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REAL HYPERSURFACES WITH ξ-PARALLEL RICCI TENSOR IN A COMPLEX SPACE FORM

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ABSTRACT. We prove that if a real hypersurface with constant mean curvature of a complex space form satisfying $\nabla_\xi S = 0$ and $S\xi = \sigma\xi$ for a smooth function σ, then the structure vector field ξ is principal, where S denotes the Ricci tensor of the hypersurface.

1. Introduction

An n-dimensional complex space form $M_n(c)$ is a Kaehlerian manifold of constant holomorphic sectional curvature c. As is well known, complete and simply connected complex space forms are isometric to a complex projective space P_nC, a complex Euclidean space C_n or a complex hyperbolic space H_nC according as $c > 0,\ c = 0$ or $c < 0$.

Let M be a real hypersurfaces of $M_n(c)$. Then M has an almost contact metric structure (ϕ, ξ, η, g) induced from the Kaehlerian metric and complex structure J of $M_n(c)$. The structure vector field ξ is said to be *principal* if $A\xi = \alpha\xi$, where A is the shape operator in the direction of the unit normal C and $\alpha = \eta(A\xi)$. We denote by ∇ and S, the Levi-Civita connection with respect to the Riemannian metric tensor g and the Ricci tensor of type $(1,1)$ on M respectively. Takagi ([12]) classified all homogeneous real hypersurfaces of P_nC as six model spaces which are said to be A_1, A_2, B, C, D and E, and Cecil-Ryan ([3]) and Kimura ([8]) proved that they are realized as the tubes of constant radius over Kaehlerian submanifolds. Namely, he proved the following

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THEOREM A. *Let M be a homogeneous real hypersurface of P_nC. Then M is a tube of radius r over one of the following Kaehlerian submanifolds:*

(A_1) *a hyperplane $P_{n-1}C$, where $0 < r < \pi/2$,*
(A_2) *a totally geodesic $P_kC(1 \leq k \leq n-2)$, where $0 < r < \pi/2$,*
(B) *a complex quadric Q_{n-1}, where $0 < r < \pi/4$,*
(C) *$P_1C \times P_{(n-1)/2}C$, where $0 < r < \pi/4$ and $n(\geq 5)$ is odd,*
(D) *a complex Grassmann $G_{2,5}C$, where $0 < r < \pi/4$ and $n = 9$,*
(E) *a Hermitian symmetric space $SO(10)/U(5)$, where $0 < r < \pi/4$ and $n = 15$.*

Also Berndt ([2]) showed that all real hypersurfaces with constant principal curvatures of a complex hyperbolic space H_nC are realized as the tubes of constant radius over certain submanifolds when the structure vector ξ is principal. Nowadays in H_nC they are said to be of type A_0, A_1, A_2 and B.

It was already known that there does not exist any Einstein real hypersurface of $M_n(c)$, $c \neq 0$ (see Cecil and Ryan [3], Montiel [11]). As a generalization of this fact Ki ([5]) has shown that there is no real hypersurface with parallel Ricci tensor $\nabla S = 0$ of $M_n(c)$, $c \neq 0$. In such a situation, let us investigate the covariant derivative of the Ricci tensor in $M_n(c), c \neq 0$ along the structure vector ξ in such a way that $\nabla_\xi S = 0$. In order to prove our result we prepare the following theorems without proof:

THEOREM C [10]. *Let M be a real hypersurface in P_nC (≥ 3) on which ξ is a principal curvature vector and the focal map φ_r has constant rank on M. If $\nabla_\xi S = 0$, then M is locally congruent to one of (A_1), (A_2), (B), (C), (D) and (E).*

THEOREM D [6]. *Let M be a real hypersurface of $H_nC(n \geq 3)$. If the structure vector ξ is principal and $\nabla_\xi S = 0$, then M is locally congruent to one of (A_0), (A_1) and (A_2).*

