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Effects of Changing Weighing Factor in a Two Stage Shrinkage Testimator for the Mean of an Exponential Distribution[1)]

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Abstract

Two stage shrinkage testimator is a kind of adaptive estimators based on a test on an initial estimate of parameter. Since weighing factor plays an important roll in assessing the properties of testimator, its choice is extremely crucial in two stage testimation. Adke, Waikar and Schuurmann(1987) proposed a testimator for the mean of an exponential distribution defined with their own weighing factor. Two alternative testimators obtained using changed weighing factors are presented, and their Mean squared error(MSE) formulae are provided in this paper. Their properties are compared with those of existing one by means of MSE.

1. Introduction

Two stage shrinkage testimation is an estimation procedure that incorporates the results of a preliminary test on an available initial estimate of parameter (Hogg(1974), Katti(1962), and Waikar and Katti(1971)). Waikar, Schuurmann and Raghunathan(1984) developed a testimation procedure for the mean of a normal distribution. Later, Adke, Waikar and Schuurmann(1987) extended their results to the testimation of the mean of an exponential distribution. In both cases mentioned above, a weighing factor is used in defining testimator and it plays an important role in determining the properties of testimator. Hence, the choice of weighing factor is an important problem in two stage testimation. This paper concentrates on that problem in the testimation of the mean of an exponential distribution. Two changed weighing factors used in defining testimators are introduced and their MSE's are compared with that of Adke, Waikar and Schuurmann.

This paper is composed of five sections including present one. Two stage shrinkage testimator for the mean of an exponential distribution, proposed by Adke, Waikar and Schuurmann, is briefly reviewed in section 2. In section 3, two changed weighing factors are introduced, and two testimators obtained using changed weighing factors are defined. Section 4 is devoted to the comparison of two alternative testimators with Adke, Waikar and

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Schuurmann's. MSE formulae of two alternative testimators are presented in section 4, and tables containing the relative efficiencies (to usual single sample mean) of three testimators (Adke, Waikar and Schuurmann's, & two alternative ones) are also contained in that section for comparison purpose. Final section is devoted to some concluding remarks.

2. Two stage testimator for the mean of an exponential distribution

This section is partly reproduced from Moon(1998, Section 2) to introduce two stage testimation procedure and some notation which is necessary in this paper. Let X be a random variable following an exponential distribution with mean θ. Adke, Waikar and Schuurmann proposed a two stage shrinkage testimator of θ based on an initial estimate θ_0 of θ, which is defined as follows:

Step 1: Obtain n_1 first stage samples(X_{1i}, $i = 1, 2, \cdots, n_1$), and test

$$H_0:\ \theta = \theta_0 \text{ vs. } H_1:\ \theta \neq \theta_0$$

at significance level α using the first stage sample mean $\overline{X}_1$.

Step 2: If H_0 is accepted, testimator is defined as

$$\hat{\theta} = w\,\overline{X}_1 + (1-w)\,\theta_0 \qquad 0 < w < 1.$$

If H_0 is rejected, obtain n_2 second stage samples(X_{2i}, $i = 1, 2, \cdots, n_2$) and define testimator as the combined sample mean, that is

$$\hat{\theta} = (n_1\,\overline{X}_1 + n_2\,\overline{X}_2)/(n_1 + n_2).$$

UMPU test for testing above hypothesis is given by:

$$\text{Reject } H_0 \text{ if } Z_1 = \sum_{i=1}^{n_1} X_{1i} < k_1 \quad \text{or if} \quad Z_1 > k_2,$$

where k_1 and k_2 are chosen to satisfy

$$\begin{aligned} 1 - \alpha &= \gamma(n_1, k_2/\theta_0) - \gamma(n_1, k_1/\theta_0) \\ &= \gamma(n_1 + 1, k_2/\theta_0) - \gamma(n_1 + 1, k_1/\theta_0) \end{aligned} \tag{2.1}$$

and $\gamma(a, x)$ represents a usual incomplete gamma function. They defined the weighing factor w as $w = |Z_1 - n_1\theta_0| / (k_2 - k_1)$. They also derived MSE of testimator and defined its relative efficiency to usual single sample mean(See Adke, Waikar and Schuurmann, p.1824-1828, for detailed formula).

