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IDEMPOTENTS IN QUASI-LATTICES

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ABSTRACT. Using idempotents in quasi-lattices, we show that the category **Latt** of lattices is both reflective and coreflective in the category **qLatt** of quasi-lattices and homomorphisms. It is also shown that a quasi-ordered set is a quasi-lattice iff its partially ordered reflection is a lattice.

0. Introduction

It is well known that lattices can be defined as algebras or partially ordered sets having joins and meets.

In order to generalize the concept of lattices in the setting of quasi-ordered sets, Chajda introduced a concept of q-lattices in 1992 ([3]) and Chajda and Kotrle characterized subdirectly irreducible and congruence distributive q-lattices ([4]).

The purpose of this paper is to introduce a slightly weaker concept of quasi-lattices and to study its relationship with lattices. To obtain the result, the lattice of idempotents of a quasi-lattice plays an important role.

Using this tool and the category theory, we show that the lattice of idempotents of a quasi-lattice gives rise to the reflection and coreflection for the quasi-lattice. Moreover, we show that a quasi-ordered set is a quasi-lattice iff its partially ordered reflection is a lattice, which is precisely the lattice of idempotents of the quasi-ordered lattice. It is also shown that a quasi-lattice L is a q-lattice iff the join and meet of every pair of elements of L are idempotents.

For the terminology not introduced in the paper, we refer to [1] for the category theory and [2] for the ordered sets.

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1. The lattice of idempotents of a quasi-lattice

We now introduce a concept of quasi-lattices.

DEFINITION 1.1. An algebra $L = (L, \vee, \wedge)$ of type $(2,2)$ is called a *quasi-lattice* if it satisfies the following conditions for all $a, b, c \in L$:

(q1) $(a \vee b) \vee c = a \vee (b \vee c)$, $(a \wedge b) \wedge c = a \wedge (b \wedge c)$;

(q2) $a \vee b = b \vee a$, $a \wedge b = b \wedge a$;

(q3) $a \vee (a \wedge b) = a \vee a$, $a \wedge (a \vee b) = a \wedge a$;

(q4) $a \vee a = a \wedge a$.

We note that the concept of quasi-lattices is self-dual. Lattices are quasi-lattices, but there are quasi-lattices which are not lattices. Indeed, let $A = \{a, b, c\}$ endowed with the operations $\vee = \wedge \colon A \times A \to A$ ($x \vee y = x \wedge y = a$ for all $x, y \in A$). Then A is clearly a quasi-lattice but not a lattice.

In the following, a quasi-lattice $L = (L, \vee, \wedge)$ will be simply denoted by L.

REMARK. By definition, it is clear that any quasi-lattice L satisfies the following equations for all $a, b \in L$:

(1) $a \vee a \vee a = a \vee a$, $a \wedge a \wedge a = a \wedge a$.

(2) $a \vee (a \wedge b) = a \wedge (a \vee b)$.

DEFINITION 1.2. An element a of a quasi-lattice is said to be an *idempotent* if $a \vee a = a$.

Using the above remark, for any quasi-lattice L, $\{a \vee a \mid a \in L\}$ is the set of all idempotents of L, which will be denoted by eL.

We recall that Chajda introduced the concept of *q-lattices* which are quasi-lattices L with the *weak idempotence condition*, namely

$a \vee (b \vee b) = a \vee b$;

$a \wedge (b \wedge b) = a \wedge b$ for $a, b \in L$ (see [3]).

We now characterize q-lattices among quasi-lattices by idempotents.

PROPOSITION 1.3. *A quasi-lattice L is a q-lattice iff for any $a, b \in L$, $a \vee b$ and $a \wedge b$ are idempotents.*

PROOF. Suppose that L is a q-lattice, then applying the weak idempotence condition twice, one has

$$\begin{aligned} a \vee b &= a \vee (b \vee b) = (a \vee a) \vee (b \vee b) \\ &= (a \vee b) \vee (a \vee b); \end{aligned}$$

hence $a \vee b$ is an idempotent. Dually $a \wedge b$ is also an idempotent.

