⊠ 연구논문

Time Series Modeling of Stochastic Failure Rates
- 추계적 고장률의 시계열 모델링 -

Sungwoon Choi*
최 성 운

Abstract

본 연구에서는 부품 및 시스템 고장률 모형에 대한 추계적 과정 접근법을 제시하고 기존의 이론 분포 중심 접근법에서 탈피하여 부품고장률을 시계열 모형으로 설정하고 이에 따른 복합 시스템 고장율의 선형결합에 대한 모델을 제시하며 주요 모델에 대한 수치예를 든다.

또한 Burn-In 테스트에 사용되는 욕조(Bathtub) 고장률 모형에 대한 기존의 혼합분포 접근법의 대체 방법으로 비선형 시계열 모형을 제안한다.

1. Introduction

The last few years have witnessed much criticism about the failure of reliability theory to have a tangible impact on the problems of modern science and technology. The lack of meaningful assessments of the integrity of composite materials and the survivability of structural elements have been cited as examples. Such criticism is justified because much of the literature in reliability tends to focus on old themes such as a characterization of classes of survival distributions or inference for parameters of failure models. Parametric families of distributions, such as the exponential and the Weibull, have been used as failure models for almost 30 years. Besides tradition and convenience, a typical reason for selecting these models has been a subjective assessment about the aging characteristics of an item and/or the model's goodness-of-fit to failure data. Often the failure data used is obtained from life-testing experiments conducted under controlled laboratory environments that are static. To many engineers and scientists, this black-box approach to model selection is unsatisfactory. A more appealing approach would be to choose a model based on the physics of failure and the characteristics of the operating environment[11].

It is well-known in the industrial community that the failure rates of newly manufactured items vary with time because of, for instance, the engineering design,

* Dept. of Industrial Engineering, KyungWon University

The present research has been conducted by the Research Grant of Kyungwon University in 1998.

manufacturing processes, maintenance & quality inspection procedures, and environmental & operational conditions. Such failure rates invariably exhibit random changes in level and slope and at times exhibit periodic patterns as well depending upon whether or not the maintenance is periodic. Therefore it would not be appropriate to model such stochastic processes using the common failure-distribution approach, especially since the fail to consider the periodicities[9].

It was found during that if the failure times of a system follow either Rayleigh, Weibull or exponential distributions and if the reliability-decay process of the system is represented by an AR(1) model with a specific time-dependent coefficient in each case, then the conditional mean of the AR(1) process, given that the initial value is equal to 1, is the same as the maximum likelihood estimate of the reliability function obtained by the classical method. Advantages of such a representation were discussed. Properties of time-dependent ARMA(1,1) processes were discussed and asymptotic results were obtained[8].

Singh [9] developed an unconventional but powerful approach to analyzing the observed and/or estimated failure rates of complex systems that operate in series and/or in parallel under varying operational & environmental conditions. Consequently such failure rates can be construed as time series which are complex in the sense that they are aggregates and/or products of two or more time series, Hence special time-series techniques are required for their analysis. Some of the results recently developed apply to the analysis of such time series. These results can also be used for the analysis of the reliability decay (growth) processes, actual failure times, times between failures, and interactions between failure times and maintenance times of complex systems.

The primary aim of the present study is to model stochastic failure rates of parts/systems that operate under varying operational & environmental conditions. Illustrative examples are discussed using simulated data. And secondary aim of this study is to model stochastic bathtub failure rates of burn-in.

Figure 1 shows the distribution of the number of failures. The plot of the number of failures in Figure 1 clearly shows that the failure times have a) a conspicuous downward trend, and b)random fluctuations. This fact is also reflected from the graph of the empirical failure rates in Figure 2 which, instead of being a constant, seems to form a stationary time series. Hence the analysis of such failure rates are better modeled by a time series approach rather than the distribution approach.

2. Stochastic Process Approach to Failure Models

Over the last few years, some literature devoted to the evolution of a relatively new class of failure models, both univariate and multivariate, for applications in reliability and biometry has begun to appear. A distinguishing feature of these new models is the fundamental theme that drives their development: they have been derived by considering stochastic processes that are presumed to describe the failure-generating mechanisms.

