Comm. Korean Math. Soc. **13** (1998), No. 4, pp. 727–733

ON MULTIPLICATION MODULES (II)

YONG HWAN CHO

ABSTRACT. In this short paper we shall find some properties on multiplication modules and prove three theorems.

1. Introduction

Let R be a commutative ring with unity, and let S be the set of non-zero divisors of R. Let R_S be the total quotient ring of R. For any ideal I of R, we put $I^{-1} = \{x \in R_S | xI \subseteq R\}$. Clearly, I^{-1} is an R-submodule of R_S. We say that I is an *invertible ideal* in R if $I^{-1}I = R$. Let M be any unitary R-module and let $T = \{t \in S | tm = 0$ for some $m \in M$ implies $m = 0\}$. Then T is a multiplicatively closed subset of S. For any submodule N of M, let $N' = \{x \in R_T | xN \subseteq M\}$. It is easily seen that N' is an R-submodule of $R_T, R \subseteq N'$ and $N'N \subseteq M$. If $N'N = M$, then N is called an *invertible submodule* of M. Note that Bourbaki, in Commutative Algebra, has a different definition of invertible submodule. The non-zero R-module M is a D_1R *module* if each non-zero cyclic submodule of M is invertible in M.

We say that a submodule N of M is *dense* in M if $M = \sum_\Phi \Phi(N)$, where the sum is over all $\Phi \in Hom(N, M)$. M is called a π *module* if each non-zero submodule of M is dense. An R-module M is called a *Prüfer module* if every finitely generated non-zero submodule of M is invertible. It is proved in section 2 that $E(M)$, the ring of endomorphism of M, is a $D_1E(M)$ module for a faithful D_1R module M such that $\Theta(M) \neq 0$.[Theorem 2.4]

In section 3 we prove the following two theorems

[Theorem 3.6] Let M be a faithful multiplication R-module. Then M is

Received November 15, 1996. Revised August 8, 1998.
1991 Mathematics Subject Classification: 13A15, 13C12.
Key words and phrases: multiplication module, multiplication ideal, faithful module, D_1R- module, Prüfer module, dense submodule.

a Prüfer module if and only if R is a Prüfer domain
[Theorem 3.7] Let M be an R module and let $\mathcal{R}, \mathcal{A}$ be some ring and its ideal,respectively. Then M is a multiplication module if and only if $\mathcal{A}$ is a multiplication ideal of $\mathcal{R}$.

2. Invertible submodules

PROPOSITION 2.1. *Let R be a ring. The following statements are equivalent.*

(1) *R is a D_1R-module*
(2) *R is an integral domain*
(3) *R is torsion free as a R-module*

PROOF. (1)$\Longleftrightarrow$(2): Suppose that R is a D_1R-module, and let $ab = 0, 0 \neq b \in R$. Put $I = bR$. Then $II' = R$ where $I' = \{x \in R_T | xI \subseteq R\}$ and $T = \{s \in S | sr = 0$ for some $r \in R$ implies $r = 0\}$.

Hence, there exist $t_1, \cdots, t_k \in T$ such that $1 = r_1 b\frac{s_1}{t_1} + r_2 b\frac{s_2}{t_2} + \cdots + r_k b\frac{s_k}{t_k}$. Thus $t_1 \cdots t_k = r_1 b s_1 t_2 \cdots t_k + r_2 b s_2 t_1 t_3 \cdots t_k + \cdots + r_k b s_k t_1 \cdots t_{k-1}$.

So, $a(t_1 \cdots t_k) = abr_1 s_1 t_2 \cdots t_k + abr_2 s_2 t_1 t_3 \cdots t_k + \cdots + abr_k s_k t_1 \cdots t_{k-1} = 0$. Since T is a multiplicatively closed subset of S, the set of non-zero divisors of R, $t_1 \cdots t_k \in T$ and $a = 0$.