Conversely, suppose that L satisfies the given condition, then for $a, b \in L$, one has

$$\begin{aligned} a \vee (b \vee b) &= \{a \vee (b \vee b)\} \vee \{a \vee (b \vee b)\} \\ &= (a \vee a) \vee (b \vee b \vee b \vee b) \\ &= (a \vee a) \vee (b \vee b) = (a \vee b) \vee (a \vee b) \\ &= a \vee b. \end{aligned}$$

Dually the remaining condition holds. Thus L is a q-lattice. □

REMARK. There is a quasi-lattice which is not a q-lattice. Indeed, let $\{0, e\}$ be the two point chain, where $0 \leq e$ and let $L = \{0, e, b\}$, while joins and meets for $0, e$ are the usual ones, but $0 \vee b = b \vee 0 = b$, $e \vee b = b \vee e = b \vee b = e$, and $0 \wedge b = b \wedge 0 = 0$, $e \wedge b = b \wedge e = b \wedge b = e$, then L is clearly a quasi-lattice but not a q-lattice, because $0 \vee b$ is not an idempotent.

LEMMA 1.4. *In a quasi-lattice L, eL is a subalgebra of L. Moreover eL is a lattice.*

PROOF. Using (q4), $eL = \{a \vee a \mid a \in L\} = \{a \wedge a \mid a \in L\}$; hence it is immediate that eL is a subalgebra of L. Furthermore, $(eL, \vee, \wedge)$ is a lattice, because for any $a, b \in eL$, $a \vee (a \wedge b) = a \vee a = a$ and $a \wedge (a \vee b) = a \wedge a = a$. □

DEFINITION 1.5. Let L and L' be quasi-lattices (lattices). A map $f \colon L \to L'$ is called a *homomorphism* if it preserves joins and meets.

It is clear that the class of quasi-lattices and homomorphisms between them forms a category, which will be denoted by **qLatt**. Since the class of quasi-lattices is equational, the category **qLatt** is an algebraic category and hence complete and cocomplete (see [1]). Moreover the category **Latt** of lattices and homomorphisms between them is a full subcategory of **qLatt**.

THEOREM 1.6. *The category* **Latt** *is coreflective in the category* **qLatt**.

PROOF. For a quasi-lattice L, eL is a sublattice of L by Lemma 1.4. Let $j\colon eL \to L$ be the inclusion homomorphism, then we show that (eL, j) is the **Latt**-coreflection for $L \in$ **qLatt**. In fact, if we take $M \in$ **Latt** and a homomorphism $f\colon M \to L$, then for each $a \in M$, $a \vee a = a$; hence $f(a) = f(a \vee a) = f(a) \vee f(a)$, so that $f(a) \in eL$. Thus $f(M) \subseteq eL$. Let $\overline{f}\colon M \to eL$ be the corestriction of f to eL. Then clearly $j \circ \overline{f} = f$ and $\overline{f}$ is a homomorphism. Since j is 1-1, the lattice homomorphism $\overline{f}$ with $j \circ \overline{f} = f$ is unique. Hence **Latt** is a coreflective subcategory of **qLatt**. □

Using the results in [1], one has:

COROLLARY 1.7. *The subcategory* **Latt** *is closed under the formation of colimits in* **qLatt**.

A lattice $L = (L, \vee, \wedge)$ defines a partial order relation $\leq$ on the set L, namely $a \leq b$ iff $a \vee b = b$ (equivalently $a \wedge b = a$). We define a quasi-order relation $\leq$ on a quasi-lattice.

NOTATION. For a quasi-lattice L, we define the map $\varepsilon_L\colon L \to eL$ by $\varepsilon_L(a) = a \vee a (a \in L)$.

PROPOSITION 1.8. *On a quasi-lattice* $(L, \vee, \wedge)$, *we define a relation* $\leq$ *as follows:* $a \leq b$ *iff* $(a \vee a) \vee (b \vee b) = b \vee b$. *Then* $(L, \leq)$ *is a quasi-ordered set.*

PROOF. We note that for any $a, b \in L$, $a \leq b$ iff $\varepsilon_L(a) \leq \varepsilon_L(b)$ in the lattice eL. Thus $\leq$ is the initial quasi-order relation on L with respect to $\varepsilon_L\colon L \to (eL, \leq)$ (see [1]). □

In the following, the relation $\leq$ on a quasi-lattice L means the above quasi-order relation and the quasi-lattice L is also denoted by $(L, \leq)$.