3. Testimators with changed weighing factors

As was mentioned in previous sections, the weighing factor plays an important role in testimator. Adke, Waikar and Schuurmann defined weighing factor of their testimator as $w = |Z_1 - n_1\theta_0| / (k_2 - k_1)$ so that a higher weight is given to θ_0 when $\overline{X}_1$ is closer to θ_0. For notational convenience, let's denote their testimator and weighing factor as $\hat{\theta}_1$ and w_1, respectively. In this section, two alternative testimators (denoted as $\hat{\theta}_2$ and $\hat{\theta}_3$) defined with changed weighing factors (denoted as w_2 and w_3) are introduced.

An weighing factor w_1 was chosen to give higher weight to $\overline{X}_1$ when the information contained in first stage sample does not support an initial estimate, and vice versa. However, higher weight is not high enough when test statistic Z_1 is within acceptance region but takes value close to critical values(say, k_1 and k_2). That is, w_1 takes value much less than 1 in this case. If w_1 takes value closer to 1 in this case, then it is expected that the better testimator is obtained since much higher weight is given to $\overline{X}_1$ when initial estimate is suspected. According to this principle, two alternative weighing factors and testimators are introduced as follows:

i) $$w_2 = \begin{cases} (n_1\theta_0 - Z_1)/(n_1\theta_0 - k_1), & \text{if } k_1 < Z_1 < n_1\theta_0, \\ (Z_1 - n_1\theta_0)/(k_2 - n_1\theta_0), & \text{if } n_1\theta_0 < Z_1 < k_2, \end{cases} \quad \text{and}$$

$$\hat{\theta}_2 = w_2\overline{X}_1 + (1 - w_2)\theta_0, \quad \text{if } H_0 \text{ is accepted.}$$

Weighing factor w_2 is linear functions of Z_1 as w_1, and slopes of w_2 are steeper than those of w_1. Weighing factor w_2 is chosen since it gives weight to $\overline{X}_1$ which is close to maximum value 1 when Z_1 is within acceptance region but close to k_1 or k_2. Hence, in this case, w_2 gives much higher weight to $\overline{X}_1$ than w_1 does which is a desirable property. On the other hand, compared with w_1, w_2 also gives higher weight to $\overline{X}_1$ and relatively lower one to θ_0 when Z_1 is close to $n_1\theta_0$(that is, when initial estimate θ_0 contains valuable information), which is a undesirable property.

ii) $$w_3 = \begin{cases} (Z_1 - n_1\theta_0)^2/(n_1\theta_0 - k_1)^2, & \text{if } k_1 < Z_1 < n_1\theta_0, \\ (Z_1 - n_1\theta_0)^2/(k_2 - n_1\theta_0)^2, & \text{if } n_1\theta_0 < Z_1 < k_2, \end{cases} \quad \text{and}$$

$$\hat{\theta}_3 = w_3 \overline{X}_1 + (1 - w_3)\theta_0, \quad \text{if } H_0 \text{ is accepted.}$$

Weighing factor w_3 is quadratic functions of Z_1. It is tried for the same reason as w_2. That is, it gives much higher weight (upto maximum 1) to $\overline{X}_1$ than w_1 does when H_0 is accepted but an initial estimate is somewhat suspected. Furthermore, w_3 gives lower weight (than w_1) to $\overline{X}_1$ and higher one (than w_1) to θ_0 when Z_1 takes value close to $n_1\theta_0$, since it is quadratic functions. The latter one is a desirable property that w_2 does not have.

4. Comparison of testimators

Two testimators $\hat{\theta}_2$ and $\hat{\theta}_3$ defined with changed weighing factors w_2 and w_3 are introduced in previous section. In this section, their properties are compared with those of $\hat{\theta}_1$ by means of MSE.

