Gibbs sampler, we need the marginal posterior densities as follows:

$$[\theta \mid \alpha_1, \beta_1, \underline{x}] \sim G[n+\alpha_1, \beta_1 - \sum_{i=1}^{n} \log(1-\exp(-x_i^2))], \tag{3.1}$$

$$[\beta_1 \mid \theta, \alpha_1, \underline{x}] \sim G(\alpha_1+\alpha_2, \theta+\beta_2) \tag{3.2}$$

and

$$[\alpha_1 \mid \theta, \beta_1, \underline{x}] \propto \exp(-c\alpha_1)\frac{1}{\Gamma(\alpha_1)}\beta_1^{\alpha_1}\theta^{\alpha_1}. \tag{3.3}$$

We place diffuse second-stage prior on β_1, that is, $\alpha_2 = 1 \times 10^{-5}$ and $\beta_2 = 1 \times 10^{-5}$.

To implement the Gibbs sampler, we consider $m=5$ independent sequences each with a sample of size $d=2000$, and with a burn-in sample of another 2000. To monitor the convergence of the Gibbs sampler for θ, the parameter of interest, we follow the method by Gelman and Rubin(1992). Let $B/2000$ be the variance between the 5 sequence means, $\overline{\theta}_{k.}$, each based on 2000 values. That is, $B/2,000 = \sum_{k=1}^{5}(\overline{\theta}_{k.} - \overline{\theta}_{..})^2/(5-1)$, where $\overline{\theta}_{..} = \sum_{k=1}^{5} \overline{\theta}_{k.}/5$ and . Also, let W denoted the average of the 5 within-sequence variance, S_k^2 each based on (2000-1) df; that is $W = \sum_{k=1}^{5} S_k^2/5$. Then find

$$\hat{V} = \frac{2,000-1}{2,000}W + \frac{1}{2,000}B + \frac{1}{5 \times 2,000}B$$

Finally, find $\hat{R} = \hat{V}/W$. If $\sqrt{\hat{R}}$ is near 1 for the scalar estimands $\hat{\theta}$ of interest, then this suggests that the desired convergence is achieved in the Gibbs sampler.

An inspection of Table 3.1 reveals that the hierarchical Bayes procedure is not sensitive to the choice of c as different choices of c can lead to almost same point estimates of $\hat{\theta}$. So we use the $c=100$ and hence $\hat{\theta}=2.9532$. For the observations, the Figures 3.1, 3.2 and 3.3 are graphs of $[\hat{\theta} \mid \underline{x}]$, $[\widehat{R(t)} \mid \underline{x}]$, $[\widehat{h(t)} \mid \underline{x}]$ at a mission time $t=1$, respectively. From the Gibbs sampler, $R_1^{(t)}, \cdots, R_m^{(t)}$ is a sample from $[R(t) \mid \underline{x}]$ and $h_1^{(t)}, \cdots, h_m^{(t)}$ is a sample from $[h(t) \mid \underline{x}]$. The 90% credible intervals are $(R_{(0.05m)}^{(t)}, R_{(0.95m)}^{(t)})$ and $(h_{(0.05m)}^{(t)}, h_{(0.95m)}^{(t)})$. $(0.05m)$ and $(0.95m)$ are the $0.05m^{th}$ and $0.95m^{th}$ order statistics. For various mission time t, tables 3.2 and 3.3 give the true values, the posterior means, the residuals and 90% credible intervals for $R(t)$ and $h(t)$ on same simulated data.

[Table 3.1] Posterior Mean and Standard Deviation of θ

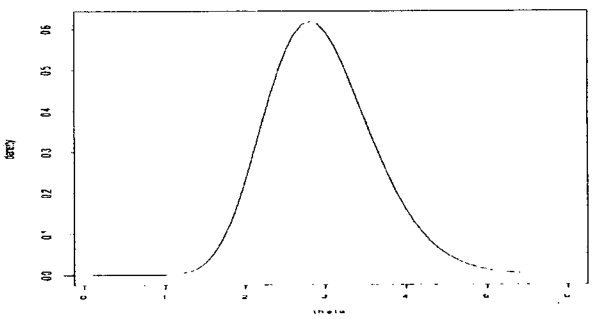
c	0.01	0.1	1	10	100
$\hat{\theta}$	2.9532	2.9526	2.9528	2.9534	2.9532
$\widehat{SD(\theta)}$	0.6578	0.6585	0.6596	0.6600	0.6614

[Table 3.2] Posterior Mean and 90% Credible Interval of $R(t)$

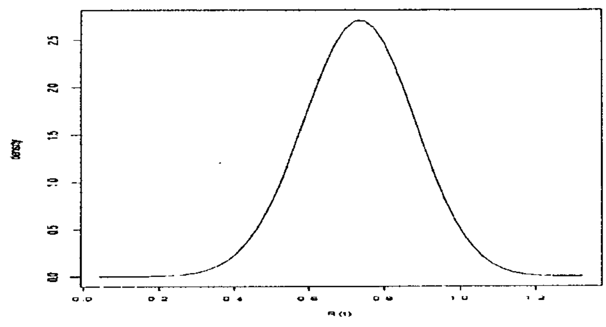
t	$R(t)$	$\widehat{R(t)}$	$R(t)-\widehat{R(t)}$	90% C.I.
0.8	0.8944	0.8746	0.0161	(0.7678,0.9541)
1.0	0.7474	0.7316	0.0159	(0.5909,0.8484)
1.2	0.5557	0.5442	0.0115	(0.4096,0.6711)
1.4	0.3658	0.3591	0.0067	(0.2561,0.4644)
1.6	0.2144	0.2110	0.0034	(0.1451,0.2817)
1.8	0.1130	0.1113	0.0016	(0.0763,0.1531)

[Table 3.3] Posterior Mean and 90% Credible Interval of $h(t)$

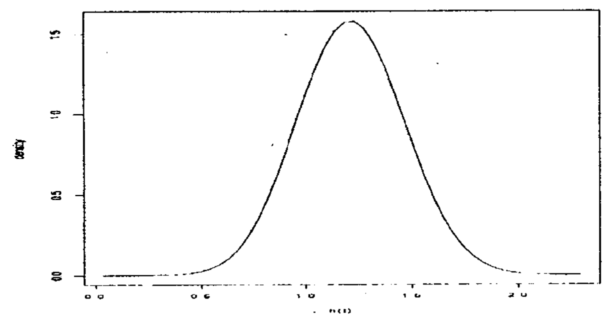
t	$h(t)$	$\widehat{h(t)}$	$h(t)-\widehat{h(t)}$	90% C.I.
0.8	0.6324	0.6737	- 0.0413	(0.3531,1.0518)
1.0	1.1800	1.2082	- 0.0282	(0.8556,1.5703)
1.2	1.7875	1.8045	- 0.0170	(1.5019,2.0932)
1.4	2.3872	2.3969	- 0.0097	(2.1774,2.5984)
1.6	2.9463	2.9516	- 0.0053	(2.8113,3.0778)
1.8	3.4572	3.4600	- 0.0028	(3.3794,3.5318)



[Figure 3.1] Estimated pdf of $[\theta \mid x]$



[Figure 3.2] Estimated pdf of $[R(t) \mid \underline{x}]$



[Figure 3.3] Estimated pdf of $[h(t) \mid \underline{x}]$

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