In this paper let us consider the condition that ξ is an eigenvector of the Ricci tensor S, which is much more general notion than $A\xi = \alpha\xi$. From this point of views, it is proved in [4] and [7] that

THEOREM E. *Let M be a real hypersurface of $M_n(c), c \neq 0$. If it satisfies $\nabla_\xi S = 0$ and $A\xi$ is principal or $S\xi = \sigma\xi$ for some constant σ on M, then ξ is a principal curvature vector.*

From different viewpoints of Theorem E we prove the following:

THEOREM. *Let M be a real hypersurface with constant mean curvature in $M_n(c), c \neq 0$. If it satisfies $\nabla_\xi S = 0$ and $S\xi = \sigma\xi$ for a smooth function σ on M, then ξ is a principal curvature vector.*

Finally we want to give a remark that the notion of $S\xi = \sigma\xi$ is much more weaker than $A\xi = \alpha\xi$. For this we want to show an example in $M_n(c), c \neq 0$ which satisfies the condition $S\xi = \sigma\xi$ but its structure vector field ξ is not principal. In fact, Kimura [9] constructed a minimal ruled real hypersurface in P_nC. Its expression of the Weingarten map is given by

$$A\xi = \alpha\xi + \mu W, \quad AW = \mu\xi \quad \text{and} \quad AX = 0$$

for any vector field X orthogonal to ξ and W, where W is a unit vector field orthogonal to ξ. Then it satisfies naturally $S\xi = \sigma\xi$ for the Ricci tensor S of ruled real hypersurfaces in P_nC. Ahn, Lee and Suh [1] also constructed a ruled real hypersurface in H_nC. All manifolds in this paper are assumed to be connected and of class C^∞ and the real hypersurfaces are supposed to be orientable.

1. Preliminaries

First of all, we review that fundamental properties of a real hypersurface of a complex space form (cf. [4], [7]).

Let $M_n(c)$ be a real $2n$-dimensional complex space form equipped with parallel almost complex structure J and a Riemannian metric tensor G which is J-Hermitian, and covered by a system of coordinate neighborhoods $\{\tilde{V}; x^A\}$.

Let M be a real $(2n-1)$-dimensional hypersurface of $M_n(c)$ covered by a system of coordinate neighborhoods $\{V; y^h\}$ and immersed isometrically in $M_n(c)$ by the immersion $i : M \to M_n(c)$. Throughout the present paper the following convention on the range of indices are used, unless otherwise stated:

$$A, B, \cdots = 1, 2, \cdots, 2n; \quad i, j, \cdots = 1, 2, \cdots, 2n-1.$$

The summation convention will be used with respect to those system of indices. We represent the immersion i locally by $x^A = x^A(y^h)$ and $B_j = (B_j{}^A)$ are also $(2n-1)$-linearly independent local tangent vectors of M, where $B_j{}^A = \partial_j x^A$ and $\partial_j = \partial/\partial y^j$. A unit normal C to M may then be chosen. The induced Riemannian metric g with components g_{ji} on M is given by $g_{ji} = G_{BA}B_j{}^B B_i{}^A$ because the immersion is isometric. For the tangent vector B_i and unit normal C to M, the following representations are obtained in each coordinate neighborhood:

$$JB_i = \phi_i{}^h B_h + \xi_i C, \qquad JC = -\xi^i B_i,$$

where we have put $\phi_{ji} = G(JB_j, B_i)$ and $\xi_i = G(JB_i, C)$, ξ^h being components of a vector field ξ associated with ξ_i and $\phi_{ji} = \phi_j{}^r g_{ri}$. By the properties of the almost Hermitian structure J, it is clear that ϕ_{ji} is skew-symmetric. A tensor field of type $(1,1)$ with components $\phi_i{}^h$ will be denoted by ϕ. By the properties of the almost complex structure J, the following relations are then given:

$$\phi_i{}^r\phi_r{}^h = -\delta_i{}^h + \xi_i\xi^h, \quad \xi^r\phi_r{}^h = 0, \quad \xi_r\phi_i{}^r = 0, \quad \xi_i\xi^i = 1,$$

that is, the aggregate (ϕ, g, ξ) defines an almost contact metric structure.