MSE formulae of $\hat{\theta}_2$ and $\hat{\theta}_3$ are derived through tedious but straightforward calculation, and are given in the below:

i) $\text{MSE}(\hat{\theta}_2 \mid \theta) = \text{E}(\hat{\theta}_2^{\,2}) - 2\theta \cdot \text{E}(\hat{\theta}_2) + \theta^2$.

The expressions for $\text{E}(\hat{\theta}_2)$ and $\text{E}(\hat{\theta}_2^{\,2})$ are,

$$E(\hat{\theta}_2) = b_{20}(\theta) + \theta \cdot b_{21}(\theta) + \theta^2 \cdot b_{22}(\theta), \text{ where}$$

$$\begin{aligned}
b_{20}(\theta) &= \theta_0 \cdot \gamma(n_1, k_1/\theta, k_2/\theta) + [n_1\theta_0^2/(k_2 - n_1\theta_0)] \cdot \gamma(n_1, n_1\theta_0/\theta, k_2/\theta) \\
&\quad - [n_1\theta_0^2/(n_1\theta_0 - k_1)] \cdot \gamma(n_1, k_1/\theta, n_1\theta_0/\theta), \\
b_{21}(\theta) &= 1 - [n_2/(n_1 + n_2)] \cdot \gamma(n_1, k_1/\theta, k_2/\theta) - [n_1/(n_1 + n_2)]\,\gamma(n_1 + 1, k_1/\theta, k_2/\theta) \\
&\quad - [2n_1\theta_0/(k_2 - n_1\theta_0)] \cdot \gamma(n_1 + 1, n_1\theta_0/\theta, k_2/\theta) \\
&\quad + [2n_1\theta_0/(n_1\theta_0 - k_1)] \cdot \gamma(n_1 + 1, k_1/\theta, n_1\theta_0/\theta), \\
b_{22}(\theta) &= [(n_1 + 1)/(k_2 - n_1\theta_0)] \cdot \gamma(n_1 + 2, n_1\theta_0/\theta, k_2/\theta) \\
&\quad - [(n_1 + 1)/(n_1\theta_0 - k_1)] \cdot \gamma(n_1 + 2, k_1/\theta, n_1\theta_0/\theta),
\end{aligned}$$

where $\gamma(n, a, b) = \gamma(n, b) - \gamma(n, a)$. And,

$$E(\hat{\theta}_2^{\,2}) = \theta^2 + [\theta^2/(n_1 + n_2)] + \sum_{j=0}^{2} c_j^* \cdot \gamma(n_1 + j, k_1/\theta, k_2/\theta)$$

$$+ \sum_{j=0}^{4} d_{2j}^{*} \cdot \gamma(n_1 + j,\ k_1/\theta,\ n_1\theta_0/\theta) + \sum_{j=0}^{4} e_{2j}^{*} \cdot \gamma(n_1 + j,\ n_1\theta_0/\theta,\ k_2/\theta),$$

where

$$c_j^{*} = n_1(n_1+1)\cdots(n_1+j-1)\theta^j c_j, \quad j = 0, 1, 2,$$

$$d_{2j}^{*} = n_1(n_1+1)\cdots(n_1+j-1)\theta^j d_{2j}, \quad j = 0, 1, \cdots, 4, \quad \text{and}$$

$$e_{2j}^{*} = n_1(n_1+1)\cdots(n_1+j-1)\theta^j e_{2j}, \quad j = 0, 1, \cdots, 4.$$

Further,

$$c_0 = [(n_1+n_2)^2\theta_0^2 - n_2\theta^2(n_2+1)]/(n_1+n_2)^2, \quad c_1 = -2n_2\theta/(n_1+n_2)^2,$$

$$c_2 = -1/(n_1+n_2)^2,$$

$$d_{20} = [n_1^4\theta_0^4 - 2n_1^3\theta_0^3(n_1\theta_0 - k_1)]/n_1^2(n_1\theta_0 - k_1)^2,$$

$$d_{21} = [-4n_1^3\theta_0^3 + 4n_1^2\theta_0^2(n_1\theta_0 - k_1)]/n_1^2(n_1\theta_0 - k_1)^2,$$

