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q-ANALOGUES OF p-ADIC GAMMA FUNCTIONS AND p-ADIC EULER CONSTANTS

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Abstract. *In this paper we will derive some new properties of q-extensions of p-adic gamma functions and p-adic Euler constants.*

1. Introduction

In [1], Jakson defined a q-analogue of the gamma function by

$$\Gamma_q(x) = \frac{(q:q)_\infty (1-q)^{1-x}}{(q^x:q)_\infty}$$

when $0 < q < 1$, $x \in \mathbb{C}$. Here the product $(a:q)_\infty$ is defined by

$$(a:q)_\infty = \prod_{n=0}^{\infty} (1-aq^n).$$

Koblitz [5] showed some properties of q-analogues $\Gamma_{p,q}$, $\gamma_{p,q}$ which extend Morita's p-adic gamma functions Γ_p [3],[4],[6],[7], and Diamond's p-adic Euler constants γ_p [2], respectively. Recall that Γ_p is defined by

$$\Gamma_p(x) = \lim_{n \to x} (-1)^n \prod_{\substack{j<n \\ (p,j)=1}} j$$

for $x \in \mathbb{Z}_p$, and $\gamma_p \in \mathbb{Q}_p$ is defined to be

$$\gamma_p = \frac{-\Gamma_p'(1)}{\Gamma_p(1)}.$$

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The purpose of this paper is to derive some new functional equation of $\Gamma_{p,q}$ and a new expression of $\gamma_{p,q}$. $\gamma_{p,q}$ is established by using Calitz's method in [7],[9] and the q-factorial function in [1]. More explicitly, our results are as follows:

$$\Gamma_{p,q}(x)\Gamma_{p,q}(1-x) = (-1)^{\ell(x)} \lim_{n\to x} \prod_{\substack{j<n \\ (p,j)=1}} q^j,$$

$$\gamma_{p,q} = \lim_{n\to\infty} p^{-n}\left\{1 - (-1)^p \frac{[p^n-1;q]!}{[p;q]^{p^{n-1}-1}[p^{n-1}-1;q^p]!}\right\}.$$

NOTATIONS. Let $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}_p$ be the ring of p-adic integers, the field of p-adic numbers, and the p-adic completion of the algebraic closure of $\mathbb{Q}_p$. If $q \in \mathbb{C}$, one usually assumes $|q| < 1$; if $q \in \mathbb{C}_p$, one usually assumes $|q-1|_p < p^{-1/(p-1)}$, so that $q^x = \exp_p(x \log_p q)$ for $|x|_p \le 1$, where $|p|_p = 1/p$. Throughout this paper we use the notation

$$[x] = [x;q] = \frac{1-q^x}{1-q}.$$

2. q-extensions of p-adic gamma functions

Let $t \in \mathbb{C}_p$, $|t|_p < 1$, $t \ne 0$. Set $q = 1+t$. Then, for any positive integer n, $\Gamma_{p,q}(n)$ is defined by

$$\Gamma_{p,q}(n) = (-1)^n \prod_{\substack{j<n \\ (p,j)=1}} [j].$$

Then $\Gamma_{p,q}$ extends to a continuous function from the p-adic integers $\mathbb{Z}_p$ to the p-adic units $\mathbb{Z}_p^*$, and $\lim_{q\to 1} \Gamma_{p,q} = \Gamma_p$.

LEMMA 1 ([3],[5],[8]). *Let p be any prime. Then $\Gamma_{p,q}$ has the following properties:*

(1) *For all $x \in \mathbb{Z}_p$, $\Gamma_{p,q}(x+1) = \epsilon_{p,q}(x)\Gamma_{p,q}(x)$ where*

$$\epsilon_{p,q}(x) = \begin{cases} -[x] & \text{if } |x|_p = 1, \\ -1 & \text{if } |x|_p < 1. \end{cases}$$

(2) (i) *If either* $p \neq 2$ *or* $p = 2$ *and* $|x-y|_p \neq \frac{1}{4}$, *then, for all* $x, y \in \mathbb{Z}_p$,

$$|\Gamma_{p,q}(x) - \Gamma_{p,q}(y)|_p \leq |x-y|_p.$$

(ii) *If* $p = 2$ *and* $|x-y|_p = \frac{1}{4}$, *then, for all* $x, y \in \mathbb{Z}_p$,

$$|\Gamma_{p,q}(x) - \Gamma_{p,q}(y)|_p \leq 2|x-y|_p.$$

(3) $\Gamma_{p,q}(1) = -1$ *and* $\Gamma_{p,q}(2) = 1$.
(4) *For all* $x \in \mathbb{Z}_p$, *we have* $|\Gamma_{p,q}(x)|_p = 1$.

