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A NOTE ON THE SAMPLE PATH-VALUED CONDITIONAL YEH-WIENER INTEGRAL

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ABSTRACT. In this paper we define a sample path-valued conditional Yeh-Wiener integral for function F of the type

$$E\big[F(x)|x(*,\frac{T}{S}*)=\psi(\blacktriangle)\big],$$

where ψ is in $C[0,\sqrt{S^2+T^2}]$ and $\blacktriangle=\frac{\sqrt{S^2+T^2}}{S^2}*^2$ and evaluate a sample path-valued conditional Yeh-Wiener integral using the result obtained.

1. Introduction

For $Q=[0,S]\times[0,T]$, let $C(Q)$ denote Yeh-Wiener space, i.e., the space of all real-valued continuous function $x(s,t)$ on Q. Yeh [3] defined a Gaussian measure m_y on $C(Q)$ such that as a stochastic process $\{x(s,t)|(s,t)\in Q\}$ has mean $E[x(s,t)]=\int_{C(Q)}x(s,t)m_y(dx)=0$ and covariance $E\,[x(s,t)x(u,v)]=\min\{s,u\}\min\{t,v\}$. Let $C[0,T]$ denote the standard Wiener space on $[0,T]$ with Wiener measure m_w and $E_w[x(s)]=\int_{C[0,T]}x(s)m_w(dx)$ denote the Wiener integral. In [2], Park and Skoug defined a sample path-valued conditional Yeh-Wiener integral of functionals for the condition $x(S,\bullet)=\psi(\bullet)$ where ψ is in $C[0,T]$ and they evaluated the conditional Yeh-Wiener integral.

The purpose of this paper is to treat conditional Yeh-Wiener integral with sample path-valued conditioning function. We first define a sample path-valued conditional Yeh-Wiener integral for function F of the type $E\big(F(x)|x(*,\frac{T}{S}*)=\psi(\blacktriangle)\big)$ where ψ is in $C[0,\sqrt{S^2+T^2}]$ and

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$\blacktriangle = \frac{\sqrt{S^2+T^2}}{S^2} *^2$. We then use this result to evaluate the sample path-valued conditional Yeh-Wiener integral of certain functional.

2. Sample path-valued conditional Yeh-Wiener integral

With a similar method as in [2], we obtain the following theorem.

THEOREM 1. *If F is a Yeh-Wiener integrable function on $C(Q)$, then we have*

$$(2.1) \qquad E_w\Big\{E\big[F(x)|x(*,T) = \sqrt{T}\psi(*)\big]\Big\} = E\big(F(x)\big)$$

for $*$ in $[0,S]$.

For a Yeh-Wiener integrable function F on $C(Q)$, consider the conditional Yeh-Wiener integral of the type

$$(2.2) \qquad E\big[F(x)|x(*,\frac{T}{S}*) = \psi(\blacktriangle)\big]$$

where ψ is in $C[0,\sqrt{S^2+T^2}]$ and

$$(2.3) \qquad \blacktriangle = \frac{\sqrt{S^2+T^2}}{S^2} *^2 .$$

Since $x(s,t) - \frac{S}{Ts}\min\{t, \frac{T}{S}s\}x(s,\frac{T}{S}s)$ are stochastically independent for (s,t) in Q, we have

$$(2.4) \quad E\Big(F(x)|x(*,\frac{T}{S}*) = \psi(\blacktriangle)\Big)$$
$$= E\Big(F\big(x(*,\bullet) - \frac{S}{T*}\min\{\bullet,\frac{T}{S}*\}x(*,\frac{T}{S}*) + \frac{S}{T*}\min\{\bullet,\frac{T}{S}*\}\psi(\blacktriangle)\big)\Big)$$

for almost all ψ in $C[0,\sqrt{S^2+T^2}]$.

Using (2,4), we obtain the following theorem.

THEOREM 2. *If F is Yeh-Wiener integrable function on $C(Q)$, then we have*

$$E_w\Big\{E\big[F(x)|x(*,\frac{T}{S}*)=\sqrt{\frac{ST}{\sqrt{S^2+T^2}}}\psi(\blacktriangle)\big]\Big\}=E\big(F(x)\big) \tag{2.5}$$

for $*$ in $[0,S]$ and $\blacktriangle$ given by (2.3).