$$d_{22} = [6n_1^2\theta_0^2 - 2n_1\theta_0(n_1\theta_0 - k_1)]/n_1^2(n_1\theta_0 - k_1)^2,$$

$$d_{23} = -4n_1\theta_0/n_1^2(n_1\theta_0 - k_1)^2, \quad d_{24} = 1/n_1^2(n_1\theta_0 - k_1)^2,$$

$$e_{20} = [n_1^4\theta_0^4 + 2n_1^3\theta_0^3(k_2 - n_1\theta_0)]/n_1^2(k_2 - n_1\theta_0)^2,$$

$$e_{21} = [-4n_1^3\theta_0^3 - 4n_1^2\theta_0^2(k_2 - n_1\theta_0)]/n_1^2(k_2 - n_1\theta_0)^2,$$

$$e_{22} = [6n_1^2\theta_0^2 + 2n_1\theta_0(k_2 - n_1\theta_0)]/n_1^2(k_2 - n_1\theta_0)^2,$$

$$e_{23} = -4n_1\theta_0/n_1^2(k_2 - n_1\theta_0)^2, \text{ and } \quad e_{24} = 1/n_1^2(k_2 - n_1\theta_0)^2.$$

ii) $\text{MSE}(\widehat{\theta}_3 \mid \theta) = \text{E}(\widehat{\theta}_3^{\,2}) - 2\theta \cdot \text{E}(\widehat{\theta}_3) + \theta^2.$

The expressions for $\text{E}(\widehat{\theta}_3)$ and $\text{E}(\widehat{\theta}_3^{\,2})$ are,

$$E(\widehat{\theta}_3) = b_{30}(\theta) + \theta\, b_{31}(\theta) + \theta^2\, b_{32}(\theta) + \theta^3\, b_{33}(\theta), \text{ where}$$

$$b_{30}(\theta) = \theta_0 \cdot \gamma(n_1, k_1/\theta, k_2/\theta) - [n_1^2\theta_0^3/(n_1\theta_0 - k_1)^2] \cdot \gamma(n_1, k_1/\theta, n_1\theta_0/\theta)$$
$$- [n_1^2\theta_0^3/(k_2 - n_1\theta_0)^2] \cdot \gamma(n_1, n_1\theta_0/\theta, k_2/\theta),$$

$$b_{31}(\theta) = 1 - [n_2/(n_1+n_2)] \cdot \gamma(n_1, k_1/\theta, k_2/\theta) - [n_1/(n_1+n_2)]\,\gamma(n_1+1, k_1/\theta, k_2/\theta)$$
$$+ [3n_1^2\theta_0^2/(n_1\theta_0 - k_1)^2] \cdot \gamma(n_1+1, k_1/\theta, n_1\theta_0/\theta)$$
$$+ [3n_1^2\theta_0^2/(k_2 - n_1\theta_0)^2] \cdot \gamma(n_1+1, n_1\theta_0/\theta, k_2/\theta),$$

$$b_{32}(\theta) = -[3\theta_0 n_1(n_1+1)/(n_1\theta_0 - k_1)^2] \cdot \gamma(n_1+2, k_1/\theta, n_1\theta_0/\theta)$$

$$- [3\,\theta_0 n_1(n_1+1)/(k_2-n_1\theta_0)^2]\cdot\gamma(n_1+2,\,n_1\theta_0/\theta,\,k_2/\theta),$$

$$b_{33}(\theta) = [\,(n_1+1)(n_1+2)/(n_1\theta_0-k_1)^2]\cdot\gamma(n_1+3,\,k_1/\theta,\,n_1\theta_0/\theta)$$

$$+ [\,(n_1+1)(n_1+2)/(k_2-n_1\theta_0)^2]\cdot\gamma(n_1+3,\,n_1\theta_0/\theta,\,k_2/\theta),$$

and

$$E(\,\widehat{\theta}_3^{\,2}) = \theta^2 + [\theta^2/(n_1+n_2)] + \sum_{j=0}^{2} c_j^* \cdot \gamma(n_1+j,\,k_1/\theta,\,k_2/\theta)$$