In [1], the q-factorial $[n;q]!$ is defined by

$$[n;q]! = [n;q][n-1;q]\cdots[2;q][1;q].$$

PROPOSITION 2. *Let* $n \in \mathbb{N}$. *Then*

$$\Gamma_{p,q}(n+1) = (-1)^{n+1} \frac{[n;q]!}{[p;q]^{[\frac{n}{p}]_g} \left[[\frac{n}{p}]_g; q^p\right]!},$$

where $[\ ]_g$ *is the greatest integer function. In particular,*

$$\Gamma_{p,q}(p^n) = (-1)^p \frac{[p^n-1;q]!}{[p;q]^{p^{n-1}-1}[p^{n-1}-1;q^p]!}.$$

PROOF. From the definition of $\Gamma_{p,q}$, we have

$$\Gamma_{p,q}(n+1) = \frac{(-1)^{n+1}[1;q][2;q]\cdots[n;q]}{[p;q][2p;q]\cdots\left[[\frac{n}{p}]_g p;q\right]}.$$

Therefore we get

$$\begin{aligned}\Gamma_{p,q}(n+1) &= \frac{(-1)^{n+1}[n;q]!}{[p;q]^{[\frac{n}{p}]_g}[1;q^p][2;q^p]\cdots\left[[\frac{n}{p}]_g;q^p\right]} \\ &= \frac{(-1)^{n+1}[n;q]!}{[p;q]^{[\frac{n}{p}]_g}\left[[\frac{n}{p}]_g;q^p\right]!}.\end{aligned}$$

□

PROPOSITION 3. *For any p, $\Gamma_{p,q}(-n)$ $(n \in \mathbb{N})$ is given by*

$$\Gamma_{p,q}(-n) = (-1)^{n+1-[\frac{n}{p}]_g} \prod_{\substack{j<n+1 \\ (p,j)=1}} q^j \left(\Gamma_{p,q}(n+1)\right)^{-1}.$$

In particular, if $p = 2$ then

$$\Gamma_{p,q}(-n) = (-1)^{n+1-[\frac{n}{p}]_g} q^{(n-[\frac{n}{p}]_g)^2} \left(\Gamma_{p,q}(n+1)\right)^{-1}.$$

PROOF. By Lemma 1, $1 = \Gamma_{p,q}(0) = \Gamma_{p,q}\big((-1)+1\big) = \epsilon_{p,q}(-1)\Gamma_{p,q}(-1)$ $= \epsilon_{p,q}(-1)\epsilon_{p,q}(-2)\Gamma_{p,q}(-2)$. Continuing this process, we get

$$1 = \Big(\prod_{j=1}^{n} \epsilon_{p,q}(-j)\Big)\Gamma_{p,q}(-n).$$

Thus we have

$$\Gamma_{p,q}(-n)^{-1} = \prod_{j=1}^{n} \epsilon_{p,q}(-j).$$

By definition of $\epsilon_{p,q}$, we have

$$\begin{aligned} \Gamma_{p,q}(-n)^{-1} &= (-1)^{[\frac{n}{p}]_g} \prod_{\substack{j<n+1 \\ (p,j)=1}} q^{-j}[j] \\ &= (-1)^{[\frac{n}{p}]_g} \prod_{\substack{j<n+1 \\ (p,j)=1}} q^{-j}(-1)^{-n-1}(-1)^{n+1} \prod_{\substack{j<n+1 \\ (p,j)=1}} [j] \\ &= (-1)^{-n-1+[\frac{n}{p}]_g} \prod_{\substack{j<n+1 \\ (p,j)=1}} q^{-j}\Gamma_{p,q}(n+1) \quad . \end{aligned}$$

□

From Proposition 3, we obtain

$$\Gamma_{p,q}(n+1)\Gamma_{p,q}(-n) = (-1)^{n+1-[\frac{n}{p}]_g} \prod_{\substack{j<n+1 \\ (p,j)=1}} q^j.$$

Replacing n by $n-1$, we have

$$\Gamma_{p,q}(n)\Gamma_{p,q}(1-n) = (-1)^{n-[\frac{n-1}{p}]_g} \prod_{\substack{j<n \\ (p,j)=1}} q^j$$

Now we define $\ell : \mathbb{Z}_p \to \{1, 2, \ldots, p\}$ by assigning to $x \in \mathbb{Z}_p$ its residue modulo $p\mathbb{Z}_p$. Let $n = a_0 + a_1 p + \cdots$ be a positive integer in base p. If $a_0 \neq 0$, then $[\frac{n-1}{p}]_g = a_1 + a_2 p + \cdots$. Therefore $n - p[\frac{n-1}{p}]_g = a_0 = \ell(n)$. If $a_0 = 0$, then $n-1 = (p-1) + b_1 p + b_2 p^2 + \cdots$. Hence

$$[\frac{n-1}{p}]_g = b_1 + b_2 p + \cdots .$$

So we have

$$n - p[\frac{n-1}{p}]_g = 1 + (p-1) = p = \ell(n).$$

Therefore we get the following.

