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A MODULUS OF CONTINUITY ON THE INCREMENTS OF A TWO-PARAMETER GAUSSIAN PROCESS

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ABSTRACT. A modulus of continuity on the increments of a two-parameter Gaussian process is obtained via estimating large deviation probability inequalities on the suprema of the Gaussian process.

1. Introduction and results

Let $\{X(x,y),\ 0 \le x \le 1,\ 0 \le y \le 1\}$ be an almost surely continuous, centered two-parameter Gaussian process with $X(0,0) = 0$. For two distinct points (x_1, y_1), (x_2, y_2) in $[0,1] \times [0,1]$, we always assume that $X(x,y)$ has

$$\text{(i)}\quad E\{X(x_1,y_1) - X(x_2,y_2)\}^2 = \sigma^2\left(\sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}\right),$$

where $\sigma(t), t > 0$, is a nondecreasing continuous, regularly varying function with exponent γ at zero for some $0 < \gamma < 1$, that is, $\sigma(t) = t^\gamma L(t)$ for some $0 < \gamma < 1$, where $L(\cdot)$ is a slowly varying function at zero (i.e., it is measurable, positive and $\lim_{t\to 0} L(\lambda t)/L(t) = 1$ for all $\lambda > 0$). Note that the process $\{X(x,y)\}$ is a generalization of the two-parameter Lévy Brownian motion. For our purpose, we assume that, for any $t > 0$, there exist constants $c_1, c_2 > 0$ such that

$$\text{(ii)}\qquad \frac{d\sigma^2(t)}{dt} \le c_1 \frac{\sigma^2(t)}{t} \qquad \text{and} \qquad \frac{d^2\sigma^2(t)}{dt^2} \le c_2 \frac{\sigma^2(t)}{t^2}.$$

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For example, for some $c_0 > 0$, let $\sigma(t) = c_0 t^\gamma$, $0 < \gamma < 1$. Then $\{X(t_1, t_2),\ 0 \le t_1 \le 1,\ 0 \le t_2 \le 1\}$ is a two-dimensional time parameter fractional Brownian motion of order 2γ, and our assumptions are satisfied.

Our main result is as follows:

THEOREM 1 (Modulus of Continuity). *Assume that the above Gaussian process* $\{X(x,y)\}$ *satisfies conditions* (i) *and* (ii). *Let* $0 < h < 1$. *Then we have*

$$\lim_{h\to 0} \sup_{0<s\le h} \sup_{0\le x\le 1-s} \sup_{0\le y\le 1} \frac{|X(x+s,y) - X(x,y)|}{\sqrt{2\sigma^2(h)\log\left\{\left(\frac{1-h}{h} \vee 1\right)\frac{1}{h}\right\}}} = 1 \qquad \text{a.s.}$$

and

$$\lim_{h\to 0} \sup_{0\le x\le 1-h} \sup_{0\le y\le 1} \frac{|X(x+h,y) - X(x,y)|}{\sqrt{2\sigma^2(h)\log\left\{\left(\frac{1-h}{h} \vee 1\right)\frac{1}{h}\right\}}} = 1 \qquad \text{a.s.},$$

where $m \vee n = \max\{m,n\}$.

2. Proofs

The proof of Theorem 1 is mainly based on the following Lemma 1, due to Fernique [2]: Let $\mathbb{D} = \{\mathbf{t} : \mathbf{t} = (t_1, \ldots, t_N), a_i \le t_i \le b_i, i = 1, 2, \ldots, N\}$ be a real N-dimensional parameter space. We assume that the space $\mathbb{D}$ has the usual Euclidean norm $\|\cdot\|$ such that

$$\|\mathbf{t} - \mathbf{s}\|^2 = \sum_{i=1}^{N} (t_i - s_i)^2.$$

Let $\{X(\mathbf{t}),\ \mathbf{t} \in \mathbb{D}\}$ be a real-valued separable Gaussian process with $EX(\mathbf{t}) = 0$. Suppose that

$$0 < \sup_{\mathbf{t}\in\mathbb{D}} E\big(X(\mathbf{t})\big)^2 =: \Gamma^2 < \infty, \qquad \Gamma > 0,$$

and

$$E\{X(\mathbf{t}) - X(\mathbf{s})\}^2 \le \varphi^2(\|\mathbf{t} - \mathbf{s}\|),$$

