Comm. Korean Math. Soc. **13** (1998), No. 4, pp. 801–810

FACTORIZATION IN KREIN SPACES

MEE HYEA YANG

ABSTRACT. Let $A(z)$, $W(z)$ and $C(z)$ be power series with operator coefficients such that $W(z) = A(z)C(z)$. Let $\mathcal{D}(A)$ and $\mathcal{D}(C)$ be the state spaces of unitary linear systems whose transfer functions are $A(z)$ and $C(z)$ respectively. Then there exists a Krein space $\mathcal{D}$ which is the state space of unitary linear system with transfer function $W(z)$. And the element of $\mathcal{D}$ is of the form

$$(f(z) + A(z)h(z), k(z) + C^*(z)g(z))$$

where $(f(z), g(z))$ is in $\mathcal{D}(A)$ and $(h(z), k(z))$ is in $\mathcal{D}(C)$.

1. Introduction and preliminaries

The invariant subspace theory can be interpreted as the theory of linear systems. There is a close relationship between invariant subspaces of the backward shift and a factorization of the transfer functions of linear systems. L. de Branges and Rovnyak generalized the Beuring's theorem using a factorization when coefficients spaces of linear systems are Hilbert spaces [1]. The factorization can be applied when multiplication by its transfer functions are contractive on the Krein space $\mathcal{C}(z)$ [4]. The factorization of transfer functions induces a multiplication of unitary linear systems. A fundamental problem is to construct a linear system with given transfer function. Complementation theory can be used to construct a conjugate isometric canonical linear system [2, 3]. Yang has shown the existence of a unitary linear system when multiplication by its transfer function is everywhere defined transformation in the Krein space of square summable power series [6].

Received March 6, 1998. Revised June 8, 1998.
1991 Mathematics Subject Classification: 47B37.
Key words and phrases: factorization, linear system.
This paper was supported by University of Inchon Research Fund.

A vector space $\mathcal{K}$ over the complex field with a scalar product is called a Krein space if $\mathcal{K}$ is an orthogonal sum of a Hilbert space $\mathcal{K}_+$ and the anti-space of a Hilbert space $\mathcal{K}_-$. Define operator J on $\mathcal{K}$ by

$$Jf = f_+ - f_- \quad \text{whenever} \quad f = f_+ + f_- , \quad f_+ \in \mathcal{K}_+, \quad f_- \in \mathcal{K}_-.$$

We call J the fundamental symmetry for the given fundamental decomposition. The space $\mathcal{K}$ can be considered as a Hilbert space $\mathcal{K}_J$ with the scalar product $\langle f, g\rangle_{\mathcal{K}_J} = \langle Jf, g\rangle_{\mathcal{K}}$. All J-norms defined by different fundamental decompositions are equivalent.

Let $\mathcal{H}$ and $\mathcal{C}$ be Krein spaces. A continuous linear transformation

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} : \mathcal{H} \oplus \mathcal{C} \longrightarrow \mathcal{H} \oplus \mathcal{C}$$

is called a linear system. The Krein space $\mathcal{H}$ is called the state space and the Krein space $\mathcal{C}$ is called the coefficient space or the external space. A linear system is said to be conjugate isometric if the adjoint matrix is isometric. The operator valued function $W(z)$ of the form

$$W(z) = D + C(zI - A)^{-1}B, \quad z \in \rho(A)$$

is called to be the transfer function of the linear system. A linear system is said to be observable if there is no nonzero element f of the state space such that $CA^n f = 0$ for every nonnegative integer n. A linear system is canonical if it is observable and every element of the state space is power series with vector coefficients. Then the main transformation A takes $f(z)$ into $[f(z) - f(0)]/z$ and the output transformation C takes $f(z)$ into $f(0)$.