Conversely, suppose that R is an integral domain. Then $S = T$. Let I be any non-zero cyclic submodule of R, say, $I = bR, b \neq 0$. Clearly I is an invertible submodule of R.
(2)$\Longleftrightarrow$ (3): $Ann_R(x) = 0$ for every non-zero $x \in R$ if and only if R has no zero divisors. □

PROPOSITION 2.2. *Let M be an R-module and N an invertible submodule of M such that $M = N \bigoplus K$. Then $M = N$.*

PROOF. Since N is invertible, $N'N = M$. For each $m \in M$, $m = \sum_{i=1}^{n} \frac{r_i}{t_i} n_i$, $t_i \in T$, $n_i \in N$ Hence $(t_1 \cdots t_n)m = r_1(t_2 \cdots t_n)n_1 + \cdots + r_n(t_1 \cdots t_{n-1})n_n$. Put $t = t_1 \cdots t_n$ and $n = r_1(t_2 \cdots t_n)n_1 + \cdots + r_n(t_1 \cdots t_{n-1})n_n$. Then $tm = n$, $n \in N$ From the fact that $M =$

$N \bigoplus K,\ m = n' + k$ for some $n' \in N,\ k \in K$. Thus $tm = t(n' + k) = n$ and so $n - tn' = tk \in N \bigcap K = \{0\}$. That is to say, $tk = 0$ and $t \in T$. Hence $k = 0$ and $M \subseteq N$. □

An R-module M is said to be a *prime module* if $ann(N) = ann(M)$ for each non-zero submodule N of M.

PROPOSITION 2.3. *π module is a prime module.*

PROOF. Let N be a non-zero submodule of a $\pi-$module M and $r \in ann(N)$. Since M is π module, N is dense in M. For any $m \in M,\ m = \sum_{i=1}^{k} \Phi_i(n_i),\ \Phi_i \in Hom_R(N, M),\ n_i \in N$. Hence $rm = \sum_{i=1}^{k} \Phi_i(rn_i) = 0$ and $ann(N) \subseteq ann(M)$. Thus $ann(N) = ann(M)$. □

Let M be any R-module, we denote the ring of R-endomorphism of M by $End(M)$ or simply $E(M)$. For any R-module M, there exists an ring monomorphism $\Phi : R/ann(M) \longrightarrow E(M)$. Thus we may think of $R/ann(M)$ as a subring of $E(M)$.

THEOREM 2.4. *Let M be an R-module. If M is a faithful D_1R module and $\Theta(M) \neq 0,\ \Theta(M) = \sum_{a\in M}[Ra : M]$, then $E(M)$ is a $D_1E(M)$ module.*

PROOF. Since $\Theta(M) \neq 0$ there exist $a \in M,\ r \in R,\ r \neq 0$ such that $rM \subseteq Ra$. We define $\Phi_r : M \longrightarrow Ra$ by $\Phi_r(m) = rm$. Then it is clear that $\Phi_r \neq 0$ and $\Phi_r \in E(M)$. We know that Φ_r is a monomorphism ([7]-Lemma 2.1) from the fact that M is a D_1R-module. Put $N = Ra$. Then N is a cyclic submodule of D_1R module M and so N is invertible in M. So, as in the proof of Proposition 2.2, there exist $t \in T,\ r_1 \in R$ such that $tm = r_1a$. Let $r \in ann(Ra),\ m \in M$. Then $trm = rr_1a = 0$ and $rm = 0$ by our definition of T. Therefore $ann(Ra) \subseteq ann(M)$ and so $ann(Ra) = ann(M)$. But $Ra \simeq R/ann(a)$. Thus $M(\simeq \Phi_r(M))$ is isomorphic to a submodule of $Ra \simeq R/ann(M)$. Note that $ann(M) = 0$ since M is faithful. So, M is isomorphic to an ideal of R. We know that $E(M)$ does not have zero divisors. In fact, let $fg = 0,\ f \neq 0,\ f, g \in E(M)$. For any $m \in M,\ fg(m) = 0$. Since M is a D_1R module, f is a monomorphism([7]-Lemma 2.1), $g(m) = 0$ and $g = 0$. Further if we define $\Phi : R \longrightarrow E(M)$ by $\Phi(r) = \Phi_r,\ \Phi_r(m) = rm$ for $m \in M$, then we know that Φ is a ring monomorphism since $ker\Phi = ann(M) = 0$.