REMARK 1.9. (1) In a quasi-lattice L, $a \leq b$ iff $(a \vee a) \wedge (b \vee b) = a \vee a$, because $\varepsilon_L(a) \leq \varepsilon_L(b)$ iff $(a \vee a) \wedge (b \vee b) = a \vee a$ in the lattice $(eL, \vee, \wedge)$.

(2) In a quasi-lattice L, $a \leq b$ and $b \leq a$ iff $a \vee a = b \vee b$, because $a \leq b$ and $b \leq a$ iff $\varepsilon_L(a) = \varepsilon_L(b)$. In particular, for any $a \in L$, $a \leq a \vee a \leq a$.

(3) We recall that the quasi-order relation $\leq$ on a q-lattice $(L, \vee, \wedge)$ is defined by $a \leq b$ iff $a \vee b = b \vee b$, or equivalently $a \wedge b = a \wedge a$ ([3]). Using the equation $a \vee (b \vee b) = a \vee b$, $a \wedge (b \wedge b) = a \wedge b$, for any $a, b \in L$, $a \vee b = b \vee b$ iff $(a \vee a) \vee (b \vee b) = b \vee b$ and hence our quasi-order relation on L is identical with $\leq$ in [3].

Using (q1), (q2) and (q4), one has the following immediately:

LEMMA 1.10. *For a quasi-lattice L, $\varepsilon_L \colon L \to eL$ is a homomorphism.*

PROPOSITION 1.11. *Let $(L, \leq)$ be a quasi-lattice and $a, b, c, d \in L$. Then one has:*

(1) $a \leq a \vee b$ and $b \leq a \vee b$.
(2) *If $a \leq c$ and $b \leq c$, then $a \vee b \leq c$.*
(3) $a \wedge b \leq a$ *and* $a \wedge b \leq b$.
(4) *If $c \leq a$ and $c \leq b$, then $c \leq a \wedge b$.*
(5) *If $a \leq c$ and $b \leq d$, then $a \vee b \leq c \vee d$ and $a \wedge b \leq c \wedge d$.*

PROOF. Since $\varepsilon_L(a), \varepsilon_L(b) \leq \varepsilon_L(a) \vee \varepsilon_L(b) = \varepsilon_L(a \vee b)$, we have $a \leq a \vee b$ and $b \leq a \vee b$. Similarly, using the above lemma, the remaining part can be proved and we omit the proof. □

LEMMA 1.12. *For any $L \in$ **qLatt**, let $\leq$ be the quasi-order on L, and $R = \{(x, y) \in L \times L \mid x \leq y$ and $y \leq x\}$. Then R is a congruence relation on L.*

PROOF. We note that $\varepsilon_L \colon L \to eL$ is a homomorphism and $ker(\varepsilon_L) = \{(x, y) \in L \times L \mid \varepsilon_L(x) = \varepsilon_L(y)\} = \{(x, y) \in L \times L \mid x \leq y$ and $y \leq x\} = R$. Thus R is a congruence relation on L. □

REMARK 1.13. For a quasi-lattice L, we consider the quotient algebra L/R of L with respect to the congruence relation $R = ker(\varepsilon_L)$ on L, then $L/R = L/ker(\varepsilon_L) \cong eL$, since $\varepsilon_L \colon L \to eL$ is an onto homomorphism.

THEOREM 1.14. *The category* **Latt** *is reflective in the category* **qLatt**.