Assume that the coefficient space $\mathcal{C}$ is a Krein space. Write $\mathcal{C}$ as the orthogonal sum of a Hilbert space $\mathcal{C}_+$ and the anti-space of a Hilbert space $\mathcal{C}_-$. Define

$$\mathcal{C}_+(z) = \{f\colon f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad a_n \in \mathcal{C}_+, \quad \sum_{n=0}^{\infty} \langle a_n, \quad a_n\rangle_{\mathcal{C}} < \infty\},$$

and

$$\mathcal{C}_-(z) = \{f\colon f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad a_n \in \mathcal{C}_-, \quad \sum_{n=0}^{\infty} \langle a_n, \quad a_n\rangle_{\mathcal{C}} > -\infty\}.$$

Then, the set of all square summable power series $\mathcal{C}(z)$ is written by the orthogonal sum of $\mathcal{C}_+(z)$ and $\mathcal{C}_-(z)$. $\mathcal{C}(z)$ is a Krein space with the scalar product

$$\langle f(z), g(z)\rangle_{\mathcal{C}(z)} = \sum_{n=0}^{\infty} \langle a_n, b_n\rangle_{\mathcal{C}}$$

where $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$. Then the space $\mathcal{C}(z)$ is the state space of conjugate isometric canonical linear system whose transfer function is identically zero.

2. Existence of unitary linear systems

Define

$$ext\ \mathcal{C}(z) = \left\{ f\colon f(z) = \sum_{-\infty}^{\infty} a_n z^n,\quad a_n \in \mathcal{C},\quad \sum_{-\infty}^{\infty} \langle J a_n, a_n\rangle_{\mathcal{C}} < \infty \right\}.$$

The space $\mathcal{C}(z)$ is considered as a Krein space with the unique scalar product such that

$$\langle f(z), f(z)\rangle_{ext\ \mathcal{C}(z)} = \sum_{-\infty}^{\infty} \langle a_n, a_n\rangle_{\mathcal{C}}.$$

Let $W(z)$ be a power series with operator coefficients such that multiplication by $W(z)$ is an everywhere defined transformation in the Krein space $\mathcal{C}(z)$. It has been shown that the adjoint of multiplication by $W(z)$ in $ext\ \mathcal{C}(z)$ is everywhere defined and it coincides with multiplication by $W(z^{-1})$ in $ext\ \mathcal{C}(z)$ [6]. This property can be used to construct a unitary linear system which has $W(z)$ as a transfer function. Let $ext\ \mathcal{G}(W)$ be the set of pairs $(u(z), v(z))$ of elements of $ext\ \mathcal{C}(z)$ such that $u(z) - W(z)z^{-1}v(z^{-1})$ and $v(z) - W^*(z)z^{-1}u(z^{-1})$ belong to $\mathcal{C}(z)$. It is a Hilbert space with respect to the scalar product

$$\begin{aligned} &\langle (u(z), v(z)), (u(z), v(z))\rangle_{ext\ \mathcal{G}(W)} \\ &= \langle J\ u(z), u(z)\rangle_{ext\ \mathcal{C}(z)} + \langle J\ v(z), v(z)\rangle_{ext\ \mathcal{C}(z)} \end{aligned}$$

holds for every element $(u(z), v(z))$ of $ext\ \mathcal{G}(W)$. Then $(u(z), z^{-1}v(z^{-1}))$ belongs to $ext\ \mathcal{G}(W)$ if the adjoint of multiplication by $W(z)$ in $\mathcal{C}(z)$ takes

$u(z)$ into $v(z)$, and $(z^{-1}u(z^{-1}), v(z))$ belongs to *ext* $\mathcal{G}(W)$ if the adjoint of multiplication by $W^*(z)$ in $\mathcal{C}(z)$ takes $v(z)$ into $u(z)$

Define the space *ext core*(W) to be the set of pairs

$$(u(z) - W(z)z^{-1}v(z^{-1}), -v(z) + W^*(z)z^{-1}u(z^{-1}))$$

with $(u(z), v(z)) \in ext\ \mathcal{G}(W)$. A unique scalar product is defined in the space so that the identity

$$\begin{aligned}
&\langle (u(z) - W(z)z^{-1}v(z^{-1}), -v(z) + W^*(z)z^{-1}u(z^{-1})), \\
&(u(z) - W(z)z^{-1}v(z^{-1}), -v(z) + W^*(z)z^{-1}u(z^{-1}))\rangle_{ext\ core(W)} \\
&= \langle u(z) - W(z)z^{-1}v(z^{-1}), u(z)\rangle_{ext\ \mathcal{C}(z)} \\
&+ \langle v(z) - W^*(z)z^{-1}u(z^{-1}), v(z))\rangle_{ext\ \mathcal{C}(z)}
\end{aligned}$$

is satisfied. The symmetry and the nondegeneracy of a scalar product can be easily verified.

A construction is made of a unitary linear system with transfer function $W(z)$.