where $\varphi(\cdot)$ is a nondecreasing continuous function such that

$$\int_0^\infty \varphi(e^{-y^2})dy < \infty.$$

LEMMA 1. *Let $\{X(\mathbf{t}),\ \mathbf{t} \in \mathbb{D}\}$ be given as the above statements. Then, for $\lambda > 0$, $x \geq 1$* and *$\mathcal{A} > 2\sqrt{N \log 2}$, we have*

$$P\Big\{\sup_{\mathbf{t}\in\mathbb{D}} X(\mathbf{t}) > x\{\Gamma + (2\sqrt{2}+2)\mathcal{A}\int_0^\infty \varphi(\sqrt{N}\lambda 2^{-y^2})dy\}\Big\} \leq (4^N + \psi)\prod_{i=1}^N (\frac{b_i - a_i}{\lambda} \vee 1)e^{-x^2/2},$$

where $\psi = \sum_{n=1}^\infty \exp\Big\{\frac{1}{2} - 2^n\Big(\frac{\mathcal{A}^2}{2} - 2N\log 2\Big)\Big\} < \infty$.

Let us estimate an upper bound of the following large deviation probabilities by using the above Lemma 1:

LEMMA 2. *For any $\epsilon > 0$, there exists a positive constant C_ϵ depending on ϵ such that the inequality*

$$P\Big\{\sup_{0<s\leq h}\ \sup_{0\leq x\leq 1-s}\ \sup_{0\leq y\leq 1} \frac{|X(x+s,y) - X(x,y)|}{\sigma(h)} > u\Big\} \leq C_\epsilon\Big(\frac{1-s}{h} \vee 1\Big)\Big(\frac{1}{h}\Big)e^{-u^2/(2+\epsilon)}$$

holds for all $u > 0$ and $0 < h < 1$.

PROOF. For $0 < h < 1$, let

$$\mathbb{D}_h = \{(s,x,y) : 0 < s \leq h, 0 \leq x \leq 1-s, 0 \leq y \leq 1\}$$

be a three dimensional space. In order to apply Lemma 1, we set

$$Z(s,x,y) = \frac{X(x+s,y) - X(x,y)}{\sigma(h)}, \qquad (s,x,y) \in \mathbb{D}_h,$$

and

$$\varphi(z) = \frac{2\sigma(\sqrt{2}z)}{\sigma(h)}, \qquad z > 0.$$

Clearly, $EZ(s,x,y)=0$ for all $(s,x,y)\in\mathbb{D}_h$. Since $\sigma(\cdot)$ is nondecreasing, it follows from the assumption (i) that

$$\Gamma^2 := \sup_{(s,x,y)\in\mathbb{D}_h} E\big(Z(s,x,y)\big)^2 = 1.$$

Using the elementary inequality $(a\pm b)^2\le 2(a^2+b^2)$, we have

$$\begin{aligned}
&E\{Z(s_1,x_1,y_1)-Z(s_2,x_2,y_2)\}^2\\
&\quad\le \frac{2}{\sigma^2(h)}\Big\{E\{X(x_1+s_1,y_1)-X(x_2+s_2,y_2)\}^2\\
&\qquad\qquad + E\{X(x_1,y_1)-X(x_2,y_2)\}^2\Big\}\\
&\quad\le \frac{4}{\sigma^2(h)}\sigma^2\left(\sqrt{2((x_1-x_2)^2+(s_1-s_2)^2+(y_1-y_2)^2)}\right).
\end{aligned}$$

Thus we get

$$\begin{aligned}
&E\{Z(s_1,x_1,y_1)-Z(s_2,x_2,y_2)\}^2\\
&\quad\le \varphi^2\big(\sqrt{2((s_1-s_2)^2+(x_1-x_2)^2+(y_1-y_2)^2)}\big).
\end{aligned}$$

Since $\sigma(\cdot)$ is a regularly varying function, we can show that for any $\epsilon>0$, there exists a small $c=c(\epsilon)>0$ such that

$$(2\sqrt{2}+2)\mathcal{A}\int_0^\infty \varphi(\sqrt{3}ch2^{-y^2})\,dy<\epsilon/8$$