Denoting by ∇_j the operator of van der Waerdern-Bortolotti covariant differentiation with respect to the induced Riemannian metric, equations of the Gauss and Weingarten for M are respectively obtained:

$$\nabla_j B_i = A_{ji}C, \quad \nabla_j C = -A_j{}^r B_r.$$

where $H = (A_{ji})$ is a second fundamental form and $A = (A_j{}^h)$, which is related by $A_{ji} = A_j{}^r g_{ri}$ is the shape operator derived from C. By means of above equations the covariant derivatives of the structure tensors are yielded:

$$\nabla_j\phi_i{}^h = -A_{ji}\xi^h + A_j{}^h\xi_i, \quad \nabla_j\xi_i = -A_{jr}\phi_i{}^r. \tag{1.1}$$

Since the ambient space is a complex space form, equations of the Gauss and Codazzi for M are respectively given by

$$\begin{aligned} R_{kjih} = {} & \frac{c}{4}(g_{kh}g_{ji} - g_{jh}g_{ki} + \phi_{kh}\phi_{ji} - \phi_{jh}\phi_{ki} - 2\phi_{kj}\phi_{ih}) \\ & + A_{kh}A_{ji} - A_{jh}A_{ki}, \end{aligned} \tag{1.2}$$

$$\nabla_k A_{ji} - \nabla_j A_{ki} = \frac{c}{4}(\xi_k \phi_{ji} - \xi_j \phi_{ki} - 2\xi_i \phi_{kj}), \tag{1.3}$$

where R_{kjih} are components of the Riemannian curvature tensor R of M.

Hereafter, to write our formulas in convention forms, let us denote by ${A_{ji}}^2 = A_{jr}{A_i}^r, h = g_{ji}A^{ji}, \alpha = A_{ji}\xi^j\xi^i$ and $\beta = {A_{ji}}^2\xi^j\xi^i$. If we put $U_j = \xi^r\nabla_r\xi_j$, then U is orthogonal to the structure vector ξ. Because of the properties of the almost contact metric structure and the second equation of (1.1), we get

$$\phi_{jr}U^r = A_{jr}\xi^r - \alpha\xi_j, \tag{1.4}$$

which shows that $g(U,U) = \beta - \alpha^2$. By the definition of U and the second equation of (1.1), we see that

$$U^r\nabla_j\xi_r = {A_{jr}}^2\xi^r - \alpha A_{jr}\xi^r. \tag{1.5}$$

On the other hand, differentiating (1.4) covariantly along M and making use of (1.1), we find

$$\xi_j A_{kr}U^r + \phi_{jr}\nabla_k U^r = \xi^r\nabla_k A_{jr} - A_{jr}A_{ks}\phi^{rs} - \alpha_k\xi_j + \alpha A_{kr}{\phi_j}^r, \tag{1.6}$$

which shows that

$$(\nabla_k A_{ji})\xi^j\xi^i = 2A_{kr}U^r + \alpha_k, \tag{1.7}$$

where $\alpha_k = \partial_k\alpha$.