THEOREM 2.1 [6]. *Assume that $W(z)$ is a power series with operator coefficients such that multiplication by $W(z)$ is an everywhere defined transformation in $\mathcal{C}(z)$. Then there exists a unitary linear system with transfer function $W(z)$ such that the state space $\mathcal{D}(W)$ contains ext core(W) isometrically and densely.*

3. Krein completion

Let $\mathcal{H}$ be a Hilbert space with a scalar product $\langle x, y\rangle_{\mathcal{H}}$ and J be a bounded self-adjoint transformation on $\mathcal{H}$. Define an indefinite scalar product such that $[x, y] = \langle Jx, y\rangle_{\mathcal{H}}$. The space $\mathcal{H}$ admits a decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_+ \oplus \mathcal{H}_-$ where $\mathcal{H}_0$ is the kernel of J, $\mathcal{H}_+$ is the largest subspace invariant under J such that $[x, x] > 0$ for $x \in \mathcal{H}$ and $\mathcal{H}_-$ is the largest subspace invariant under J such that $[x, x] < 0$ for $x \in \mathcal{H}$. Denote $\hat{\mathcal{H}}_+$ ($\hat{\mathcal{H}}_-$) by the Krein completion of $\mathcal{H}_+$ ($\mathcal{H}_-$, respectively) with respect to J. Then the spaces $\hat{\mathcal{H}}_+$ with scalar product $[x, y]$ and $\hat{\mathcal{H}}_-$ with the scalar product $-[x, y]$ become the Hilbert spaces. The space

$\hat{\mathcal{H}} = \hat{\mathcal{H}}_+ \oplus \hat{\mathcal{H}}_-$ is a Krein space which contains $\mathcal{H}$ as a dense vector subspace with the scalar product

$$[x,y]_{\hat{\mathcal{H}}} = [x_+,y_+]_{\hat{\mathcal{H}}_+} + [x_-,y_-]_{\hat{\mathcal{H}}_-}$$

where $x = x_+ + x_-$, $y = y_+ + y_-$, $x_\pm$ and $y_\pm \in \hat{\mathcal{H}}_\pm$, respectively.

A construction of continuous transformations in Krein spaces is due to Mark Krein [5].

THEOREM 3.1 [5]. *Let $\mathcal{A}$ and $\mathcal{B}$ be Hilbert spaces with the scalar product $\langle , \rangle_{\mathcal{A}}$ and $\langle , \rangle_{\mathcal{B}}$ respectively. Assume that $J_{\mathcal{A}}$ is a self-adjoint transformation of $\mathcal{A}$ into itself, that $J_{\mathcal{B}}$ is a self-adjoint transformation of $\mathcal{B}$ into itself, that U is a continuous transformation of $\mathcal{A}$ into $\mathcal{B}$ and that V is a continuous transformation of $\mathcal{B}$ into $\mathcal{A}$ such that*

$$\langle J_{\mathcal{B}}\, U\, a, b\rangle_{\mathcal{B}} = \langle J_{\mathcal{A}}\, a, V\, b\rangle_{\mathcal{A}}$$

for all elements a of $\mathcal{A}$ and b of $\mathcal{B}$. Let $\hat{\mathcal{A}}$ be the Krein completion of $\mathcal{A}$ with respect to $J_{\mathcal{A}}$ and let $\hat{\mathcal{B}}$ be the Krein completion of $\mathcal{B}$ with respect to $J_{\mathcal{B}}$. Then unique adjoint transformations $\hat{U}$ of $\hat{\mathcal{A}}$ into $\hat{\mathcal{B}}$ and $\hat{V}$ of $\hat{\mathcal{B}}$ into $\hat{\mathcal{A}}$ exist such that the identity

$$\langle J_{\mathcal{B}}\, \hat{U}\, a, b\rangle_{\mathcal{B}} = \langle J_{\mathcal{A}}\, a, \hat{V}\, b\rangle_{\mathcal{A}}$$

for all elements a of $\hat{\mathcal{A}}$ and b of $\hat{\mathcal{B}}$.