$$+ \sum_{j=0}^{6} d_{3j}^* \cdot \gamma(n_1+j,\,k_1/\theta,\,n_1\theta_0/\theta) + \sum_{j=0}^{6} e_{3j}^* \cdot \gamma(n_1+j,\,n_1\theta_0/\theta,\,k_2/\theta),$$

where

$$d_{3j}^* = n_1(n_1+1)\cdots(n_1+j-1)\,\theta^j d_{3j}, \quad j=0,1,\cdots,6, \quad \text{and}$$

$$e_{3j}^* = n_1(n_1+1)\cdots(n_1+j-1)\,\theta^j e_{3j}, \quad j=0,1,\cdots,6.$$

Further,

$d_{30} = [\,n_1^6\theta_0^6 - 2\,n_1^4\theta_0^4(n_1\theta_0-k_1)^2]\,/\,n_1^2(n_1\theta_0-k_1)^4,$

$d_{31} = [-6\,n_1^5\theta_0^5 + 6\,n_1^3\theta_0^3(n_1\theta_0-k_1)^2]\,/\,n_1^2(n_1\theta_0-k_1)^4,$

$d_{32} = [15\,n_1^4\theta_0^4 - 6\,n_1^2\theta_0^2(n_1\theta_0-k_1)^2]\,/\,n_1^2(n_1\theta_0-k_1)^4,$

$d_{33} = [-20\,n_1^3\theta_0^3 + 2\,n_1\theta_0(n_1\theta_0-k_1)^2]\,/\,n_1^2(n_1\theta_0-k_1)^4,$

$d_{34} = 15\,n_1^2\theta_0^2\,/\,n_1^2(n_1\theta_0-k_1)^4, \qquad d_{35} = -6\,n_1\theta_0\,/\,n_1^2(n_1\theta_0-k_1)^4,$

$d_{36} = 1\,/\,n_1^2(n_1\theta_0-k_1)^4,$

$e_{30} = [\,n_1^6\theta_0^6 - 2\,n_1^4\theta_0^4(k_2-n_1\theta_0)^2]\,/\,n_1^2(k_2-n_1\theta_0)^4,$

$e_{31} = [-6\,n_1^5\theta_0^5 + 6\,n_1^3\theta_0^3(k_2-n_1\theta_0)^2]\,/\,n_1^2(k_2-n_1\theta_0)^4,$

$e_{32} = [15\,n_1^4\theta_0^4 - 6\,n_1^2\theta_0^2(k_2-n_1\theta_0)^2]\,/\,n_1^2(k_2-n_1\theta_0)^4,$

$e_{33} = [-20\,n_1^3\theta_0^3 + 2\,n_1\theta_0(k_2-n_1\theta_0)^2]\,/\,n_1^2(k_2-n_1\theta_0)^4, \qquad e_{34} = 15\,n_1^2\theta_0^2\,/\,n_1^2(k_2-n_1\theta_0)^4,$

$e_{35} = -6\,n_1\theta_0\,/\,n_1^2(k_2-n_1\theta_0)^4$, and $e_{36} = 1\,/\,n_1^2(k_2-n_1\theta_0)^4.$

To compare the properties of three testimators with different weighing factors, their MSE are calculated for various values of n_1(10, 20, 30 and 50) and n_2(10 ~60(10)). Three true mean values ($\theta = 1.00,\ 0.90,\ 0.80$) are used with fixed initial estimate $\theta_0 = 1$. Two significance levels (0.05, 0.10) are included. The values of k_1 and k_2 are obtained from (2.1).

Table 1 contains relative efficiency (to usual single sample mean) calculation results. From these tables, the following phenomena are found for each value of θ.

(1) When θ_0 is very close to θ (that is, when $\theta \approx \theta_0 = 1.0$), it follows that

$$eff_{\bar{X},\, \widehat{\theta_1}}(\theta) \ > \ eff_{\bar{X},\, \widehat{\theta_3}}(\theta) \ > \ eff_{\bar{X},\, \widehat{\theta_2}}(\theta)$$

for all cases. That is, $\widehat{\theta}_1$ is the best and $\widehat{\theta}_3$ is the second, regardless of n_1, n_2 and α values.

