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NEUMANN SERIES EXPANSION OF THE INVERSE OF A FRAME OPERATOR

JAE KUN LIM

ABSTRACT. We present a proof that, among all complex numbers, Duffin-Schaeffer's choice in the Neumann series expansion of the inverse of a frame operator has the best possible convergence rate.

1. Introduction

Ever since frames were introduced in [9] as a part of the analysis of non-harmonic Fourier series, their wide range of applications has been recognized by many authors in numerous contexts. See, for example, [8], [11], and the references therein. One of the most important problems in the theory of frames is to find the dual frame of a given frame. Finding the dual frame of a frame is tantamount to inverting the frame operator which is bounded, positive and invertible. One way of doing this is to find finite dimensional approximations of the inverse of the frame operator [2], [4], [6], [3]. Another way is to express the inverse of the frame operator by a Neumann series [9], [8], [1], [12] and [10]. For the frame operator S of a frame with frame bounds A and B Duffin and Schaeffer [9] showed that $S^{-1} = 2/(A+B)\sum_{n=0}^{\infty}(I - 2/(A+B)S)^n$, and that $\|I - 2/(A+B)S\| \leq (B-A)/(B+A)$. We show that among all complex number z for which $S^{-1} = z\sum_{n=0}^{\infty}(I-zS)^n$ converges the choice of $z = 2/(A+B)$ by Duffin and Schaeffer has the best possible convergence rate $\|I-2/(A+B)S\| = (B-A)/(B+A)$ if A and B are optimal frame bounds. Previously, Li [12] and Gröchenig [10] showed that the same choice is best among real numbers. If z is real, one can prove the result by considering the partial order relations among self-adjoint operators; If z is a complex number, then $I - zS$ is normal but no longer self-adjoint. Hence a finer analysis of spectrum is

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needed to generalize the result to the complex numbers. Our method of proof is a simple application of the spectral mapping theorem (actually the polynomial mapping theorem would suffice) and plane analytic geometry.

2. Preliminary

All of the results on frames collected in this section can be found in [14], [8], [11] and [5]. The elementary functional analysis that is needed in our discussion can be found, for example, in [13] or [7].

Let $\mathcal{H}$ denote a separable Hilbert space over $\mathbb{C}$. A sequence of vectors $\{f_n\}_{n\in\mathbb{N}}$ is a *frame* if there exist positive numbers A and B such that for any $f \in \mathcal{H}$

$$A\|f\|^2 \leq \sum_{n\in\mathbb{N}} |\langle f, f_n\rangle|^2 \leq B\|f\|^2.$$

A and B are called a *lower frame bound* and an *upper frame bound*, respectively. The supremum of the lower frame bounds is said to be the *optimal lower frame bound*, and the infimum of the upper frame bounds is said to be the *optimal upper frame bound.*

Let $Sf := \sum_{n\in\mathbb{N}}\langle f, f_n\rangle f_n$. Then S is a well-defined bounded positive invertible operator from $\mathcal{H}$ into $\mathcal{H}$, and S satisfies that $AI \leq S \leq BI$, where I is the identity operator. It is easy to see that the optimal lower frame bound is the minimal spectrum of S and that the optimal upper frame bound is the maximal spectrum of S. Since S is positive and invertible, for any $f \in \mathcal{H}$ $f = SS^{-1}f = \sum_{n\in\mathbb{N}}\langle f, S^{-1}f_n\rangle f_n$. We call $\{S^{-1}f_n\}_{n\in\mathbb{N}}$ the dual frame of the frame $\{f_n\}_{n\in\mathbb{N}}$. This basis-like feature is the most important property of a frame, and therefore the need to find the inverse of the frame operator.

3. Main results

Let S be a positive invertible operator with $a := \min(\sigma(S))$ and $b := \max(\sigma(S))$, where $\sigma(S)$ denotes the spectrum of S which is a compact subset of $\mathbb{R}$. Suppose we have estimates of the minimum and maximum spectrum of S, i.e., there exist $0 < A \leq B < \infty$ such that $AI \leq aI \leq S \leq bI \leq BI$. This means that A and B are estimations of the frame bounds of a frame whose frame operator is S. If $A = B$, then obviously $S = AI$.