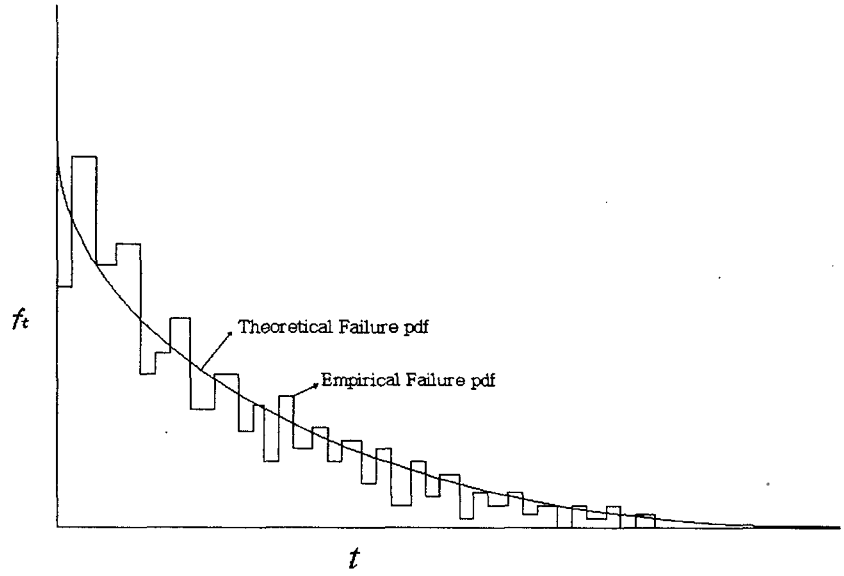

Figure 1. Theoretical and Empirical Failure pdf

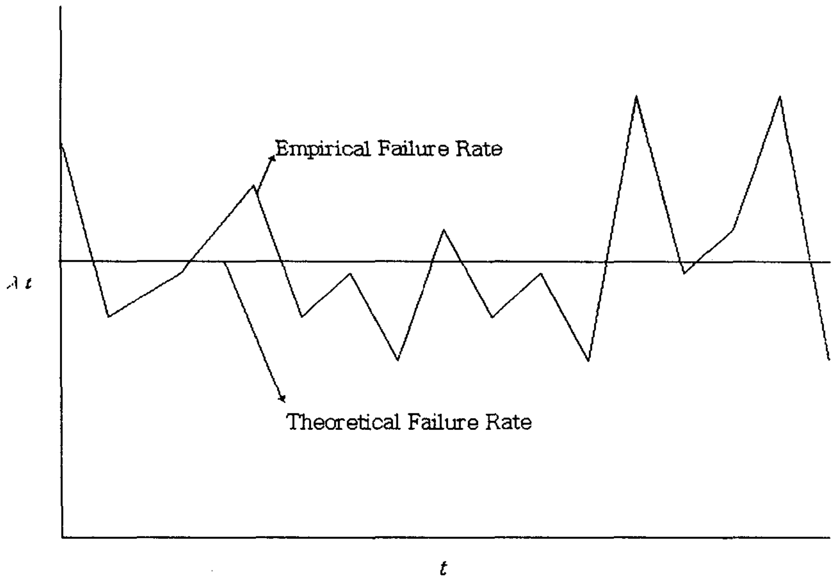

Figure 2. Theoretical and Empirical Failure Rate

Describing failure-generating mechanisms by stochastic processes is particularly germane when the environment under which the items operate is dynamic, that is, when the induced stresses(or covariates) vary over time. This is because dynamic environments induce internal stresses in an item that change the rates and the modes by which the item degrades to failure.

Because dynamic environments induce changes in the physics of failure, a stochastic-process approach to failure modelling provides flexibility with respect to describing the failure-generating mechanisms. This flexibility results in a better description of the failure data and an improved assessment of item survivability. This is perhaps the main reason why reliability engineers should be interested in the approach of this paper. Futhermore, the approach also raises the general level of the state of the art in reliability theory and survival analysis, especially in the ability to describe the survivability of multistate items. A disadvantage of the stochastic-process-based approach to failure modelling is that the resulting expressions for the survival function take unmanageable forms and can only be expressed via their Laplace transforms. However, with the rapid advances in the use of computer and simulation-based technologies in the statistical sciences, together with a widespread use of numerical techniques such as saddle-point approximations, the disadvantage of not having closed-form expressions will gradually disappear. Consequently, future research in reliability will also have to focus on computation and computability.

In developing an approach to failure modelling based on a stochastic process, it appears that four strategies have evolved; the two predominant ones are emphasized here. With the first strategy, the item state(or, equivalently, its wear) has been described by a diffusion process: typically a Wiener process, a gamma process or a deterministic diffusion. It can be shown that deterministic diffusions give rise to some of the well-known failure models that are in use today. Diffusion processes are stochastic processes with continuous sample paths. With the second strategy, it is the failure rate(also known as the hazard rate) of the item that is described by a stochastic process: typically a gamma process; a shot-noise process; functionals of a Wiener process; or, in general, a Lévy process. Lévy processes have stationary independent increments, and their marginal distributions are infinitely divisible. A more recent trend has been the development of failure models based on a consideration of two processes, one for the item state or wear and the other for a covariate that drives it. Covariate processes that drive the wear are referred to in the engineering literature as excitation processes. A study of models derived from excitation processes may signal a new philosophy of life-testing experiments wherein one must also monitor the conditions of the test. The third strategy for developing failure models focuses on describing the damage-causing environment by a stochastic process, typically a shock-inflicting Poisson process; the resulting failure models are known as shock models. The fourth and final strategy, and one for which there has been little development, is that in which a response variable that is strongly correlated with the lifelength, such as temperature, is described by a stochastic process, typically a stationary, continuous-time Gaussian process.

The flow chart depicted in Figure 3 shows stochastic process approach to failure models[11].