(2) When θ_0 is somewhat close to θ (that is, when $\theta \approx 0.90$), it also follows that

$$eff_{\bar{X},\, \widehat{\theta_1}}(\theta) \ > \ eff_{\bar{X},\, \widehat{\theta_3}}(\theta) \ > \ eff_{\bar{X},\, \widehat{\theta_2}}(\theta)$$

for all cases. But, the differences among efficiencies are much less than those of (1).

(3) When θ_0 is relatively far from θ (that is, when $\theta \approx 0.80$) and $n_1 = 10$, it also follows that

$$eff_{\bar{X},\, \widehat{\theta_1}}(\theta) \ > \ eff_{\bar{X},\, \widehat{\theta_3}}(\theta) \ > \ eff_{\bar{X},\, \widehat{\theta_2}}(\theta).$$

But for other larger values of n_1 $(=20, 30, 50)$, we have the following different result.

$$eff_{\bar{X},\, \widehat{\theta_2}}(\theta) \ > \ eff_{\bar{X},\, \widehat{\theta_3}}(\theta) \ > \ eff_{\bar{X},\, \widehat{\theta_1}}(\theta).$$

(4) The differences among efficiencies when $\alpha = 0.10$ are reduced a little bit when compared with those of $\alpha = 0.05$ in all cases included in Table 1.

Table 1 contains the relative efficiencies only for $\theta_0 = 1$. Finally, one theorem that extends the applicability of above results (1) ~ (4) for various combinations of θ and θ_0 is provided.

Theorem. $Eff_{\bar{X},\, \widehat{\theta_i}}(\theta)$, $i = 1, 2, 3$, are invariant to the ratio θ / θ_0.

Proof. It is proved in Moon(1998) for $\widehat{\theta}_1$. Proofs for $\widehat{\theta}_2$ and $\widehat{\theta}_3$ can be made similarly.

5. Concluding Remarks

Two testimators with changed weighing factors are introduced with the hope that they are superior to existing Adke, Waikar and Schuurmann's testimator in MSE sense. Their MSE formulae are derived. And their MSE values are calculated for various values of θ, n_1, n_2 and α with fixed $\theta_0 = 1$. From the comparison of efficiencies (relative to usual single sample mean) of three testimators, the following conclusions are made. Note that with fixed

n_1, n_2 and α, conclusions hold for various combinations of θ and θ_0 values if θ / θ_0 is the same since the relative efficiencies are invariant to θ / θ_0.

(1) When we have a valuable initial estimate (that is, when θ_0 is relatively close to θ), the behavior of $\widehat{\theta}_1$ is the best and that of $\widehat{\theta}_3$ is the second in all cases included in Table 1. On the other hand, if we have an initial information that is relatively far from θ and relatively larger $n_1 (= 20, 30, 50)$, then the behavior of $\widehat{\theta}_2$ is the best and that of $\widehat{\theta}_3$ is the second. But for a relatively smaller $n_1 (= 10)$ in this case, the result is the same as that of a valuable initial estimate case.

(2) As the significance level α gets larger, the differences among three efficiencies decrease.

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Table 1. Efficiency of testimators w.r.t. $\overline{X}$ when $\theta_0 = 1$.