So $S^{-1} = (1/A)I$. In this case the inversion is trivial. So throughout the discussion we assume that $A < B$.

Consider $p(t) := 1 - zt, z \in \mathbb{C}$. Notice that if $\|p(S)\| = \|I - zS\| < 1$, then $(zS)^{-1} = (I - (I - zS))^{-1} = \sum_{n=0}^{\infty}(I - zS)^n$. Hence we have a Neumann series expansion of S^{-1}:

$$S^{-1} = z\sum_{n=0}^{\infty}(I - zS)^n. \tag{3.1}$$

(3.1) converges in the operator norm topology of $\mathcal{B}(\mathcal{H})$, the Banach space of all the bounded linear operators from $\mathcal{H}$ into $\mathcal{H}$, and its rate of convergence is $r_z := \|1 - zS\|$.

It is now natural to look for the conditions on z under which $\|1-zS\| < 1$. That is, the conditions under which the series (3.1) converges. Let $z := x + iy$, $x, y \in \mathbb{R}$. Then $p(t) = 1 - zt = 1 - xt - iyt$. $p([A, B]) = \{p(t) : A \leq t \leq B\}$ is a line segment in $\mathbb{C}$.

$$\begin{aligned}|p(t)|^2 &= (1 - xt)^2 + y^2t^2\\ &= (x^2 + y^2)t^2 - 2xt + 1\\ &= (x^2 + y^2)\left\{\left(t - \frac{x}{x^2 + y^2}\right)^2 + \frac{y^2}{(x^2 + y^2)^2}\right\}.\end{aligned}$$

Hence $\{|p(t)|^2 : A \leq t \leq B\}$ is a piece of the parabola with its axis of symmetry $t = x/(x^2 + y^2)$. Notice that by the spectral mapping theorem $\sigma(I - zS) = \sigma(p(S)) = p(\sigma(S)) \subset p([A, B])$.

Case 1: $x \leq 0$. The axis of the parabola is on the left side of $t = 0$. $\|I - zS\|^2 \geq \min\{|p(t)|^2 : a \leq t \leq b\} \geq \min\{|p(t)|^2 : A \leq t \leq B\} \geq (1 - xA)^2 \geq 1$, where we have used the fact that $\|T\| = \max\{|\lambda| : \lambda \in \sigma(T)\}$ if T is a bounded normal operator. Hence (3.1) does not converge.

Case 2: $0 < x$ and $x/(x^2 + y^2) \leq A$. The axis of the parabola lies in between $t = 0$ and $t = A$. The range of $z = x + iy$ in this case is given by

$$\begin{aligned}R_2 &:= \left\{z = x + iy : \frac{x}{x^2 + y^2} \leq A, \quad x > 0\right\}\\ &= \left\{z : \left(x - \frac{1}{2A}\right)^2 + y^2 \geq \left(\frac{1}{2A}\right)^2, \quad x > 0\right\}.\end{aligned}$$

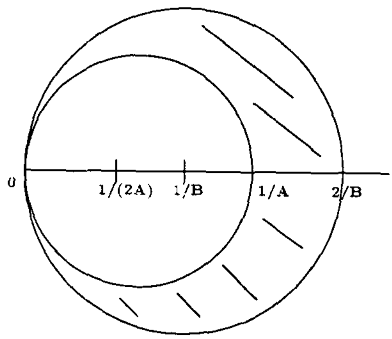

FIGURE 1. R_2^1

$\|1 - zS\|^2 \le \max\{|p(t)|^2 : A \le t \le B\} = |p(B)|^2 = (x^2+y^2)B^2 - 2xB + 1$. Hence $\|1 - zS\| < 1$ if $(x^2+y^2)B^2 - 2xB < 0$, i.e., $(x - 1/B)^2 + y^2 < 1/B^2$. Let

$$R_2^1 := \left\{ z : x > 0, \quad (x - \frac{1}{2A})^2 + y^2 \ge (\frac{1}{2A})^2, \quad (x - 1/B)^2 + y^2 < 1/B^2 \right\}$$