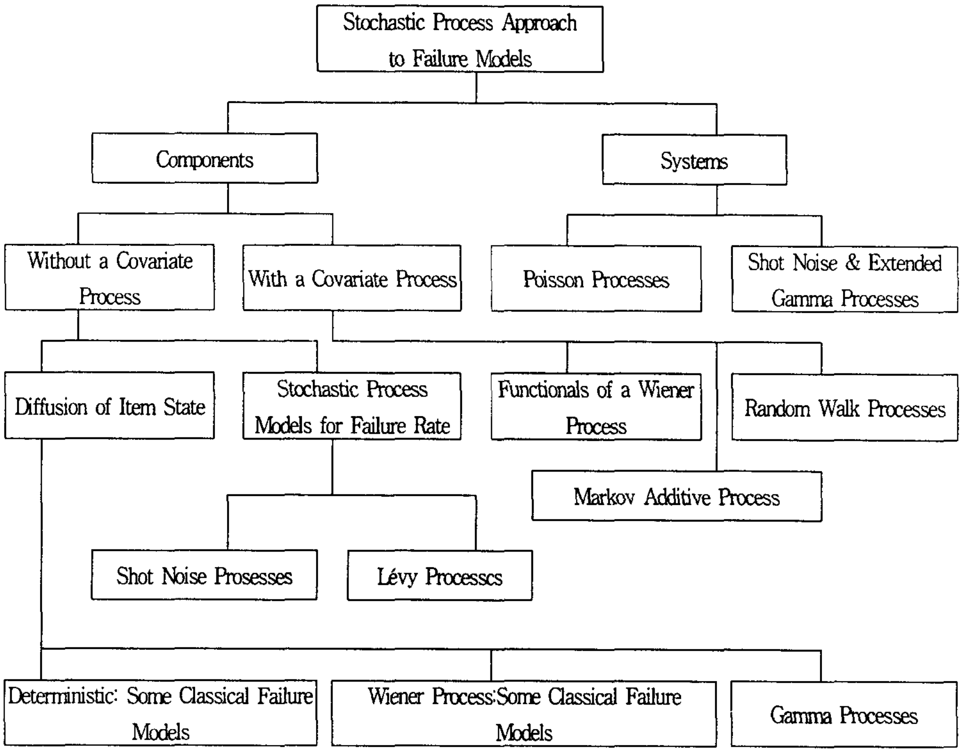

Figure 3. Stochastic Process Approach to Failure Models

3. Time Series Modeling

3.1 Stationary Time Series Models

3.1.1 Autoregressive Processes

It is given by

$$\dot{Z} = \phi_1 \dot{Z}_{t-1} + \cdots + \phi_p \dot{Z}_{t-p} + a_t$$

or

$$\phi_p(B)\dot{Z}_t = a_t ,$$

where $\phi_p(B) = (1 - \phi_1 B - \cdots - \phi_p B^p)$.

3.1.2 Moving Average Processes

It is given by

$$\dot{Z}_t = a_t - \theta_1 a_{t-1} - \cdots - \theta_q a_{t-q}$$

or

$$\dot{Z}_t = \theta_q(B)\, a_t$$

where

$$\theta_q(B) = (1 - \theta_1 B - \cdots - \theta_q B^q).$$

3.1.3 The General Mixed ARMA(p,q) Processes

In model building, it may be necessary to include both autoregressive and moving average terms in a model. This leads to the following useful mixed autoregressive moving average (ARMA) process:

$$\phi_p(B)\dot{Z}_t = \theta_q(B)a_t,$$

where

$$\phi_p(B) = 1 - \phi_1 B - \cdots - \phi_p B^p,$$

and

$$\theta_q(B) = 1 - \theta_1 B - \cdots - \theta_q B^q.$$

3.2 Autoregressive Integrated Moving Average(ARIMA) Models

We have

$$\phi_p(B)(1-B)^d Z_t = \theta_0 + \theta_q(B)a_t,$$

where the stationary AR operator $\phi_p(B) = (1 - \phi_1 B - \cdots - \phi_p B^p)$ and the invertible MA operator $\theta_q(B) = (1 - \theta_1 B - \cdots - \theta_q B^q)$ share no common factors.

3.3 Seasonal ARIMA Models

We get the following well-known Box-Jenkins multiplicative seasonal ARIMA model:

$$\Phi_p(B^s)\phi_p(B)(1-B)^d(1-B^s)^D\dot{Z}_t = \theta_q(B)\Theta_Q(B^s)a_t$$

where

$$\dot{Z}_t = \begin{cases} Z_t - \mu, & \text{if } d = D = 0, \\ Z_t, & \text{otherwise.} \end{cases}$$

For convenience, we often call $\phi_p(B)$ and $\theta_q(B)$ the regular autoregressive and moving average factors(polynomials) and $\Phi_p(B^s)$ and $\Theta_Q(B^s)$ the seasonal autoregressive and moving average factors(or polynomials), respectively. The model is often denoted as ARIMA $(p,d,q)\times(P,D,Q)_s$, where the subindex s refers to the seasonal period.

3.4 Intervention and Outlier Models

3.4.1 Intervention Model

For multiple intervention inputs, we have the following general class of models;

$$Z_t = \sum_{j=0}^{k} \frac{w_j(B)B^{b_j}}{\delta_j(B)} I_{jt} + \frac{\theta(B)}{\phi(B)} a_t$$

where $I_{jt}, j = 1, 2, \ldots, k$ are intervention variables.

3.4.2 Outlier Models

For a given stationary or properly deduced stationary process, let Z_t be the observed series and X_t be the outlier-free series. Assume that $\{X_t\}$ follows a general ARMA(p,q) model

$$\phi(B)X_t = \theta(B)a_t$$

where $\phi(B)=(1-\phi_1 B-\cdots-\phi_p B^p)$ and $\theta(B)=(1-\theta_1 B-\cdots-\theta_q B^q)$ are stationary and invertible operators sharing no common factors, and $\{a_t\}$ is a sequence of white noise, identically and independently distributed as $N(0,\sigma_a^2)$. An additive outlier (AO) is defined as

$$Z_t=\begin{cases}X_t, & t\neq T\\ X_t+\omega, & t=T\end{cases}$$

$$=X_t+\omega I_t^{(T)}$$

$$=\frac{\theta(B)}{\phi(B)}a_t+\omega I_t^{(T)}$$

where

$$I_t^{(T)}=\begin{cases}1, & t=T,\\ 0, & t\neq T,\end{cases}$$

is the indicator variable representing the presence or absence of an outlier at time T. An innovational outlier (IO) model is defined as

$$Z_t=X_t+\frac{\theta(B)}{\phi(B)}\omega I_t^{(T)}$$

$$=\frac{\theta(B)}{\phi(B)}(a_t+\omega I_t^{(T)}).$$

Hence, an additive outlier affects only the level of the Tth observation, whereas an innovational outlier affects all observations $Z_T, Z_{T+1},\cdots$, beyond time T through the memory of the system described by $\theta(B)/\phi(B)$.