Hence R is isomorphic to a subring of $E(M)$. Therefore R is an integral domain; thus $E(M)$ is commutative by [3]-Corollary 1.3 and $E(M)$ is an integral domain. Hence, $E(M)$ is a $D_1E(M)$ module by Proposition 2.1. □

3. Multiplication modules

An R-module M is called a *multiplication module* provided for each submodule N of M there exists an ideal I of R such that $N = IM$. Note that $I \subseteq (N : M)$ and $N = IM \subseteq (N : M)M \subseteq N$ so that $N = (N : M)M$. It follows that an R-module M is a multiplication module if and only if $N = (N : M)M$ for all submodules N of M. An ideal A of R which is a multiplication module is called a *multiplication ideal.*

PROPOSITION 3.1. *A faithful multiplication module M over a domain R is finitely generated.*

PROOF. By [4]-Proposition 3.4, it is clear. □

COROLLARY 3.2. *A faithful multiplication module M over a Dedekind domain is Noetherian.*

LEMMA 3.3. *Let M be a faithful multiplication R-module and I any invertible ideal in R. Then IM is an invertible submodule of M.*

PROOF. Since M is a faithful multiplication module, M is torsion free. In fact, if there is a regular element $c \in R$, $0 \neq m \in M$ such that $cm = 0$, then $Rm = IM$ for some ideal I of R. Hence $cRm = cIM = 0$. Since $I \neq 0$, there exist an element $i \neq 0 \in I$ and $ci \neq 0$, i.e, $cI \neq 0$. Thus $ann(M) \neq 0$ and we get a contradiction. So, M is torsion free. Now we put $N = IM$ and show that $I^{-1} \subseteq N' = (IM)'$. Let $x \in I^{-1}$. Then $x \in R_S$ such that $xI \subseteq R$, S is the set of non-zero divisors in R. But since M is torsion free, $S = T$, where $T = \{t \in S | tm = 0$ for some $m \in M$ implies $m = 0$ $\}$. $xI \in R$ implies $xIM \subseteq RM$, i.e, $xN \subseteq M$ and $x \in R_T$. This means that $x \in N'$. Thus, $M = RM = I^{-1}IM \subseteq N'(IM) = N'N \subseteq M$ and $N'N = M$. Therefore $N = IM$ is invertible in M. □

PROPOSITION 3.4. *A non zero dense submodule of a faithful multiplication module over an integral domain is invertible.*

PROOF. Let M be a faithful multiplication module over an integral domain R and N a non zero dense submodule. Then M is finitely generated by Proposition 3.1 and since M is multiplication R-module, there exists an ideal I of R such that $N = IM$. Since N is dense in M, N is a multiplication module by [8]-Proposition 3.2. Therefore I is a multiplication ideal of R ([9]-Theorem 10) since M is finitely generated and $N = IM$ is a multiplication module. Select a non-zero element $i \in I$. Then i is regular since R is a domain. Let J be an ideal of R generated by i. Since I is a multiplication ideal, there exists an ideal K of R such that $J = KI$. Since J is invertible, I is invertible in R. Thus, by Lemma 3.3, $N = IM$ is invertible. □

LEMMA 3.5. *Let M be a multiplication R-module. If N is a finitely generated submodule of M, then there exists a finitely generated ideal I of R such that $N = IM$.*