Transforming (1.6) by ${\phi_i}^j$ and taking account of (1.1) and (1.5), we have

$$\nabla_k U_i + \xi_i {A_{kr}}^2\xi^r + \xi^r(\nabla_k A_{sr}){\phi_i}^s = (\nabla_k\xi^r)(\nabla_r\xi_i) + \alpha A_{ki}. \tag{1.8}$$

By the definition of U, (1.1), (1.7) and (1.8) it is verified that

$$\xi^r\nabla_r U_j = -3U^s A_{rs}{\phi_j}^r + \alpha A_{jr}\xi^r - \beta\xi_j - \phi_{jr}\alpha^r. \tag{1.9}$$

We put

$$A_{jr}\xi^r = \alpha\xi_j + \mu W_j, \tag{1.10}$$

where W is a unit vector field orthogonal to ξ. Then from (1.4) we see that $U = -\mu\phi W$, and W is also orthogonal to U. We assume that $\mu \neq 0$ on M, that is, ξ is not a principal curvature vector field and we put $\Omega = \{p \in M \mid \mu(p) \neq 0\}$. Then Ω is an open subset of M and hereafter we discuss our argument on Ω otherwise stated.

By (1.2), the Ricci tensor S of M is given by

$$S_{ji} = \frac{c}{4}\{(2n+1)g_{ji} - 3\xi_j\xi_i\} + hA_{ji} - {A_{ji}}^2, \tag{1.11}$$

where $h = trA$. Since ξ and W are mutually orthogonal, we see, using (1.1), (1.4) and (1.10), that

$$\mu\xi_r\nabla_j W^r = A_{jr}U^r. \tag{1.12}$$

2. Real hypersurfaces satisfying $\nabla_\xi S = 0$

Let M be a real hypersurface with constant mean curvature of $M_n(c)$, $c \neq 0$. Then by (1.11) we have

$$\nabla_k S_{ji} = -\frac{3}{4}c(\xi_i\nabla_k\xi_j + \xi_j\nabla_k\xi_i) + h\nabla_k A_{ji} - (\nabla_k A_{jr}){A_i}^r - (\nabla_k A_{ir}){A_j}^r.$$

In the sequel we suppose that the Ricci tensor is parallel in ξ-direction, that is , $\nabla_\xi S = 0$. Then the last equation gives

$$\begin{aligned} &\frac{3}{4}c(U_j\xi_k + U_k\xi_j) + \frac{c}{4}(A_{kr}{\phi_j}^r + A_{jr}{\phi_k}^r) - \frac{c}{4}h\phi_{jk} \\ &= h(\nabla_j A_{kr})\xi^r - A_{kr}(\nabla_j {A_t}^r)\xi^t - A_{jr}(\nabla_k {A_t}^r)\xi^t. \end{aligned} \tag{2.1}$$

Furthermore, we suppose that the structure vector field ξ is an eigenvector of the Ricci tensor, namely

$$S_{jr}\xi^r = \sigma\xi_j \tag{2.2}$$

for some smooth function σ on M. Then (1.11) implies that

$${A_{jr}}^2\xi^r = hA_{jr}\xi^r + (\beta - h\alpha)\xi_j, \tag{2.3}$$

where we have put

$$\beta - h\alpha = \frac{c}{2}(n-1) - \sigma. \tag{2.4}$$

Thus (1.5) is reduced to

$$U^r \nabla_j \xi_r = (h-\alpha) A_{jr} \xi^r + (\beta - h\alpha)\xi_j. \tag{2.5}$$

From (1.10) and (2.3), it is seen that

$$A_{jr} W^r = \mu \xi_j + (h-\alpha) W_j \tag{2.6}$$

on Ω, where $\mu^2 = \beta - \alpha^2$ and hence

$$A_{jr}{}^2 W^r = h A_{jr} W^r + (\beta - h\alpha) W_j. \tag{2.7}$$

Differentiating (2.6) covariantly along Ω, we find

$$\begin{aligned}(\nabla_k A_{jr}) W^r + A_{jr} \nabla_k W^r \\ = \mu_k \xi_j + \mu \nabla_k \xi_j + (h_k - \alpha_k) W_j + (h-\alpha)\nabla_k W_j.\end{aligned} \tag{2.8}$$