More generally, a time series might contain several, say k, outliers of different types, and we have the following general outlier model:

$$Z_t=\sum_{j=0}^{k}\omega_j\nu_j(B)I_t^{(T_j)}+X_t$$

where $X_t=\frac{\theta(B)}{\phi(B)}a_t$, $\nu_j(B)=1$ for an AO and $\nu_j(B)=\theta(B)/\phi(B)$ for an IO at time $t=T_j$.

3.5 Spectral Models

The given sequence of n numbers, $\{Z_t\}$, can be written as a linear combination of the orthogonal trigonometric functions. That is,

$$Z_t=\sum_{k=0}^{[n/2]}[a_k\cos(2\pi kt/n)+b_k\sin(2\pi kt/n)],\qquad t=1,2,\ldots,n.$$

Equation is called the Fourier series of the sequence Z_t. The a_k and b_k are called Fourier coefficients.

3.6 Transfer Function Models

We have the following entertained transfer function model:

$$y_t=\frac{\omega(B)}{\delta(B)}x_{t-b}+\frac{\theta(B)}{\phi(B)}a_t$$

More generally, the output series may be influenced by multiple input series, and we have the following multiple-input causal model:

$$y_t=\sum_{j=0}^{k}\nu_j(B)x_{jt}+n_t,$$

or

$$y_t = \sum_{i=0}^{k} \frac{\omega_j(B)}{\delta_j(B)} B^{b_j} x_{jt} + \frac{\theta(B)}{\phi(B)} a_t$$

where $\nu_j(B) = \omega_j(B) B^{bj} / \delta_j(B)$ is the transfer function for the jth input series x_{jt} are a_t are assumed to be independent of each of the input series $x_{jt}, j=1,2,\ldots,k$.

3.7 The Vector ARMA Processes

A useful class of parsimonious models is the vector autoregressive moving average ARMA (p,q) process

$$\Phi_p(B)\dot{Z}_t = \Theta_q(B) a_t,$$

where

$$\Phi_p(B) = \Phi_0 - \Phi_1 B - \Phi_2 B^2 - \cdots - \Phi_p B^p$$

and

$$\Theta_q(b) = \Theta_0 - \Theta_1 B - \Theta_2 B^2 - \cdots - \Theta_q B^q$$

are the autoregressive and moving average matrix polynomials of orders p and q, respectively, and Φ_0 and Θ_0 are nonsingular $m \times m$ matrices.

3.8 State Space Models

For a linear time-invariant system, its state space representation is described by the state equation

$$Y_{t+1} = AY_t + GX_{t+1}$$

and the output equation

$$Z_t = HY_t,$$

where Y_t is a state vector of dimension k, A is a $k \times k$ transition matrix, G is a $k \times n$ input matrix, X_t is an $n \times 1$ vector of the input to the system, Z_t is an $m \times 1$ vector of the output, and H is an $m \times k$ output or observation matrix. If both the input X_t and the output Z_t are stochastic processes, then the state space representation is given by

$$Y_{t+1} = AT_t + Ga_{t+1}$$

$$Z_t = HY_t + b_t$$

where $a_{t+1} = X_{t+1} - E(X_{t+1} \mid X_t, t \le n)$ is the $n \times 1$ vector of one-step-ahead forecast error of the input process X_t and b_t is an $m \times 1$ vector of disturbances assumed to be independent of a_t. The vector a_{t+1} is also known as the innovation of the input X_t at time $(t+1)$[3,13].

3.9 Serially Correlated Disturbances

A linear regression model with an ARMA(p,q) disturbance term may be written as

$$y_t = x_t'\beta + u_t, \qquad t = 1, \ldots, T,$$

$$u_t = \phi_1 u_{t-1} + \cdots + \phi_p u_{t-p} + a_{t-1} + \cdots + \theta_q a_{t-q} \text{ [4]},$$

3.10 Upward or Downward Trend

A time series such as the failure rates can have an upward trend as anticipated for many systems[9]. If so, it can be induced in model by incorporating a deterministic polynomial of degree d by including a non-zero constant θ in model

$$\phi_p(B)\nabla^d Z_t = \theta + \theta_q(B)\, a_t.$$

4. Linear Combinations of Time Series

4.1 Contemporaneous Aggregation

If X_t and Y_t are two independent, zero-mean, stationary series, then in this section statements of the following kind will be considered:

$$\text{if} \quad X_t \sim ARMA(p, m), \quad Y_t \sim ARMA(q, n)$$

$$\text{and} \quad Z_t = X_t + Y_t$$

$$\text{then} \quad Z_t \sim \mathrm{ARMA}(x, y).$$

Such a statement will be denoted by

$$ARMA(p, m) + ARMA(q, n) = ARMA(x, y).$$

Lemma.

$$MA(m) + MA(n) = MA(y),$$

where

$$y \leq \max(m, n).$$

Theorem.

$$ARMA(p, m) + ARMA(q, n) = ARMA(x, y),$$

where

$$x \leq p + q \quad \text{and} \quad y \leq \max(p+n, q+m).$$

It is interesting to consider a number of special cases of the basic theorem, concentrating on those situations in which coincidental reductions of parameters fo not occur. In the cases considered the interpretations given will assume that aggregates are of independent components, an assumption almost certainly not true in practice, and that the observational error is white noise and independent of the true process, an assumption of debatable reality but on which little empirical work is available from which to form an opinion.