(see Figure 1). Notice that $R_2^1 \neq \emptyset$ only if $1/(2A) < 1/B$. So if the estimates of the spectrum of S satisfies $B < 2A$, then (3.1) converges for any $z \in R_2^1$, and the rate of convergence $\|I - zS\|^2 = r_z^2 \le {r'_z}^2 := (x^2 + y^2)B^2 - 2xB + 1 = B^2\{(x - 1/B)^2 + y^2\}$. It is easily seen that by looking at Figure 6 r'_z is minimized for $z \in R_2^1$ when $z = 1/A$. Now suppose that the estimates of the spectrum of S are exact, i.e., $A = a$ and $B = b$, then $r_z = r'_z$, since $A, B \in \sigma(S)$. So the best possible rate of convergence is $\sqrt{B^2\{(1/A - 1/B)^2 + y^2\}} = (B - A)/A \ (> (B - A)/(B + A))$.

Case 3: $x > 0$ and $x/(x^2 + y^2) \ge B$. The axis lies on the right of $t = B$. The range of z in this case is

$$R_3 := \left\{ z = x + iy : \frac{x}{x^2 + y^2} \ge B, \quad x > 0 \right\}$$
$$= \left\{ z : \left(x - \frac{1}{2B}\right)^2 + y^2 \le \left(\frac{1}{2B}\right)^2, \quad x > 0 \right\}.$$

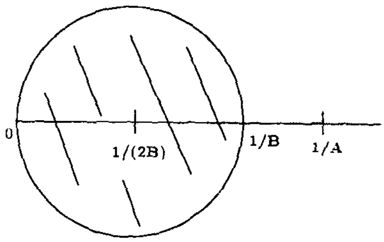

FIGURE 2. R_3

$\|1-zS\|^2 \leq |p(A)|^2 = (x^2+y^2)A^2-2xA+1 = A^2\{(x-1/A)^2+y^2\}$. Hence $\|1-zS\|^2 < 1$ if $(x^2+y^2)A^2-2xA+1 < 1$, i.e., if $(x^2+y^2)A^2-2xA < 1/A^2$. Notice that $\{z : (x^2+y^2)A^2-2xA < 1/A^2\} \cap R_3 = R_3$, since $1/(2B) < 1/A$. Hence for any $z \in R_3$, (3.1) converges and the rate of convergence is $r_z \leq r'_z := \sqrt{A^2\{(x-1/A)^2+y^2\}}$. r'_z is minimized for $z \in R_3$ when $z = 1/B$ (See Figure 2), and the minimum is $\sqrt{A^2\{(1/B-1/A)^2+y^2\}}$ $= (B-A)/B$ ($> (B-A)/(B+A)$). It is the best possible rate of convergence that can be expected for $z \in R_3$, if $A = a$ and $B = b$. Let us notice that Case 3 has some practical value that the ordinary Duffin-Schaeffer's method lacks. Note that R_3 depends only on B and not on A. It is well-known that upper frame bounds of frames may easily be estimated whereas lower frame bounds may not. Hence if we are in the situation that a given sequence is a frame with some upper frame bound B and that we do not have an explicit lower frame bound, then by taking any $z \in R_3$ we have a Neumann series expansion of the inverse of the frame operator. Of course, we do not have its rate of convergence explicitly, but certainly it is less than 1.

Case 4: $0 < x$ and $A \leq x/(x^2+y^2) \leq B$. The axis lies between $t = A$ and $t = B$. The range of z in this case is

R_4 :

$$= \left\{z : 0 < x,\ \ A \leq \frac{x}{x^2+y^2} \leq B\right\}$$

$$= \left\{z : 0 < x,\ \ \left(x-\frac{1}{2A}\right)^2+y^2 \leq \left(\frac{1}{2A}\right)^2,\ \ \left(x-\frac{1}{2B}\right)^2+y^2 \geq \left(\frac{1}{2B}\right)^2\right\}.$$

Note that

$$\begin{aligned}\|1-zS\|^2 &\le \max(|p(A)|^2,|p(B)|^2)\\ &= \max\left(A^2\left\{\left(x-\frac{1}{A}\right)^2+y^2\right\},\quad B^2\left\{\left(x-\frac{1}{B}\right)^2+y^2\right\}\right).\end{aligned}$$