4. Factorization

Let $A(z)$, $W(z)$ and $C(z)$ be power series with operator coefficients such that $W(z) = A(z)C(z)$. Assume that multiplication by $A(z)$ and $C(z)$ are contractive transformations in $\mathcal{C}(z)$. Then multiplication by $W(z)$ is contractive transformation in $\mathcal{C}(z)$. L. de Branges [1] showed that there is a Krein space $\mathcal{D}(W)$ which is the state space of canonical linear system with transfer function $W(z)$ and that there is a partial isometry of Cartesian product Krein space $\mathcal{D}(A) \otimes \mathcal{D}(C)$ into $\mathcal{D}(W)$ which takes $((f(z), g(z)), (h(z), k(z)))$ into

$$(f(z) + A(z)h(z), k(z) + C^*(z)g(z)).$$

A construction can be generalized.

THEOREM 4.1. *Let $A(z)$ and $C(z)$ be power series with operator coefficients and $W(z) = A(z)C(z)$. Let $\mathcal{D}(A)$ and $\mathcal{D}(C)$ are the state spaces of unitary linear systems whose transfer functions are $A(z)$ and $C(z)$. Then the Cartesian product of $\mathcal{D}(A)$ and $\mathcal{C}(C)$ is a state space of a unitary linear system whose transfer function is $W(z)$.*

PROOF. Let $(f(z), g(z))$ be in $\mathcal{D}(A)$ and $(h(z), k(z))$ be in $\mathcal{D}(C)$. A continuous transformation A of $\mathcal{D}(A) \times \mathcal{D}(C)$ into itself is defined by taking $((f(z), g(z)), (h(z), k(z)))$ into $((p(z), q(z)), (u(z), v(z)))$ where

$$\begin{aligned} p(z) &= [f(z) - f(0)]/z + [A(z) - A(0)]h(0)/z \\ q(z) &= zg(z) - A^*(z)f(0) + [1 - A^*(z)A(0)]h(0) \\ u(z) &= ([h(z) - h(0)]/z \\ v(z) &= zk(z) - C^*(z)h(0). \end{aligned}$$

Then the adjoint transformation of A takes $((f(z), g(z)), (h(z), k(z)))$ into $((p(z), q(z)), (u(z), v(z)))$ where

$$\begin{aligned} p(z) &= zf(z) - A(z)g(0) \\ q(z) &= [g(z) - g(0)]/z \\ u(z) &= zh(z) - C(z)k(0) + [1 - C(z)C^*(0)]g(0) \\ u(z) &= [k(z) - k(0)]/z. \end{aligned}$$

A continuous transformation B of $\mathcal{C}$ into $\mathcal{D}(A) \times \mathcal{D}(C)$ is defined by taking c into $((p(z), q(z)), (u(z), v(z)))$ where

$$\begin{aligned} p(z) &= [A(z) - A(0)]C(0)c/z \\ q(z) &= [1 - A^*(z)A(0)]C(0)c \\ u(z) &= [C(z) - C(0)]c/z \\ v(z) &= [1 - C^*(z)C(0)]c. \end{aligned}$$

Then the adjoint transformation of B takes $((f(z), g(z)), (h(z), k(z)))$ into $k(0) + C^*(0)g(0)$.

A continuous transformation C of $\mathcal{D}(A) \times \mathcal{D}(C)$ into $\mathcal{C}$ is defined by taking $((f(z), g(z)), (h(z), g(z)))$ into $f(0) + A(0)h(0)$. Then the adjoint transformation of C takes c into $((p(z), q(z)), (u(z), v(z)))$ where

$$\begin{aligned} p(z) &= [1 - A(z)A^*(0)]C^*(0)c \\ q(z) &= [A^*(z) - A^*(0)]c/z \\ u(z) &= [1 - C(z)C^*(0)]c \\ u(z) &= [C^*(z) - C^*(0)]c/z. \end{aligned}$$

The external operator D is $A(0)C(0)$. Then the matrix transformation $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is unitary and

$$W(z) = D + \sum_{n=0}^{\infty} CA^n B z^n.$$

This completes the proof of the theorem. □

A construction will be made of an unitary linear system with transfer function $W(z)$.

THEOREM 4.2. *Let $A(z)$ and $C(z)$ be power series with operator coefficients and $W(z) = A(z)C(z)$. Let $\mathcal{D}(A)$ and $\mathcal{D}(C)$ are the state spaces of unitary linear systems whose transfer functions are $A(z)$ and $C(z)$. Assume that the set of all elements of the form*

$$((A(z)f(z), -g(z))), (-f(z), C^*(z)g(z)))$$

in $\mathcal{D}(A) \times \mathcal{D}(C)$ is a Krein space which is contained continuously and isometrically in $\mathcal{D}(A) \times \mathcal{D}(C)$. Then there is a Krein space $\mathcal{D}$ which is the state space of unitary linear space whose transfer function is $W(z)$ such that for $((f(z), g(z), (h(z), k(z)))$ in $\mathcal{D}(A) \times \mathcal{D}(C)$,

$$(f(z) + A(z)h(z), k(z) + C^*(z)g(z))$$

is in $\mathcal{D}$.