(i) $AR(p)$ + white noise = ARMA(p, p)

This corresponds to an AR(p) signal as true process plus a simple white noise observational error series.

(ii) $AR(p) + AR(q) = ARMA(p+q, \max(p, q))$ and in particular $AR(1) + AR(1) = ARMA(2, 1)$

This situation might correspond to a series which is aggregated from two independent AR series. A further case of possible interest is : the sum of k AR(1) series is $ARMA(k, k-1)$.

(iii) $MA(p) + MA(q) = MA(\max(p, q))$ and in particular $MA(p)$ + white noise = $MA(p)$.

Thus if a true process follows an MA model then the addition of a white noise observation error will not alter the class of model, although the parameter values will change.

(iv) $ARMA(p,m)+$ white noise $=ARMA(p,p)$ if $p>m$ and $ARMA(p,m)$ if $p<m$.
Thus the addition of an observational error may alter the order of an ARMA model but need not do so.

(v) $AR(p)+MA(n)=ARMA(p,p+n)$.
Again, this is possibly relevant to the aggregation case, or to the observation error case with noise not being white noise.

Three other situations that might occur in practice lead either exactly, or approximately, to ARMA models. These are:

(i) A variable that obeys a simple model such as $AR(1)$ if it were recorded at an interval of K units of time but which is actually observed at an interval of M units. The variable considered has to be an accumulated one, that is of the type called a stock variable by economists.

(ii) If X_t is an instantaneously recorded variable, called a flow variable by economists, and suppose that it obeys the model

$$X_t-aX_{t-b}=\varepsilon_t, \quad |a|<1$$

where b may be a non-integer multiple or fraction of the observation period, then it is easily shown that this is equivalent to the AR(∞) model

$$\sum_{j=0}^{\infty} h_j X_{t-j}=\varepsilon_t,$$

where $h_j=\{\sin(j-b)\pi\}/(j-b)\pi$ and doubtless this model could well be approximated by an ARMA model.

(iii) If X_t, Y_t are generated by the bivariate autoregressive scheme with feedback;

$$a(B)X_t+b(B)Y_t=\varepsilon_t, \quad c(B)X_t+d(B)Y_t=\eta_t,$$

where ε_t, η_t are uncorrelated white noise series and $b(0)=c(0)=0$, then the model obeyed by X_t alone is found by eliminating Y_t in the equations to be

$$[a(B)d(B)-c(B)b(B)]X_t=d(B)\varepsilon_t+b(B)\eta_t$$

and so the ARMA(p,q) model occurs once more, and it is easily shown that generally $p>q$ [4,5].

4.2 Contemporaneous Product of Time Series

If X_t & Y_t are s-independent stationary $AR(1)$ processes, then their product,

$$Z_t=X_tY_t \text{ is an } AR(1).$$

Similarly, if X_t is an MA(1)[9].

If,

$$X_{i,t} \sim ARMA(p_i,q_i), \quad i\in[1,k] \text{ then,}$$

$$\prod_{i=1}^{k} X_{i,t} \sim ARMA(p,q),$$

$$p \leq \prod_{i=1}^{l} p_i,$$

$$q \leq p + \max_{i} \in [l, k] (q_i - p_i).$$

In particular,

$$AR(p_1)AR(p_2) = ARMA(p_1 \cdot p_2, p_1 \cdot p_2 - \min(p_1, p_2)),$$

$$AR(p)AR(p) = ARMA(p^2, p^2 - p),$$

$$ARMA(p_1, q_1)MA(q_2) = MA(q_2).$$

4.3 Composite $(\Sigma + \Pi)$ ARMA Models

The main result of this paper is stated in the following:

Proposition. Let X and Y be two zero-mean dependent Gaussian ARMA processes of orders (p_1, q_1) and (p_2, q_2) respectively and let $Z = X + Y + XY$. Let there exist a polynomial $\phi(B)$ of degree p_3 with all zeroes lying outside the unit circle such that

$$\phi(B)\{\gamma_{XY}(k) + \gamma_{YX}(k) + \gamma_{XY}(k)\gamma_{YX}(k)\} = 0, \quad k > q_3, \quad \text{say}$$

then Z is ARMA (p, q) with $p \leq p_1 + p_2 + p_3 + p_1 p_2$, $q \leq p + \max(q_i - p_i, i = 1, 2, 3)$, where p_3 and q_3 are some numbers which can always be determined in a specific situation[10].

4.4 Temporal Aggregation of the ARIMA Process

Assume that the observed time series Z_T is the m-period nonoverlapping aggregates of z_t defined as

$$Z_T = \sum_{t=m(T-1)+1}^{mT} z_t$$

$$= (1 + B + \ldots + B^{m-1}) z_{mT},$$

where T is the aggregate time unit, and m is fixed and is called the order of aggregation.

4.5 Systematic Sampling of the ARIMA Process

We examine the relationship between the underlying process z_t and the model based on the sampled subseries y_T, where $y_T = z_{mT}$ and m is the systematic sampling interval[13].