PROOF. Using (2.4), the left side of (2.5) becomes

$$\begin{aligned}E_w\Big\{E\big[F\big(x(*,\bullet)&-\frac{S}{T*}\min\{\bullet,\frac{T}{S}*\}x(*,\frac{T}{S}*)\\&+\frac{S}{T*}\min\{\bullet,\frac{T}{S}*\}\sqrt{\frac{ST}{\sqrt{S^2+T^2}}}\psi(\blacktriangle)\big)\big]\Big\}.\end{aligned} \tag{2.6}$$

Let

$$\begin{aligned}y(s,t)=x(s,t)&-\frac{S}{Ts}\min\{t,\frac{T}{S}s\}x(s,\frac{T}{S}s)\\&+\frac{S}{Ts}\min\{t,\frac{T}{S}s\}\sqrt{\frac{ST}{\sqrt{S^2+T^2}}}\psi(u)\end{aligned} \tag{2.7}$$

where $u=\sqrt{S^2+T^2}s^2/S^2$. The mean of y is $E\big(y(s,t)\big)=0$ and the covariance of y is

$$E\big(y(s,t)y(u,v)\big)=\min\{s,u\}\min\{t,v\}. \tag{2.8}$$ □

Here $\frac{S}{Ts}\min\{t,\frac{T}{S}s\}$ is equal to 1 and less than 1 for the cases $\frac{T}{S}s\le t$ and $\frac{T}{S}s\ge t$, respectively. Thus $y(s,t)$ is the Yeh-Wiener process, and so (2.6) becomes $E\big(F(x)\big)$.

In the following example, we verify that (2.5) in Theorem 2 holds for the function $F(x)=\int_Q x^2(s,t)dsdt$. (2.1) in Theorem 1 can be obtained by the similar method as in [2].

EXAMPLE. Let $F(x) = \int_Q x^2(s,t)dsdt$. Then We have, using (2.4) and Fubini Theorem,

(2.9)
$$I \equiv E\Big(\int_Q x^2(s,t)dsdt \Big| x(s,\frac{T}{S}s) = w(\frac{\sqrt{S^2+T^2}}{S^2}s^2)\Big)$$
$$= \int_Q E\Big\{\big[(x(s,t) - \frac{S}{Ts}\min\{t,\frac{T}{S}s\}x(s,\frac{T}{S}s)$$
$$+ \frac{S}{Ts}\min\{t,\frac{T}{S}s\}w(\frac{\sqrt{S^2+T^2}}{S^2}s^2)\big]^2\Big\}dsdt$$

where ψ is in $C[0,\sqrt{S^2+T^2}]$. Since $x(s,t)$ is a Yeh-Wiener process, we have

(2.10)
$$I = \int_Q \Big\{st - \frac{S}{T}\big(\min\{t,\frac{T}{S}s\}\big)^2$$
$$+ \big(\frac{S}{Ts}\min\{t,\frac{T}{S}s\}w(\frac{\sqrt{S^2+T^2}}{S^2}s^2)\big)^2\Big\}dsdt.$$

If we replace $w\big(\sqrt{S^2+T^2}s^2/S^2\big)$ by

(2.11)
$$\sqrt{\frac{ST}{\sqrt{S^2+T^2}}}\psi\Big(\frac{\sqrt{S^2+T^2}}{S^2}s^2\Big)$$

and integrate in ψ over $C[0,\sqrt{S^2+T^2}]$, we have

(2.12)
$$E_w(I) = \frac{S^2T^2}{4} - \int_Q \Big\{\frac{S}{T}\big(\min\{t,\frac{T}{S}s\}\big)^2$$
$$- \big(\frac{S}{Ts}\min\{t,\frac{T}{S}s\}\big)^2\frac{ST}{\sqrt{S^2+T^2}}E_w\big((\psi(\frac{\sqrt{S^2+T^2}}{S^2}s^2))^2\big)\Big\}dsdt$$
$$= \frac{S^2T^2}{4}$$

where the last equality in (2,12) comes from the fact that

$$E_w\left(\left(\psi\left(\frac{\sqrt{S^2+T^2}}{S^2}s^2\right)\right)^2\right) = \frac{\sqrt{S^2+T^2}}{S^2}s^2. \tag{2.13}$$

Since $E\left[\int_Q x^2(s,t)dsdt\right] = S^2T^2/4$, we justify (2.5) in Theorem 2.

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