PROOF. Let $N = Rn_1 + \cdots, +Rn_k$. Since M is a multiplication module R-module, $N = (N : M)M$. Hence there exist $a_{ij} \in (N : M)$, $m_{ij} \in M$ such that $n_i = a_{i1}m_{i1} + \cdots + a_{is}m_{is}$ for $i = 1, \cdots, k$ and $j = 1, \cdots, s$. Let I be an ideal of R generated by $\{a_{11}, \cdots, a_{ks}\}$. Clearly, $I \subseteq (N : M)$ and $IM \subseteq (N : M)M$. On the other hand, Since $n_i = a_{i1}m_{i1} + \cdots + a_{is}m_{is} \in IM$, $N \subseteq IM$. Therefore $N \subseteq IM \subseteq (N : M)M \subseteq N$. Hence $N = IM$ and I is finitely generated. □

THEOREM 3.6. *Let M be a faithful multiplication R-module. Then M is a Prüfer module if and only if R is a Prüfer domain.*

PROOF. Let I be a f.g non-zero ideal of R. Since M is a Prüfer module, M is a D_1R-module. So $R/ann(M) \simeq R$ is an integral domain as in the proof of Theorem 2.4. By Proposition 3.1, M is a f.g R-module. Let $M = Rm_1 + \cdots + Rm_s$. Then $IM = I(Rm_1 + \cdots + Rm_s) = Im_1 + \cdots + Im_s$ is a finitely generated R-submodule of M. Since M is a Prüfer module, IM is invertible in M. By [7]-Lemma 3.3, I is an invertible ideal of R.

Conversely, let N be any non-zero finitely generated R-submodule of M. Then, by Lemma 3.5, there exists a finitely generated ideal I of R

such that $N = IM$. Since R is a Prüfer domain, I is an invertible ideal of R. So, by Lemma 3.3, $N = IM$ is an invertible submodule of M and M is a Prüfer module. $\square$

THEOREM 3.7. *Let M be an R-module and $\mathcal{R}$ the ring of matrices of the form $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}$ where $r \in R$ and $m \in M$. Let $\mathcal{A}$ be an ideal of $\mathcal{R}$ consisting of all matrices of the above form with $r = 0$. Then, The R-module M is a multiplication module if and only if $\mathcal{A}$ is a multiplication ideal of $\mathcal{R}$*

PROOF. Suppose that M is a multiplication R-module and let $\mathcal{B}(\subseteq \mathcal{A})$ be an ideal of $\mathcal{R}$. Put $N = \left\{ m \in M \,\middle|\, \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \in \mathcal{B} \right\}$ Then we can easily prove that N is an R-submodule of M. Hence there exists an ideal I of R such that $N = IM$. Let $\mathcal{C} = \left\{ \begin{pmatrix} i & m \\ 0 & i \end{pmatrix} \,\middle|\, i \in I, m \in M \right\}$. Then $\mathcal{C}$ is an ideal of $\mathcal{R}$. For an matrix $\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \in \mathcal{B}$, Since $m \in M$, there exist $i_k \in I$ and $m_k \in M$ for $k = 1, \cdots, n$ such that $m = \sum_{k=1}^{n} i_k m_k$. Thus $\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sum i_k m_k \\ 0 & 0 \end{pmatrix} = \sum_{k=1}^{n} \begin{pmatrix} 0 & i_k m_k \\ 0 & 0 \end{pmatrix} = \sum_{k=1}^{n} \left[\begin{pmatrix} i_k & 0 \\ 0 & i_k \end{pmatrix} \begin{pmatrix} 0 & m_k \\ 0 & 0 \end{pmatrix} \right]$. Thus $\mathcal{B} \subseteq \mathcal{CA}$. On the other hand, $\sum_{i=1}^{s} \left[\begin{pmatrix} c_i & m_i \\ 0 & c_i \end{pmatrix} \begin{pmatrix} 0 & m_i' \\ 0 & 0 \end{pmatrix} \right] = \begin{pmatrix} 0 & \sum c_i m_i' \\ 0 & 0 \end{pmatrix}$ where $m_i, m_i' \in M$ and $c_i \in I$. Since $\sum c_i m_i' \in IM = N$, $\mathcal{CA} \subseteq \mathcal{B}$. Therefore $\mathcal{B} = \mathcal{CA}$ and $\mathcal{A}$ is a multiplication ideal of $\mathcal{R}$.