Let the r.v. in the model,

$$Z_t = \phi Z_{t-1} + a_t,$$

be failure-time observed at the end of each semester. A yearly series can be formulated by recording the values corresponding to only one of the semesters. Define,

$$Y_T = Z_{2T}, \quad t = 0, \pm 1, \pm 2, \ldots .$$

Since,

$$Z_{2T} = \phi Z_{2T-1} + a_{2T} = \phi(\phi Z_{2T-2} + a_{2T-1}) + a_{2T}$$

$$Y_T = \phi^2 Y_{T-1} + \beta_T$$

$\beta_T = a_{2T} + \phi a_{2T-1}$, $T = 0, \pm 1, \pm 2, \ldots$ form an i.i.d. sequence with mean 0 and variance $\sigma_a^2(1 + \phi^2)$. Thus model is an AR(1).

5. Time Series Modeling of Failure Rates

This section is concerned with the modeling of stochastic failure rates(observed or estimated failure rates). Given the stochastic failure times, the stochastic failure rates can be estimated at equidistant points of time. Three interesting, important characterizations are: let the stochastic failure rates form a nonseasonal and

i) stationary time series then the failure times are exponentially distributed.
ii) nonstationary time series having exponentially increasing or decreasing trend, then the failure times are Weibull distributed.
iii) nonstationary time series having hyperbolically increasing or decreasing trend, then the failure times are Gamma distributed[9].

This section models the stochastic failure rates of a system with n items connected in series, given the estimated failure rates of each item at equidistant points of time. The failure times of each item fluctuate randomly and are generated by a stochastic process. In a series system of s-independent items, the system failure-rate is the sum of the item failure-rate.

One of the pleasant properties of the exponential life distribution is that it is preserved under the formation of series system. Specifically, we show the following theorem.

Theorem. Let Y be the life length of a series system of n independent components. Let Y_i, the life length of component i, have exponential distribution $G_{\lambda_i}(t)=1-e^{-\lambda_i t}$, $i=1,\ldots,n$. Then Y has exponential distribution $G_\lambda(t)=1-e^{-\lambda t}$, where $\lambda=\sum_1^n \lambda_i$ [1,7].

For simplicity let a series system have 2 s-independent subsystems, $s_1 \& s_2$. The estimated failure rate of s_i follows on ARMA (p_i, q_i) process. The system failure-rate is an ARMA (p, q) process.

These cases may be enumerated as follows[5]:

$$ARMA(p_1, q_1)+ARMA(p_2, q_2)=ARMA(p,q), \quad p \le p_1+p_2, \quad q \le \max_{j=1,2}(p-p_j+q_j)$$

$$AR(p) + \text{white noise} = \text{ARMA}(p, p)$$

$$AR(p_1)+AR(p_2)=ARMA[p_1+p_2, \max(p_1, p_2)]$$

$$MA(q_1)+MA(q_2)=ARMA[0, \max(q_1, q_2)]$$

$$ARMA(p, q) + \text{white noise} = \text{ARMA}[p, \max(p, q)]$$

$$AR(p)+MA(q)=ARMA(p, p+q)$$

Table.1 shows various numerical examples of time series modeling and analysis for failure rates of systems that operate in series.

The analysis performed by PROC ARIMA is divided into three stages, corresponding to the stages. The IDENTIFY, ESTIMATE, and FORECAST statements perform these stages[6].

Table 1(a). Numerical Examples of Stochastic Failure Rates

Model / t	λ_1 Process: ARMA(1,1) ϕ_1= 0.1 θ_1= 0.1	λ_2 Process: AR(1) ϕ_1= -0.1	λ Process: ARMA(1,2) ϕ_1= 0.79 θ_1= 0.87 θ_2= 0.13	λ_1Process: AR(1) ϕ_1= 0.2	λ_2Process: WhiteNoise σ_a^2= 0.2	λ Process: ARMA(1,1) ϕ_1= 0.02 θ_1= 0.01	λ_1Process: AR(1) ϕ_1= 0.2	λ_2 Process: AR(1) ϕ_1= 0.2	λ Process: ARMA(2,1) ϕ_1= 0.03 θ_1=-0.01 θ_2= 0.01
1	0.726	0.451	1.177	0.425	-0.105	0.320	0.425	0.496	0.921
2	0.774	0.711	1.485	0.470	0.204	0.674	0.470	0.525	0.995
3	0.764	0.111	0.875	0.591	-0.089	0.502	0.591	0.514	1.105
4	0.656	0.239	0.895	0.553	0.113	0.666	0.553	0.456	1.009
5	0.677	0.475	1.152	0.537	-0.311	0.226	0.537	0.479	1.016
6	0.455	0.379	0.834	0.390	-0.219	0.171	0.390	0.516	0.906
7	0.500	0.314	0.814	0.530	-0.080	0.450	0.530	0.501	1.031
8	0.710	0.571	1.281	0.371	0.201	0.572	0.371	0.655	1.026
9	0.668	0.576	1.244	0.585	-0.131	0.454	0.585	0.594	1.179
10	0.332	0.627	0.959	0.452	-0.015	0.437	0.452	0.308	0.760
11	0.399	0.457	0.796	0.603	0.071	0.674	0.603	0.422	1.025
12	0.430	0.502	0.932	0.437	0.426	0.863	0.437	0.637	1.074
13	0.424	0.152	0.576	0.382	-0.374	0.008	0.382	0.551	0.933
14	0.536	0.398	0.934	0.342	-0.058	0.284	0.342	0.556	0.898
15	0.514	0.359	0.873	0.521	0.040	0.561	0.521	0.554	1.075
16	0.887	0.450	1.337	0.538	-0.087	0.454	0.538	0.496	1.034
17	0.812	0.376	1.188	0.528	-0.137	0.391	0.528	0.519	1.047
18	0.554	0.331	0.885	0.342	0.184	0.526	0.342	0.356	0.698
19	0.606	0.550	1.156	0.470	-0.014	0.456	0.470	0.421	0.891
20	0.346	0.223	0.569	0.723	-0.128	0.595	0.723	0.422	1.145
21	0.398	0.822	1.220	0.663	0.166	0.829	0.663	0.422	1.085
22	0.755	0.524	1.279	0.636	0.197	0.833	0.636	0.272	0.908
23	0.684	0.200	0.884	0.423	0.284	0.704	0.423	0.332	0.755
24	0.306	0.508	0.814	0.538	0.139	0.677	0.538	0.671	1.209
25	0.382	0.911	1.293	0.511	0.315	0.826	0.511	0.535	1.046
26	0.662	0.698	1.360	0.415	0.028	0.443	0.415	0.622	1.037
27	0.606	0.762	1.368	0.522	-0.250	0.274	0.522	0.588	1.110
28	0.372	0.103	0.475	0.545	0.400	0.940	0.545	0.255	0.800
29	0.419	0.640	1.059	0.477	-0.129	0.812	0.477	0.388	0.865
30	0.334	0.405	0.739	0.635	0.177	0.812	0.635	0.554	1.189