By transvecting W^j and taking account of (1.12) and (2.6), we obtain

$$(\nabla_k A_{rs}) W^r W^s = -2A_{kr} U^r + h_k - \alpha_k. \tag{2.9}$$

We transvect also ξ^j to (2.8) and making use of (2.6), we can get

$$(\nabla_k A_{rs}) W^r \xi^s + \alpha \xi_r \nabla_k W^r = \mu_k + (h-\alpha)\xi_r \nabla_k W^r$$

on Ω. Since we have by the definition of μ

$$\mu \mu_k = \frac{1}{2}\beta_k - \alpha\alpha_k, \tag{2.10}$$

the above equation turns out to be

$$\mu(\nabla_k A_{rs}) \xi^r W^s = (h - 2\alpha) A_{kr} U^r + \frac{1}{2}\beta_k - \alpha\alpha_k, \tag{2.11}$$

where we have used (1.12).

Differentiating (2.2) covariantly along Ω, we find

$$(\nabla_k S_{jr})\xi^r + S_{jr}\nabla_k\xi^r = \sigma_k\xi_j + \sigma\nabla_k\xi_j.$$

Since we have $\nabla_\xi S = 0$, it follows that

$$S_{jr}U^r = (\sigma_t\xi^t)\xi_j + \sigma U_j,$$

which together with (2.2) yields

$$\sigma_t\xi^t = 0. \tag{2.12}$$

Thus the last equation becomes

$$A_{jr}{}^2U^r = hA_{jr}U^r + (\beta - h\alpha + \frac{3}{4}c)U_j, \tag{2.13}$$

because we have (1.11) and (2.4).

If we differentiate (2.5) covariantly and take account of (1.1), (1.4) and (1.10), we have

$$\begin{aligned}&(\nabla_k U_r)(\nabla_j\xi^r) + (A_{jr}\xi^r)(A_{ks}U^s) + \mu(\nabla_k A_{jr})W^r + \alpha_k A_{jr}\xi^r\\ &= (h-\alpha)(\nabla_k A_{jr})\xi^r - (h-\alpha)A_{jr}A_{ks}\phi^{rs}\\ &\quad + (\beta - h\alpha)_k\xi_j + (\beta - h\alpha)\nabla_k\xi_j,\end{aligned}$$

which implies

$$hA_{jr}U^r + 2(\beta - h\alpha + c)U_j + A_{jr}\alpha^r = h\alpha_j - \frac{1}{2}\beta_j, \tag{2.14}$$

where we have used (1.3), (1.7), (1.9), (2.9), (2.11), (2.12) and (2.13).

Taking the inner product (2.1) with $A_s{}^k\xi^s$ and using (1.3), (1.7), (2.3) and (2.9), we find

$$\begin{aligned}&\left(h^2 + 2\beta - 2h\alpha - \frac{c}{4}\right)A_{jr}U^r + \left\{h\beta - h^2\alpha + \frac{3}{4}c(h+\alpha)\right\}U_j\\ &+ \frac{1}{2}A_{jr}\beta^r + (\beta - h\alpha)\alpha_j = 0.\end{aligned} \tag{2.15}$$

Combining (2.14) with (2.15), we obtain

$$A_{jr}{}^2\alpha^r - hA_{jr}\alpha^r - (\beta - h\alpha)\alpha_j = \frac{3}{4}c(\alpha U_j - 3A_{jr}U^r). \tag{2.16}$$

Thus, the last three equations give the following:

$$\begin{aligned}&\frac{1}{2}\{A_{jr}{}^2\beta^r - hA_{jr}\beta^r - (\beta - h\alpha)\beta_j\}\\ &= \frac{3}{4}c\{(\beta + \frac{c}{4})U_j - (h+\alpha)A_{jr}U^r\}.\end{aligned} \tag{2.17}$$