PROOF. Let $\mathcal{E}$ be the set of pairs $(f(z), g(z))$ of power series with vector coefficients such that $(A(z)f(z), -g(z))$ belongs to $\mathcal{D}(A)$ and $(-f(z), C^*(z)g(z))$ belongs to $\mathcal{D}(C)$. The space $\mathcal{E}$ becomes a Krein space with scalar product defined by the identity

$$\begin{aligned} &\langle (f(z), g(z)), (f(z), g(z))\rangle_{\mathcal{E}} \\ &= \langle (A(z)f(z), -g(z)), (A(z)f(z), -g(z))\rangle_{\mathcal{D}(A)} \\ &+ \langle (-f(z), C^*(z)g(z)), (-f(z), C^*(z)g(z))\rangle_{\mathcal{D}(C)}. \end{aligned}$$

Define a Krein space $\mathcal{D}$ be the set of pairs

$$(f(z) + A(z)h(z), k(z) + C^*(z)g(z))$$

where $((f(z), g(z)), (h(z), k(z)))$ in $\mathcal{D}(A) \times \mathcal{D}(C)$ with unique scalar product such that the transformation which takes $((f(z), g(z)), (h(z), k(z)))$ into $(f(z) + A(z)h(z), k(z) + C^*(z)g(z))$ acts as a partial isometry of $\mathcal{D}(A) \times \mathcal{D}(C)$ into $\mathcal{D}$. A similar argument in the proof of the Theorem 4.1 shows that $\mathcal{D}$ is a state space of a unitary linear system whose transfer function is $W(z)$.

This completes the proof of the theorem. □

A characterization of a state space can be made using Krein completion.

THEOREM 4.3. *Let $A(z)$, $W(z)$ and $C(z)$ be power series with operator coefficients such that $W(z) = A(z)C(z)$. Assume that multiplication by $A(z)$ and $C(z)$ are everywhere defined transformations in $\mathcal{C}(z)$. Then there is an isometric transformation of $\mathcal{D}(W)$ into $\mathcal{D}(A) \times \mathcal{D}(C)$ which takes $(p(z), q(z))$ into $((f(z), g(z)), (h(z), k(z))$ where*

$$(p(z), q(z)) = (f(z) + A(z)h(z), k(z) + C^*(z)g(z)).$$

PROOF. Denote the Hilbert spaces $ext\mathcal{G}(A)$, $ext\mathcal{G}(W)$ and $ext\mathcal{G}(C)$ by $\mathcal{G}(A)$, $\mathcal{G}(W)$ and $\mathcal{G}(C)$. Let $J_{\mathcal{G}(A)}$ be the unique self adjoint transformation of $\mathcal{G}(A)$ into itself such that the identity

$$\begin{aligned} &\langle J_{\mathcal{G}(A)}(u(z), v(z)), (u(z), v(z))\rangle_{\mathcal{G}(A)} \\ &= \langle u(z) - A(z)z^{-1}v(z^{-1}), u(z)\rangle_{ext\mathcal{C}(z)} \\ &+ \langle v(z) - A^*(z)z^{-1}u(z^{-1}), v(z)\rangle_{ext\mathcal{C}(z)} \end{aligned}$$

holds for every element $(u(z), v(z))$ of $\mathcal{G}(A)$. Similarly define the self adjoint transformations $J_{\mathcal{G}(W)}$ and $J_{\mathcal{G}(C)}$. Define T be a transformation from the Hilbert $\mathcal{G}(A)$ into the Hilbert space $\mathcal{G}(C)$ which takes $(u(z), v(z))$ of $\mathcal{G}(A)$ into $(z^{-1}v(z^{-1}), w(z))$ of $\mathcal{G}(C)$. By the closed graph theorem, the graph of T is closed hence it becomes to be a Krein space. Denote the graph of T by $\mathcal{G}(T)$. Define S to be a transformation from $\mathcal{G}(W)$ into $\mathcal{G}(T)$ which takes $((u(z), w(z))$ into