Hence $\|1-zS\|^2<1$, if $z\in R_4^1:=\{z:(x-1/A)^2+y^2<1/(A^2),(x-1/B)^2+y^2<1/(B^2)\}$. Since $1/B<1/A$, $R_4\cap R_4^1=R_4^2$, where

$$\begin{aligned}R_4^2:=\Bigg\{z:0<x,\quad &\left(x-\frac{1}{2A}\right)^2+y^2\le\left(\frac{1}{2A}\right)^2,\\ &\left(x-\frac{1}{2B}\right)^2+y^2\ge\left(\frac{1}{2B}\right)^2,\\ &\left(x-\frac{1}{B}\right)^2+y^2<\left(\frac{1}{B}\right)^2\Bigg\}.\end{aligned}$$

If $1/(2A)\le 1/B$, R_4^2 is illustrated in Figure 3, and if $1/B\le 1/(2A)$, then R_4^2 is illustrated in Figure 4.

Now suppose $1/(2A)\le 1/B$ (Figure 3). First we find the region $R_4^3\subset\mathbb{C}$ in which $|p(A)|^2\le|p(B)|^2$. It is easy to see that

$$R_4^3=\left\{z:\left(x-\frac{1}{A+B}\right)^2+y^2\ge\frac{1}{(A+B)^2}\right\}.$$

Notice that $1/(2B)<1/(A+B)<1/(2A)$. Hence

$$\begin{aligned}R_4^4:=R_4^2\cap R_4^3=\Bigg\{z:0<x,\quad &\left(x-\frac{1}{2A}\right)^2+y^2\le\left(\frac{1}{2A}\right)^2,\\ &\left(x-\frac{1}{A+B}\right)^2+y^2\ge\left(\frac{1}{A+B}\right)^2,\\ &\left(x-\frac{1}{B}\right)^2+y^2<\left(\frac{1}{B}\right)^2\Bigg\}.\end{aligned}$$

See Figure 5. If $z\in R_4^4$, then (3.1) converges and the rate of convergence $r_z\le r_z':=|p(B)|=\sqrt{B^2\{(x-1/B)^2+y^2\}}$. Then in R_4^4 r_z' is minimized when $z=2/(A+B)$ (See Figure 5), and the minimum is

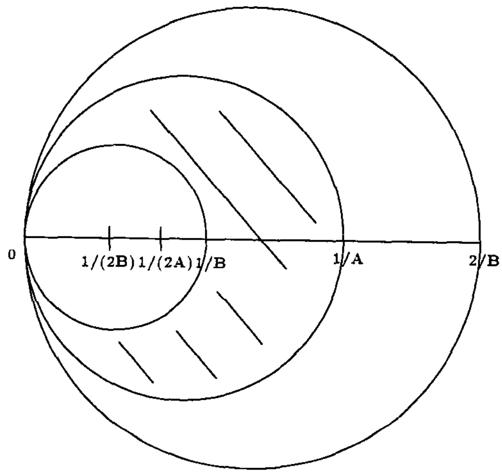

FIGURE 3. R_4^2 $(1/(2A) \leq 1/B)$

$\sqrt{B^2\{(2/(A+B)-1/B)^2+y^2\}} = (B-A)/(B+A)$. It is, as usual, the best possible rate of convergence if $A = a$ and $B = b$.

Similarly, we find the region $R_4^5 \subset \mathbb{C}$ such that $|p(A)|^2 \geq |p(B)|^2$.

$$R_4^5 = \left\{ z : \left(x - \frac{1}{A+B} \right)^2 + y^2 \leq \frac{1}{(A+B)^2} \right\}.$$

Let

$$\begin{aligned} R_4^6 := R_4^2 \cap R_4^5 = \Bigg\{ z : 0 < x, \quad & \left(x - \frac{1}{2A} \right)^2 + y^2 \leq \frac{1}{4A^2}, \\ & \left(x - \frac{1}{A+B} \right)^2 + y^2 \leq \frac{1}{(A+B)^2}, \\ & \left(x - \frac{1}{B} \right)^2 + y^2 < \frac{1}{B^2} \Bigg\}. \end{aligned}$$