Differentiating (2.3) covariantly along Ω and making use of (1.1), we get

$$\begin{aligned}&(\nabla_k A_j{}^s)A_{sr}\xi^r + A_{js}(\nabla_k A_r{}^s)\xi^r - A_{jr}{}^2 A_{ks}\phi^{rs}\\ &= h(\nabla_k A_{jr})\xi^r - hA_{jr}A_{ks}\phi^{rs} + (\beta_k - h\alpha_k)\xi_j - (\beta - h\alpha)A_{kr}\phi_j{}^r\end{aligned}$$

because the mean curvature of M is constant, from which taking the skew-symmetric part with respect to indices k and j and using (1.3),

(2.18)

$$\begin{aligned}&\frac{c}{4}(U_k\xi_j - U_j\xi_k) + \frac{c}{2}(h-\alpha)\phi_{kj} + A_{kr}{}^2A_{js}\phi^{rs} - A_{jr}{}^2A_{ks}\phi^{rs}\\ &+ 2hA_{jr}A_{ks}\phi^{rs} + (\beta - h\alpha)(A_{kr}\phi_j{}^r - A_{jr}\phi_k{}^r)\\ &= A_{ks}(\nabla_j A_r{}^s)\xi^r - A_{js}(\nabla_k A_r{}^s)\xi^r + (\beta_k - h\alpha_k)\xi_j - (\beta_j - h\alpha_j)\xi_k.\end{aligned}$$

If we take the inner product this with μW^j and use (1.3), (2.6), (2.7), (2.11), (2.13), (2.16) and (2.17), then we can obtain (for detail see [4])

$$\left\{2(\beta - \alpha^2) - \frac{c}{4}\right\}A_{jr}U^r = \{h(\beta - \alpha^2) + c(\alpha - h)\}U_j. \tag{2.19}$$

3. Proof of theorem

In the first place, we prove that

LEMMA 1. *Let M be a real hypersurface with constant mean curvature in $M_n(c), c \neq 0$. If it satisfies $\nabla_\xi S = 0$ and (2.2), then*

$$A_{jr}U^r = \lambda U_j, \tag{3.1}$$

where λ is defined by

$$(\beta - \alpha^2)(2\lambda - h) = \frac{c}{4}(\lambda + 4\alpha - 4h). \tag{3.2}$$

PROOF. Let M_0 be the set of points in Ω such that U is not principal. By (2.19) we then have $\beta - \alpha^2 = \frac{c}{8}$ on M_0. Thus we may only consider the case where $c = 4$ and hence $\beta - \alpha^2 = \frac{1}{2}$ on M_0 because $\beta - \alpha^2$ is non-negative. Therefore (2.18) implies $h = \frac{8}{7}\alpha$ and consequently α and β are constant on M_0 since the mean curvature of M is constant. Thus (2.16) and (2.17) are reduced respectively to

$$3A_{jr}U^r - \alpha U_j = 0, \quad (h + \alpha)A_{jr}U^r - (\beta + 1)U_j = 0$$

on M_0, which produce a contradiction. Hence (2.19) implies (3.1) and (3.2). This completes the proof. □

On the other hand, because of (2.13) and (3.1) we have

$$\lambda^2 = h\lambda + \beta - h\alpha + \frac{3}{4}c \tag{3.3}$$

on Ω, which implies

$$(2\lambda - h)\lambda_j = \beta_j - h\alpha_j. \tag{3.4}$$

Differentiating (3.2) covariantly along Ω, we find

$$(2\lambda - h)\beta_j - \{2\alpha(2\lambda - h) + c\}\alpha_j + \left\{2(\beta - \alpha^2) - \frac{c}{4}\right\}\lambda_j = 0. \tag{3.5}$$

Since h is constant, it follows, using (2.4) and (2.12), that

$$\beta_t\xi^t = h\alpha_t\xi^t. \tag{3.6}$$

REMARK. The function $2\lambda - h$ has no zero point on Ω. In fact, let Ω_0 be a set of points in Ω such that $2\lambda - h = 0$. Then $\beta_j - h\alpha_j = 0$ on the connected components of Ω_0 because of (3.4), and hence σ is constant by virtue of (2.4). Therefore the set Ω is empty because of Theorem E.