Conversely, Let N be an R-submodule of M. Put $\mathcal{B} = \left\{ \begin{pmatrix} 0 & n \\ 0 & 0 \end{pmatrix} \,\middle|\, n \in N \right\}$. Then $\mathcal{B}$ is an ideal of $\mathcal{R}$ and $\mathcal{B} \subseteq \mathcal{A}$. Since $\mathcal{A}$ is a multiplication ideal of $\mathcal{R}$, there exists an ideal $\mathcal{C}$ of $\mathcal{R}$ such that $\mathcal{B} = \mathcal{CA}$. Note that any element of $\mathcal{C}$ is of the form $\begin{pmatrix} s & m \\ 0 & s \end{pmatrix}$, $s \in R, m \in M$.

Let $I = \left\{ s_i \in R \middle| \begin{pmatrix} s_i & m_i \\ 0 & s_i \end{pmatrix} \in \mathcal{C} \right\}$. For any element $n \in N$, $\begin{pmatrix} 0 & n \\ 0 & 0 \end{pmatrix}$ $\in \mathcal{B}$. Hence $\begin{pmatrix} 0 & n \\ 0 & 0 \end{pmatrix} = \sum_{i=1}^{t} \left[\begin{pmatrix} s_i & m_i \\ 0 & s_i \end{pmatrix} \begin{pmatrix} 0 & m_i' \\ 0 & 0 \end{pmatrix} \right] = \begin{pmatrix} 0 & \sum s_i m_i' \\ 0 & 0 \end{pmatrix}$, i.e, $n = \sum_i s_i m_i' \in IM$. Therefore $N \subseteq IM$.

Let $\sum_{i=1}^{l} r_i m_i \in IM$, $r_i \in I$, $m_i \in M$. Then there exist matrices, $\begin{pmatrix} r_i & m_i' \\ 0 & r_i \end{pmatrix} \in \mathcal{C}$ where, $m_i' \in M$ and $\begin{pmatrix} 0 & m_i \\ 0 & 0 \end{pmatrix} \in \mathcal{A}$.

Hence, $\sum_{i=1}^{l} \left[\begin{pmatrix} r_i & m_i' \\ 0 & r_i \end{pmatrix} \begin{pmatrix} 0 & m_i \\ 0 & 0 \end{pmatrix} \right] \in \mathcal{CA} = \mathcal{B}$. Thus $\begin{pmatrix} 0 & \sum r_i m_i \\ 0 & 0 \end{pmatrix}$ $\in \mathcal{B}$, $\sum r_i m_i \in N$, i.e, $IM \subseteq N$ and M is a multiplication module. □

References

[1] A. Barnard, *Multiplication modules*, Journal of Algebra **71** (1981), 174-178.

[2] Y. Cho, *On multiplication modules (I)*, Bull. Honam Math. **13** (1996).

[3] S. H. Cox, *Commutative endomorphism rings*, Pacific J. Math. **45** (1973), 87-91.

[4] Z. A. El-Bast and P. F. Smith, *Multiplication modules*, Comm. in Algebra **16** (1988), 755-779.

[5] F. Kasch, *Modules and rings*, Academic Press, London, 1982.

[6] G. M. Low and P. F. Smith, *Multiplication modules and ideals*, Comm. in Algebra **18** (1990), 4353-4375.

[7] A. G. Naoum and F. H. Al-Alwan, *Dedekind modules*, Comm. in Algebra **24** (1996), no. 2, 397-412.

[8] G. M. Naoum and F. H. Al-Alwan, *Dense submodules of multiplication modules*, Comm. in Algebra **24** (1996), no. 2, 413-424.

[9] P. F. Smith, *Some remarks on multiplication modules*, Arch. Der Math. **50** (1988), 223-225.

[10] J. Zelmanowitz, *Commutative endomorphism rings*, Can. J. Math. **XXIII** (1971), no. 1, 69-76.

Department of Mathematics Education
Chonbuk National University
Chonju 561-756, Korea