$n_1 = 10, \quad \alpha = 0.05, \quad k_1 = 4.97890, \quad k_2 = 17.61340.$

$\hat{\theta}$	$\hat{\theta}_1$			$\hat{\theta}_2$			$\hat{\theta}_3$		
n_2 \ θ	1.00	0.90	0.80	1.00	0.90	0.80	1.00	0.90	0.80
10	5.58120	3.97365	1.75482	2.28459	2.09691	1.55984	3.29221	2.81277	1.65386
20	6.93123	4.40497	1.73743	2.40892	2.14881	1.53054	3.63932	2.96812	1.62986
30	7.49062	4.48089	1.65816	2.40002	2.10682	1.45465	3.70459	2.94885	1.55212
40	7.67743	4.42528	1.56767	2.34895	2.03843	1.37204	3.66817	2.87382	1.46565
50	7.68252	4.31703	1.47936	2.28336	1.96326	1.29285	3.59113	2.78022	1.38199
60	7.59445	4.18817	1.39714	2.21350	1.88817	1.21975	3.49764	2.68186	1.30452

$n_1 = 10, \quad \alpha = 0.10, \quad k_1 = 5.62920, \quad k_2 = 16.19890.$

$\hat{\theta}$	$\hat{\theta}_1$			$\hat{\theta}_2$			$\hat{\theta}_3$		
n_2 \ θ	1.00	0.90	0.80	1.00	0.90	0.80	1.00	0.90	0.80
10	4.36230	3.22885	1.56084	2.17364	1.95639	1.44522	2.85548	2.44627	1.50993
20	5.73503	3.67568	1.51590	2.34644	2.02121	1.39148	3.26447	2.62057	1.46090
30	6.37156	3.74433	1.41126	2.32701	1.95687	1.28983	3.33448	2.58365	1.35745
40	6.60106	3.66597	1.30238	2.24585	1.85879	1.18729	3.27401	2.47996	1.25128
50	6.61431	3.53131	1.20227	2.14619	1.75549	1.09418	3.16333	2.35803	1.15425
60	6.51405	3.37812	1.11331	2.04393	1.65635	1.01202	3.03540	2.23525	1.06825

$n_1 = 20, \quad \alpha = 0.05, \quad k_1 = 12.43910, \quad k_2 = 30.136710.$

$\hat{\theta}$	$\hat{\theta}_1$			$\hat{\theta}_2$			$\hat{\theta}_3$		
n_2 \ θ	1.00	0.90	0.80	1.00	0.90	0.80	1.00	0.90	0.80
10	4.42137	2.69535	1.03681	2.08104	1.79785	1.16621	2.94496	2.18563	1.08486
20	5.66076	3.01932	1.03772	2.28721	1.91194	1.17815	3.41508	2.37628	1.08956
30	6.51292	3.17131	1.01049	2.37917	1.94633	1.15354	3.67024	2.45050	1.06314
40	7.07846	3.23121	0.97343	2.41411	1.94342	1.11519	3.80334	2.46725	1.02547
50	7.44816	3.24015	0.93370	2.41868	1.92185	1.07228	3.86490	2.45380	0.98446
60	7.67877	3.21956	0.89426	2.40619	1.89065	1.02882	3.88441	2.42383	0.94352

$n_1 = 20, \quad \alpha = 0.10, \quad k_1 = 13.49340, \quad k_2 = 28.32260.$

$\hat{\theta}$	$\hat{\theta}_1$			$\hat{\theta}_2$			$\hat{\theta}_3$		
n_2 \ θ	1.00	0.90	0.80	1.00	0.90	0.80	1.00	0.90	0.80
10	3.29542	2.22126	0.98193	1.90670	1.65080	1.10661	2.49020	1.90759	1.02180
20	4.39450	2.53744	0.96297	2.17830	1.78714	1.09804	3.02722	2.11501	1.00590
30	5.22068	2.68568	0.91964	2.30636	1.82574	1.05633	3.34452	2.19414	0.96282
40	5.81062	2.73832	0.87008	2.35472	1.81648	1.00437	3.51814	2.20570	0.91237
50	6.21430	2.73596	0.82089	2.35842	1.78342	0.95098	3.60069	2.18167	0.86175
60	6.47978	2.70271	0.77450	2.33671	1.73869	0.89963	3.62563	2.13831	0.81372

Table 1. (Continued)

$n_1 = 30$, $\alpha = 0.05$, $k_1 = 20.482370$, $k_2 = 42.089230$.