Table 1(b). Numerical Examples of Stochastic Failure Rates

Model / t	λ_1 Process: MA(1) θ_1= 0.3	λ_2 Process: MA(1) θ_1= 0.3	λ Process: MA(1) θ_1=-0.36	λ_1 Process: ARMA(1,1) ϕ_1= 0.2 θ_1= 0.1	λ_2 Process: AR(1) σ_a^2= 0.2	λ Process: ARMA(1,1) ϕ_1= 0.01 θ_1= 0.01	λ_1 Process: AR(1) ϕ_1= 0.2	λ_2 Process: MA(1) θ_1= 0.3	λ Process: ARMA(1,2) ϕ_1= 0.47 θ_1= 0.47 θ_2= 0.53
1	0.588	0.459	1.047	0.437	0.134	0.571	0.425	0.588	1.013
2	0.370	0.183	0.553	0.485	-0.295	0.190	0.470	0.370	0.840
3	0.501	0.348	0.849	0.475	0.121	0.596	0.591	0.501	1.092
4	0.568	0.598	1.166	0.594	0.032	0.626	0.553	0.568	1.121
5	0.527	0.448	0.975	0.570	-0.040	0.530	0.537	0.527	1.064
6	0.566	0.801	1.367	0.602	-0.175	0.427	0.390	0.566	0.956
7	0.543	0.589	1.132	0.596	-0.194	0.402	0.530	0.543	1.073
8	0.374	0.451	0.825	0.431	0.334	0.765	0.371	0.374	0.745
9	0.475	0.534	1.009	0.464	0.119	0.583	0.585	0.475	1.060
10	0.702	0.485	1.187	0.615	0.355	0.970	0.452	0.702	1.154
11	0.566	0.514	1.080	0.585	-0.048	0.537	0.603	0.566	1.169
12	0.596	0.399	0.995	0.468	-0.147	0.321	0.437	0.596	1.033
13	0.578	0.468	1.046	0.491	-0.167	0.324	0.382	0.578	0.960
14	0.143	0.559	0.702	0.047	-0.264	-0.217	0.342	0.143	0.485
15	0.404	0.505	0.909	0.136	-0.215	-0.079	0.521	0.404	0.925
16	0.888	0.373	1.261	0.622	0.237	0.859	0.538	0.888	1.426
17	0.598	0.452	1.050	0.524	0.204	0.728	0.528	0.598	1.126
18	0.430	0.576	1.006	0.405	-0.174	0.231	0.342	0.430	0.772
19	0.531	0.501	1.032	0.429	-0.023	0.406	0.470	0.531	1.001
20	0.535	0.291	0.826	0.415	-0.131	0.284	0.723	0.535	1.258
21	0.532	0.417	0.949	0.418	0.054	0.472	0.663	0.532	1.195
22	0.443	0.172	0.615	0.422	0.385	0.807	0.636	0.443	1.168
23	0.496	0.319	0.815	0.421	0.156	0.577	0.423	0.496	0.919
24	0.732	0.750	1.482	0.469	-0.067	0.402	0.538	0.732	1.270
25	0.591	0.491	1.082	0.460	0.084	0.544	0.511	0.591	1.102
26	0.376	0.623	0.999	0.481	-0.132	0.349	0.415	0.376	0.791
27	0.505	0.544	1.049	0.477	0.078	0.555	0.522	0.505	1.027
28	0.537	0.313	0.850	0.861	0.121	0.982	0.545	0.537	1.082
29	0.517	0.451	0.968	0.784	-0.371	0.413	0.477	0.517	0.994
30	0.553	0.982	0.153	0.502	0.424	0.926	0.635	0.553	1.188