See Figure 6. If $z \in R_4^6$, then (3.1) converges and the rate of convergence $r_z \leq r'_z := |p(A)| = \sqrt{A^2\{(x-1/A)^2+y^2\}}$. In R_4^6, r'_z is minimized when

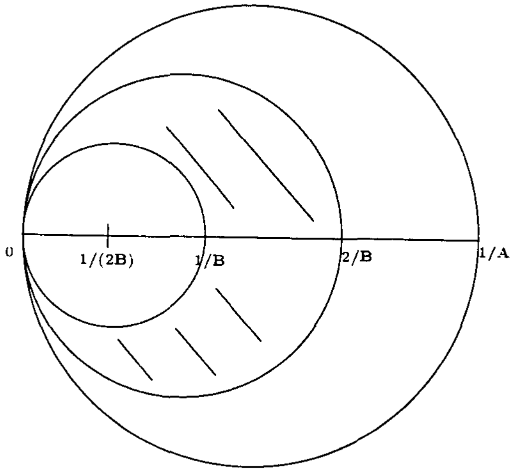

FIGURE 4. R_4^2 $(1/B \leq 1/(2A))$

$z = 2/(A+B)$ (See Figure 6), and the minimum is $\sqrt{A^2\{(2/(A+B) - 1/A)^2 + y^2\}} = (B-A)/(B+A)$. If $A = a$ and $B = b$, then it is the best possible rate of convergence. The case of $1/B \leq 1/(2A)$ can be handled similarly by looking at Figure 4 and by noticing that $1/B < 2/(A+B) < 2/B$.

We summarize our results in the following theorem.

THEOREM 3.1. *Suppose S is a bounded positive invertible operator satisfying $AI \leq S \leq BI$ for some positive numbers A and B. Then the Neumann series for S^{-1} in* (3.1) *converges if we take any $z \in R_2^1$ (if $B < 2A$) or R_3 or R_4^2. The best possible 'safe' rate of convergence of* (3.1) *is $(B-A)/A$ for $z \in R_2^1$, $(B-A)/B$ for $z \in R_3$, and $(B-A)/(B+A)$ for $z = 2/(A+B) \in R_4^2$.*

COROLLARY 3.2. *In the Neumann series expansion of the inverse of a frame operator, the choice that $z = 2/(A+B)$ is best possible among all complex numbers in the sense of the rate of convergence.*

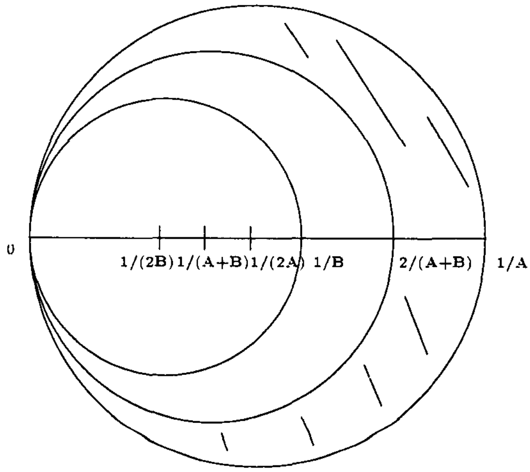

FIGURE 5. R_4^4

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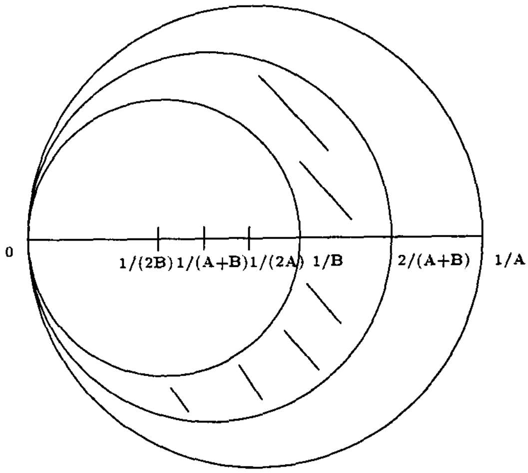

FIGURE 6. R_4^6

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Department of Mathematics
Korea Advanced Institute of Science and Technology
Taejon 305-701, Korea
E-mail: jaekun@ftn.kaist.ac.kr