PROOF. Let L be a quasi-lattice, then (ε_L, eL) is the **Latt**-reflection for $L \in$ **qLatt**. In fact, take any $M \in$ **Latt** and a homomorphism

$f\colon L \to M$, then $ker(\varepsilon_L) \subseteq ker(f)$. Indeed, take any $(x,y) \in ker(\varepsilon_L)$, i.e., $x \vee x = y \vee y$. Then $f(x) = f(x) \vee f(x) = f(x \vee x) = f(y \vee y) = f(y) \vee f(y) = f(y)$, for every element of M is an idempotent; therefore $(x,y) \in ker(f)$. So, by the Fundamental Theorem of Factorization ([6]), there is a unique homomorphism $\overline{f}\colon eL \to M$ with $\overline{f} \circ \varepsilon_L = f$. Hence **Latt** is a reflective subcategory of **qLatt**. □

COROLLARY 1.15. *The subcategory* **Latt** *is closed under the formation of limits in* **qLatt**.

REMARK. It is well known that the category **POS** of partially ordered sets and order preserving maps between them is a reflective subcategory of **QOS** of quasi-ordered sets and order preserving maps ([1]). For a quasi-ordered set $(X, \leq)$, the **POS**-reflection is given by the quotient map $q\colon X \to X/R$, where $R = \{(x,y) \in X \times X \mid x \leq y \text{ and } y \leq x\}$ and the order relation $\leq$ on X/R is given by $[x] \leq [y]$ iff $x \leq y$ $(x, y \in X)$. Thus $eL \cong L/ker(\varepsilon_L)$ is precisely the **POS**-reflection of the quasi-ordered set $(L, \leq)$, since $ker(\varepsilon_L) = \{(x,y) \mid x \leq y \text{ and } y \leq x\}$.

We now show that the converse of the above remark also holds as follows.

THEOREM 1.16. *Let L be a quasi-ordered set. Then L is a quasi-lattice iff the* **POS**-*reflection for L is a lattice.*

PROOF. By the above remark, it remains to show that the converse holds. Assume that the **POS**-reflection for L is a lattice. Let $R = \{(x,y) \in L \times L \mid x \leq y \text{ and } y \leq x\}$, then the quotient map $q\colon L \to L/R$ is the **POS**-reflection for L, and $x \leq y$ iff $q(x) \leq q(y)$ for all $x, y \in L$. Since q is onto, there is a map $m\colon L/R \to L$ with $q \circ m = 1_{L/R}$. Define binary operations $\vee, \wedge\colon L \times L \to L$ by

$\vee = m \circ \vee_{L/R} \circ (q \times q)$;

$\wedge = m \circ \wedge_{L/R} \circ (q \times q)$.

In other words,

$a \vee b = m(q(a) \vee q(b))$ and $a \wedge b = m(q(a) \wedge q(b))$ for all $a, b \in L$, where $q(a) \vee q(b)$ and $q(a) \wedge q(b)$ denote for the simplicity $q(a) \vee_{L/R} q(b)$ and $q(a) \wedge_{L/R} q(b)$, respectively. Then using the fact that $q \circ m = 1_{L/R}$, we can prove that $(L, \vee, \wedge)$ is a quasi-lattice. In fact, for all $a, b, c \in L$,

$$\begin{aligned}(a \vee b) \vee c &= m(q(a) \vee q(b)) \vee c \\ &= m(q(m(q(a) \vee q(b))) \vee q(c)) = m((q(a) \vee q(b)) \vee q(c)) \\ &= m(q(a) \vee (q(b) \vee q(c))) = m(q(a) \vee q(m(q(b) \vee q(c)))) \\ &= m(q(a) \vee q(b \vee c)) = a \vee (b \vee c).\end{aligned}$$

Moreover, $a \vee b = m(q(a) \vee q(b)) = m(q(b) \vee q(a)) = b \vee a$. Furthermore,

$$\begin{aligned}a \vee (a \wedge b) &= m(q(a) \vee q(m(q(a) \wedge q(b)))) \\ &= m(q(a) \vee (q(a) \wedge q(b))) = m(q(a)) \\ &= m(q(a) \vee q(a)) = a \vee a.\end{aligned}$$

Finally,

$$\begin{aligned}a \vee a &= m(q(a) \vee q(a)) \\ &= m(q(a)) = m(q(a) \wedge q(a)) = a \wedge a.\end{aligned}$$

Dually, one has the remaining conditions. This completes the proof. □

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