From (3.4) and (3.6), we see that $(2\lambda - h)\lambda_t\xi^t = 0$ and consequently $\lambda_t\xi^t = 0$ because $2\lambda - h$ has no zero point. Thus if we take the inner product (3.5) with ξ^j and make use of (3.6), then we obtain $\{(h - 2\alpha)(2\lambda - h) - c\}\alpha_t\xi^t = 0$ and hence $(2\lambda - h)\alpha_t\xi^t = 0$ on Ω. Accordingly we have

$$\alpha_t\xi^t = 0. \tag{3.7}$$

In the next place we prove the following:

LEMMA 2. *Under the same assumptions as those stated in* Lemma 1, *we have*

$$(\beta-\alpha^2)\lambda_j=(\lambda_t U^t)U_j. \tag{3.8}$$

PROOF. Taking the inner product (2.1) with U^k and making use of (1.4), (1.10) and (3.1), we find

$$\begin{aligned}&(h-\lambda)(\nabla_j A_{rs})U^r\xi^s-A_j{}^r(\nabla_k A_{rs})U^k\xi^s\\&=\frac{3}{4}c(\beta-\alpha^2)\xi_j+\frac{c}{4}\mu(\lambda-h)W_j-\frac{c}{4}\mu A_{jr}W^r.\end{aligned}$$

Transvecting (2.18) with U^k and using (1.4), (1.10), (2.6) and (2.7), we also obtain

$$\begin{aligned}&\lambda(\nabla_j A_{rs})\xi^r U^s-A_j{}^r(\nabla_k A_{rs})U^k\xi^s\\&=\{\frac{c}{4}(\beta-\alpha^2)-\beta_t U^t+h\alpha_t U^t\}\xi_j-\frac{c}{2}(h-\alpha)W_j-\frac{3}{4}c\mu A_{jr}W^r.\end{aligned}$$

Combining the last two equations, it follows that

$$\begin{aligned}&(h-2\lambda)(\nabla_j A_{rs})\xi^r U^s\\&=\{\beta_t U^t-h\alpha_t U^t+c(\beta-\alpha^2)\}\xi_j+\frac{c}{4}\mu(\lambda+3h-4\alpha)W_j\end{aligned} \tag{3.9}$$

since we have (2.6).

On the other hand, differentiating the second equations of (1.1) covariantly and using (1.1), we have

$$\nabla_k\nabla_j\xi_i=(A_{jr}\xi^r)A_{ki}-A_{jk}{}^2\xi_i-(\nabla_k A_{jr})\phi_i{}^r,$$

which together with (1.3) and (3.1) yields

$$U^k\xi^j\nabla_k\nabla_j\xi_i=(\alpha\lambda+\frac{c}{2})U_i-(\nabla_j A_{st})\xi^s U^t\phi_i{}^j.$$

From this and (3.9) we get

$$(h-2\lambda)U^k\xi^j\nabla_k\nabla_j\xi_i=\{(\alpha\lambda+\frac{c}{2})(h-2\lambda)+\frac{c}{4}(\lambda+3h-4\alpha)\}U_j. \tag{3.10}$$

Since we have $U_i = \xi^r \nabla_r \xi_i$, differentiation covariantly gives

$$\nabla_j U_i = (\nabla_j \xi^r)(\nabla_r \xi_i) + \xi^r \nabla_j \nabla_r \xi_i,$$

which implies

$$U^j \nabla_j U_i = -\lambda(h-\alpha)U_i + U^j \xi^r \nabla_j \nabla_r \xi_i,$$

because we have (1.1), (1.4), (1.10), (2.6) and (3.1). Substituting this into (3.10), we find

$$(h-2\lambda)U^j \nabla_j U_i = \left\{(h-2\lambda)\left(2\alpha\lambda - \lambda^2 + \frac{c}{2}\right) + \frac{c}{4}(\lambda + 3h - 4\alpha)\right\} U_i,$$

which together with (3.1) implies that

$$A_{kr}U^j \nabla_j U^r - \lambda U^j \nabla_j U_k = 0 \tag{3.11}$$

because $2\lambda - h$ has no zero point.