THEOREM 4. *If $p \neq 2$, then for all $x \in \mathbb{Z}_p$*

$$\Gamma_{p,q}(x)\Gamma_{p,q}(1-x) = (-1)^{\ell(x)} \lim_{n \to x} \prod_{\substack{j<n \\ (p,j)=1}} q^j.$$

REMARK. If we take $x = \frac{1}{2}$ $(p \neq 2)$, then we can easily obtain

$$\ell(\frac{1}{2}) = \ell(\frac{p+1}{2}) = \frac{p+1}{2}.$$

By Theorem 4, we have

$$\begin{aligned} \Gamma_{p,q}(\tfrac{1}{2})^2 &= (-1)^{\ell(\frac{1}{2})} \lim_{n \to 1/2} \prod_{\substack{j<n \\ (p,j)=1}} q^j. \\ &= \begin{cases} \lim_{n \to 1/2} \prod_{\substack{j<n \\ (p,j)=1}} q^j & \text{if } p \equiv 3 (\mod 4), \\ -\lim_{n \to 1/2} \prod_{\substack{j<n \\ (p,j)=1}} q^j & \text{if } p \equiv 1 (\mod 4). \end{cases} \end{aligned}$$

3. q-extensions of p-adic Euler constants

We can easily see that $\Gamma_{p,q}$ is locally analytic function by Lemma 1. Also, from Lemma 1, we have

$$\frac{\Gamma'_{p,q}(x+1)}{\Gamma_{p,q}(x+1)} = \frac{\epsilon'_{p,q}(x)}{\epsilon_{p,q}(x)} + \frac{\Gamma'_{p,q}(x)}{\Gamma_{p,q}(x)}.$$

So we have the following.

PROPOSITION 5. *For $m \in \mathbb{N}$, we have*

$$\frac{\Gamma'_{p,q}(m)}{\Gamma_{p,q}(m)} = \frac{\Gamma'_{p,q}(1)}{\Gamma_{p,q}(1)} - \frac{\log_p q}{1-q} \sum_{j=1}^{m-1} \frac{q^j}{[j]}.$$

This formula may be considered as a q-extension of p-adic analogue

$$\frac{\Gamma'_p(m)}{\Gamma_p(m)} = \frac{\Gamma'_p(1)}{\Gamma_p(1)} + \sum_{\substack{j<m \\ (p,j)=1}} \frac{1}{j}$$

of the one for the gamma function, which is

$$\frac{\Gamma'(m)}{\Gamma(m)} = \frac{\Gamma'(1)}{\Gamma(1)} + \sum_{j<m} \frac{1}{j}.$$

Now we define q-extension of p-adic Euler constants $\gamma_{p,q}$ to be

$$\gamma_{p,q} = -\frac{\Gamma'_{p,q}(1)}{\Gamma_{p,q}(1)} = \Gamma'_{p,q}(1) = -\Gamma'_{p,q}(0).$$

From Proposition 2, we get

$$\Gamma_{p,q}(p^n) = (-1)^p \frac{[p^n-1;q]!}{[p;q]^{p^{n-1}-1}[p^{n-1}-1;q^p]!}.$$

Therefore we obtain

$$\begin{aligned} &\lim_{n\to\infty} p^{-n}\left\{1-(-1)^p \frac{[p^n-1;q]!}{[p;q]^{p^{n-1}-1}[p^{n-1}-1;q^p]!}\right\}. \\ &= \lim_{n\to\infty} p^{-n}(1-\Gamma_{p,q}(p^n)) \\ &= \lim_{n\to\infty} \frac{1-\Gamma_{p,q}(p^n)}{p^n} = -\Gamma'_{p,q}(0) = \gamma_{p,q}. \end{aligned}$$

So we have the following.

THEOREM 6. *A q-extension $\gamma_{p,q}$ of p-adic Euler constant γ_p satisfies*

$$\gamma_{p,q} = \lim_{n\to\infty} p^{-n}\left\{1 - (-1)^p \frac{[p^n - 1; q]!}{[p;q]^{p^{n-1}-1}[p^{n-1} - 1; q^p]!}\right\}.$$

Finally, from Lemma 1 we see that $|\Gamma'_{p,q}(1)| \leq 1$. So the p-adic absolute value of $\gamma_{p,q}$ is less than or equal to 1.

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