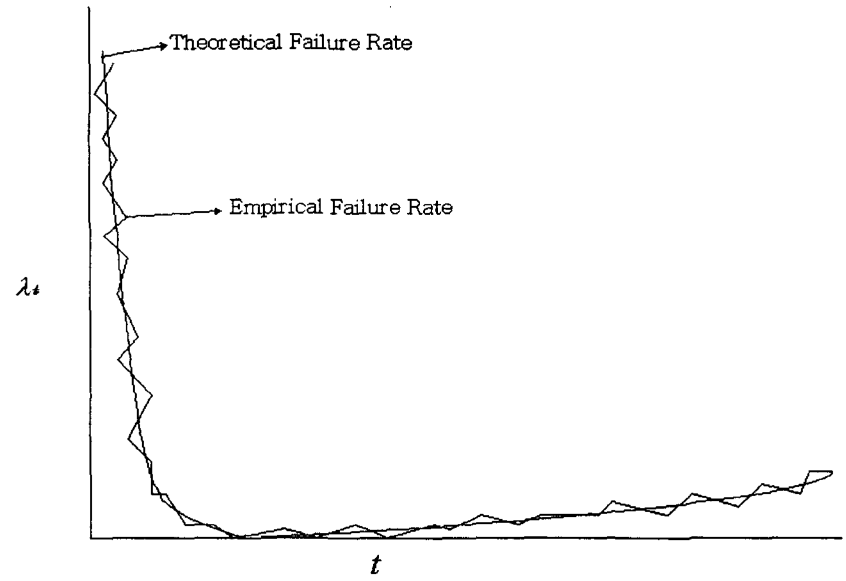

Figure 4. Theoretical and Empirical Bathtub Failure Rate of Burn-In

6. Non-linear Time Series Modeling of Failure Rates

For which components or systems is burn-in effective? Another way of posing this question is by asking, "For which distributions (which model the lifetimes of components or systems) is burn-in effective?" First, it seems resonable to rule out classes of distributions which model wearout. The reason for this is that objects which become more prone to failure throughout their life will not benefit from burn-in since burn-in stochastically weakens the residual lifetime. Consequently, distributions which have increasing failure rate or other similar aging properties are generally not candidates for burn-in.

For burn-in to be effective, lifetimes should have high failure rates initially and then improve. Since those items which survive burn-in have the same failure rate as the original, but shifted to the left, burn-in, in effect, eliminates that part of the lifetime where there is a high initial chance of failure. The class of lifetimes having bathtub-shaped failure rates has this property. For this type of distribution the failure rate starts high(the infancy period), then decreases to approximately a constant(the middle life) and then increases as it wears out (old age). As suggested, by the parenthetical remarks, this distribution is thought to describe human life and other biological lifetimes. Certain other mechanical and electronic lifetimes also can be appropriated by these distributions. This type of distribution would seem to be appropriate for burn-in, since burn-in eliminates the high-failure infancy period, leaving a lifetime which begins near its former middle life.

Most definitions of bathtub-shaped failure rates assume the failure rate decreases to some change point (t_1), then remains constant to a second change point (t_2), then increases. The case $t_1 = t_2$(i.e., no constant portion) is often adequate as an assumption in some theoretical results. We give the definition below.

Definition. A random lifetime X with distribution function $F(t)$, survival function $\overline{F}(t) = 1 - F(t)$, density $f(t)$ and failure rate $\lambda(t) = f(t)/\overline{F}(t)$ is said to have a bathtub-shaped failure rate if there exist points $0 \le t_1 \le t_2 \le \infty$, called change points, such that

$$\lambda(t) \ is \begin{cases} \text{decreasing for } 0 \le t < t_1, \\ \text{constant for } t_1 \le t < t_2, \\ \text{increasing for } t_2 \le t < \infty \ [2]. \end{cases}$$

Figure 4 shows the theoretical and empirical bath-tub failure rate of burn-in. Hence the analysis of such bath-tub failure rate are better modeled by a non-linear time series approach[12] rather than the distribution approach.

7. Summary

This paper presents a new approach to the modeling of failure rate of parts/systems that operate under varying operational & environmental conditions. The analysis of such failures are better modeled by a time series approach rather than the distribution approach. Illustration examples are discussed using simulated data.

Bathtub failure rates of burn-in can be also represented by a non-linear time series modelling.

It is our hope that this paper will be playing an important role in the developments of new and better failure rate models considering real world situations.

REFERENCES

[1] Barlow, R.E. and Proschan, F., *Statistical Theory of Reliability and Life Testing : Probability Models*, Holt, Rinenhart and Winston , Inc., NY, 1975.

[2] Block, H.W. and Savits , T.H., " Burn-in ," *Statistical Science*, 12, 1-19 , 1997.

[3] Box , G.E.P. and Jenkins, G.M. and Reinsel , G.C., *Time Series Analysis : Forecasting and Control* . 3rd ed. Prentice-Hall, Inc., NJ, 1994.

[4] Granger, C.W., and Morris, N.J., "Time Series Modeling and Interpretation, "*J.R. Statist. Soc.* A, 139, 246-257, 1976.

[5] Harvey, A.C., *Time Series Models*, Philip Allan Publishers Limited, Oxford, 1981.

[6] *SAS/ETS User's Guide*, Ver.6, 2nd Ed., SAS Institute Inc., NC, 1993.

[7] Shooman, M.L., *Probabilistic Reliability : An Engineering Approach*, McGraw-Hill, Inc., NY, 1968.

[8] Singh, N., " An Applicaton of Time-Dependent ARMA Models to Reliability -Decay Processes," *Microelectronics & Reliability*,33, 327-330 , 1993.

[9] Singh, N., " Stochastic Modeling of Aggregates & Products of Variable Failure Rates, " *IEEE Transactions on Reliability*, 44,279-284 , 1995.
[10] Singh, N. and Nirmalan, T., " Identification of Composite ($\Sigma + \Pi$) ARMA Models," *Commun. Statist. -Theory Math.*, 18, 1067-1083, 1989.
[11] Singpurwalla, N.D., " Survival in Dynamic Environments," *Statistical Science*, 10, 86-103, 1995.
[12] Tong, H., *Non-linear Time Series: A Dynamical System Approach*, Oxford University Press , Oxford, 1990.
[13] Wei, W.W.S., *Time Series Analysis: Univariate and Multivariate Methods*, Addison-Wesley Publishing Company, 1990.