Differentiating (3.1) covariantly along Ω, we get

$$(\nabla_k A_{jr})U^r + A_j{}^r \nabla_k U_r = \lambda_k U_j + \lambda \nabla_k U_j,$$

from which, taking the skew-symmetric part and making use of (1.3),

$$\begin{aligned}&\frac{c}{4}(\xi_k \phi_{jr} U^r - \xi_j \phi_{kr} U^r) + A_j{}^r \nabla_k U_r - A_k{}^r \nabla_j U_r \\ &= \lambda_k U_j - \lambda_j U_k + \lambda(\nabla_k U_j - \nabla_j U_k).\end{aligned}$$

If we take the inner product U^k to this and use (3.1) and (3.11), then we obtain (3.8). Hence we arrive at the conclusion. □

Finally we are going to prove, using Lemma 1 and Lemma 2, that the structure vector field ξ is principal.

Transvecting (2.16) and (2.17) with U^j and taking account of (2.13) and (3.1) we have respectively

$$\alpha_t U^t = (\alpha - 3\lambda)(\beta - \alpha^2), \quad \beta_t U^t = \left\{2\beta + \frac{c}{2} - (h+\alpha)\lambda\right\}(\beta - \alpha^2), \tag{3.12}$$

where we have used (2.3) and (3.1). From these and (3.4), it is seen that

$$(2\lambda - h)\lambda_t U^t = \left(\lambda h - 3\alpha\lambda + 2\beta + \frac{c}{2}\right)(\beta - \alpha^2). \tag{3.13}$$

Thus, (3.8) turns out to be

$$(2\lambda - h)\lambda_j = (\lambda h - 3\alpha\lambda + 2\beta + \frac{c}{2})U_j. \tag{3.14}$$

Differentiating (3.14) covariantly along Ω, we find

$$\begin{aligned} &2\lambda_k\lambda_j + (2\lambda - h)\nabla_k\lambda_j \\ &= (h\lambda_k - 3\lambda\alpha_k - 3\alpha\lambda_k + 2\beta_k)U_j + (\lambda h - 3\alpha\lambda + 2\beta + \frac{c}{2})\nabla_k U_j, \end{aligned}$$

from which, taking the skew-symmetric part and using (3.14),

$$\begin{aligned} &(2\beta_k - 3\lambda\alpha_k)U_j - (2\beta_j - 3\lambda\alpha_j)U_k \\ &+ (\lambda h - 3\alpha\lambda + 2\beta + \frac{c}{2})(\nabla_k U_j - \nabla_j U_k) = 0. \end{aligned}$$

Transvecting this with ξ^k and making use of (3.6), (3.7) and (3.13), we obtain

$$(2\lambda - h)(\lambda_t U^t)\xi^k(\nabla_k U_j - \nabla_j U_k) = 0.$$

Thus, it is, owing to Theorem E and Remark, seen that

$$\xi^k(\nabla_k U_j - \nabla_j U_k) = 0$$

because of (3.4), (3.12) and (3.13). By using (1.1), (1.4), (1.9), (1.10), (2.6) and (3.1), the above equation is reduced to

$$\phi_{jr}\alpha^r = \mu(h - 3\lambda)W_j.$$

If we take the inner product W^j to this, then we have $\alpha_t U^t = \mu^2(h-3\lambda)$, which together with the first equation of (3.12) gives $h = \alpha$ and hence α is constant. Thus λ is constant on Ω. Therefore β is constant on Ω because of (3.4) and consequently $\sigma = const.$ since we have (2.4). By Theorem E, we see that ξ is a principal curvature vector. This completes the proof.

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