$\hat{\theta}$	$\hat{\theta}_1$			$\hat{\theta}_2$			$\hat{\theta}_3$		
n_2 \ θ	1.00	0.90	0.80	1.00	0.90	0.80	1.00	0.90	0.80
10	3.91843	2.09719	0.77115	1.96827	1.60360	0.96727	2.63980	1.79571	0.84800
20	4.90054	2.30635	0.76587	2.16902	1.71103	0.97556	3.04239	1.93704	0.84868
30	5.68743	2.42675	0.74741	2.28859	1.76353	0.96195	3.30292	2.00925	0.83489
40	6.30122	2.49199	0.72346	2.35858	1.78441	0.93796	3.48466	2.05436	0.80801
50	6.77060	2.52207	0.69761	2.39753	1.78654	0.90936	3.61639	2.06028	0.78131
60	7.12868	2.52943	0.67154	2.41642	1.77714	0.87908	3.66571	2.04666	0.75799

$n_1 = 30$, $\alpha = 0.10$, $k_1 = 21.848860$, $k_2 = 39.962770$.

$\hat{\theta}$	$\hat{\theta}_1$			$\hat{\theta}_2$			$\hat{\theta}_3$		
n_2 \ θ	1.00	0.90	0.80	1.00	0.90	0.80	1.00	0.90	0.80
10	2.87105	1.77519	0.76759	1.76517	1.47143	0.94135	2.16127	1.59494	0.85581
20	3.69572	1.97405	0.74563	2.01663	1.59092	0.92854	2.58625	1.73306	0.83429
30	4.40088	2.08766	0.71419	2.17614	1.64832	0.89912	2.85573	1.82246	0.80171
40	4.98469	2.14591	0.68015	2.27281	1.66842	0.86319	3.04778	1.86207	0.76394
50	5.45400	2.16801	0.64645	2.32710	1.66582	0.82550	3.17554	1.86955	0.72573
60	5.82847	2.16681	0.61434	2.35270	1.64935	0.78835	3.30310	1.84578	0.69823

$n_1 = 50$, $\alpha = 0.05$, $k_1 = 37.37230$, $k_2 = 65.19630$.

$\hat{\theta}$	$\hat{\theta}_1$			$\hat{\theta}_2$			$\hat{\theta}_3$		
n_2 \ θ	1.00	0.90	0.80	1.00	0.90	0.80	1.00	0.90	0.80
10	3.47508	1.51087	0.56470	1.85906	1.35148	0.79043	2.45108	1.39652	0.66354
20	4.13709	1.60937	0.55172	2.01800	1.42784	0.78631	2.74128	1.47999	0.65426
30	4.73204	1.67586	0.53566	2.13431	1.47938	0.76877	2.98256	1.53629	0.63529
40	5.25516	1.71920	0.51832	2.21561	1.50024	0.75770	3.13803	1.56319	0.62317
50	5.70766	1.74566	0.50068	2.28370	1.52178	0.73491	3.29372	1.58506	0.60259
60	6.10021	1.76025	0.48330	2.32526	1.53354	0.70807	3.38022	1.60140	0.58111

$n_1 = 50$, $\alpha = 0.10$, $k_1 = 39.23560$, $k_2 = 62.57060$.

$\hat{\theta}$	$\hat{\theta}_1$			$\hat{\theta}_2$			$\hat{\theta}_3$		
n_2 \ θ	1.00	0.90	0.80	1.00	0.90	0.80	1.00	0.90	0.80
10	2.52245	1.34513	0.60731	1.62666	1.25586	0.80847	1.94935	1.28190	0.70582
20	3.05854	1.43535	0.58123	1.81830	1.33654	0.78318	2.23770	1.36867	0.67879
30	3.55932	1.49438	0.55493	1.96454	1.38333	0.75331	2.47787	1.41593	0.64901
40	4.01834	1.53079	0.52952	2.09599	1.42063	0.72834	2.70644	1.45765	0.62558
50	4.43495	1.55104	0.50547	2.17530	1.43001	0.70091	2.85525	1.47020	0.60041
60	4.80761	1.55915	0.48288	2.24331	1.43018	0.67324	2.97499	1.45926	0.57140