$$((u(z), v(z)), (z^{-1}v(z^{-1}), w(z)))$$

where $(u(z), v(z))$ is in $\mathcal{G}(A)$ and $(z^{-1}v(z^{-1}), w(z))$ is in $\mathcal{G}(C)$. Let $J_{\mathcal{G}(T)}$ be the unique self adjoint transformation of $\mathcal{G}(T)$ into itself such that the identity

$$\begin{aligned}
&\langle J_{\mathcal{G}(T)}((u(z), v(z)), (z^{-1}v(z^{-1}), w(z))), \\
&\qquad\qquad ((u(z), v(z)), (z^{-1}v(z^{-1}), w(z)))\rangle_{\mathcal{G}(T)} \\
&= \langle u(z) - A(z)z^{-1}v(z^{-1}), u(z)\rangle_{ext\mathcal{C}(z)} \\
&+ \langle v(z) - A^*(z)z^{-1}u(z^{-1}), v(z)\rangle_{ext\mathcal{C}(z)} \\
&+ \langle z^{-1}v(z^{-1}) - C(z)z^{-1}w(z^{-1}), z^{-1}v(z^{-1})\rangle_{ext\mathcal{C}(z)} \\
&+ \langle q(z) - C^*(z)v(z), w(z)\rangle_{ext\mathcal{C}(z)}
\end{aligned}$$

holds for $(u(z), v(z))$ in $\mathcal{G}(A)$ and $(z^{-1}v(z^{-1}), w(z))$ in $\mathcal{G}(C)$. Then the identity

$$\begin{aligned}
&\langle J_{\mathcal{G}(T)}S(u(z), w(z)), ((u(z), v(z)), (z^{-1}v(z^{-1}), w(z))\rangle_{\mathcal{G}(T)} \\
&= \langle J_{\mathcal{G}(W)}(u(z), w(z)), (u(z), w(z))\rangle_{\mathcal{G}(W)}
\end{aligned}$$

holds for every $(u(z), w(z))$ in $\mathcal{G}(W)$. Let $\mathcal{G}(\hat{W})$ be the Krein completion of $\mathcal{G}(W)$ with respect to $J_{\mathcal{G}(W)}$ and $\mathcal{G}(\hat{T})$ the Krein completion of $\mathcal{G}(T)$ with respect to $J_{\mathcal{G}(T)}$. Let $\pi_{\mathcal{G}(W)}$ be the projection of $\mathcal{G}(W)$ into $\mathcal{G}(\hat{W})$. By the theorem 3 unique adjoint transformation $\hat{S}$ and $\hat{S}^*$ such that the identity $\hat{S}\pi_{\mathcal{G}(W)} = \pi_{\mathcal{G}(T)}S$ and $\hat{S}^*\pi_{\mathcal{G}(T)} = \pi_{\mathcal{G}(B)}S^*$ are satisfied.

For every element $(u(z), w(z))$ in $\mathcal{G}(W)$

$$\pi S(u(z), w(z)) = (f(z) + A(z)h(z), k(z) + C^*(z)g(z))$$

where $\pi(u(z), v(z)) = (f(z), g(z))$ and $\pi(z^{-1}v(z), w(z)) = (h(z), k(z))$

This completes the proof of the theorem. □

References

[1] L. de Branges, *Canonical models in quantum scattering theory*, Perturbation Theory and its applications in Quantum Mechanics, Wiley, New York (1966), 295-391.

[2] ———, *Complementation in Krein spaces*, Trans. Amer. Math. Soc. **305** (1988), 277-291.

[3] ———, *A construction of Krein spaces of analytic functions*, J. Functional Analysis **98** (1991), 1-41.

[4] ———, *Factorization in Krein spaces*, J. Functional Analysis **124** (1994), 228-262.

[5] A. Dijksma, H. Langer and H. de Snoo, *Unitary colligations in Krein spaces and their role in the extension theory of isometries and symmetric linear relations in Hilbert spaces, Functional Analysis II, Proceedings Dubrovnic 1985*, Lecture Notes 1242, Springer-Verlag, Heidelberg, 1982, pp. 1-42.

[6] M. Yang, *A construction of unitary linear system*, Integral equa. and oper. Th. **19** (1994), 477-499.

Department of Mathematics
University of Inchon
Inchon 402-749, Korea