(cf. Lemma 4.5 in Choi [1]). Let $u=\omega(1+(\epsilon/8))$, $\omega\ge 1$, then we have, by Lemma 1,

$$\begin{aligned}
&P\Big\{\sup_{0<s\le h}\ \sup_{0\le x\le 1-s}\ \sup_{0\le y\le 1}\frac{|X(x+s,y)-X(x,y)|}{\sigma(h)}>u\Big\}\\
&\le 2P\Big\{\sup_{(s,x,y)\in\mathbb{D}_h} Z(s,x,y)\\
&\qquad\qquad > \omega\Big(1+(2\sqrt{2}+2)\mathcal{A}\int_0^\infty \varphi(\sqrt{3}ch2^{-y^2})\,dy\Big)\Big\}\\
&\le C_\epsilon\Big(\frac{1-s}{h}\vee 1\Big)\Big(\frac{1}{h}\Big)e^{-u^2/(2+\epsilon)},
\end{aligned}$$

where C_ϵ is a positive constant depending on $\epsilon>0$. In case $0<u\le 1$, the result is obvious if we take C_ϵ large enough. □

The following lemma is easily obtained by a combined modification of Corollary 4.2.4 in Leadbetter et al. ([3], p. 84) and Lemma 4.4 in Choi ([1], p. 199):

LEMMA 3. *Let* $\{Y_{ij}, i, j = 1, 2, \ldots, n\}$ *be jointly standardized normal random variables with* covariance $(Y_{ij}, Y_{i'j'}) = \Lambda_{ij}^{i'j'}$ *such that*

$$\delta := \max_{(i,j)\neq(i',j')} |\Lambda_{ij}^{i'j'}| < 1.$$

Let the subsequences $\{l_i; i = 1, 2, \cdots, k\}$ *and* $\{l_j; j = 1, 2, \cdots, h\}$ *of integers be such that* $1 \le l_1 < l_2 < \cdots < l_k \le n$ *and* $1 \le l_1 < l_2 < \cdots < l_h \le n$ *with* $h,\ k \le n$. *Denote* $r_{ij}^{i'j'} = \Lambda_{l_i l_j}^{l_{i'} l_{j'}}$. *Assume that*

$$|r_{ij}^{i'j'}| < |i - i'|^{-\nu} |j - j'|^{-\nu} < 1, \quad i \neq i',\ j \neq j'$$

for some $\nu > 0$, *and let* $u = \{(2-\eta)\log(hk)\}^{1/2}$, *where* $0 < \eta < (1-\delta)\nu/(1+\nu+\delta)$. *Then for any real number* u *we have*

$$P\Big\{\max_{1\le i\le k}\max_{1\le j\le h} Y_{l_i l_j} \le u\Big\} \le \big\{\Phi(u)\big\}^{kh} + c(kh)^{-\delta_0},$$

where $\delta_0 = \{\nu(1-\delta) - \eta(1+\delta+\nu)\}/\{(1+\nu)(1+\delta)\} > 0$ *and* $c = c(\delta)$ *is a constant independent of* n, u, k *and* h, *and* $\Phi(u) = \int_{-\infty}^{u} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\, dy$.

The next lemma is easily verified by the same techniques as in the proof of Lemma 4.6 in Choi ([1], p. 204):

LEMMA 4. *Let* h be *in* $0 < h_{n+1} \le h < h_n < 1$, $n \in \mathbb{N}$, *where* $\mathbb{N}$ *denotes the set of positive integers and* h_n *is decreasing. Then we have*

$$\begin{aligned}
&\liminf_{h\to 0} \sup_{0\le x\le 1-h} \sup_{0\le y\le 1} \frac{|X(x+h,y) - X(x,y)|}{\sqrt{2\sigma^2(h)\log\{(\frac{1-h}{h} \vee 1)\frac{1}{h}\}}} \\
&\quad \ge \liminf_{n\to\infty} \sup_{0\le x\le 1-h_n} \sup_{0\le y\le 1} \frac{|X(x+h_n,y) - X(x,y)|}{\sqrt{2\sigma^2(h_n)\log\{(\frac{1-h_n}{h_n} \vee 1)\frac{1}{h_n}\}}} \qquad \text{a.s.}
\end{aligned}$$

Now we are ready to prove Theorem 1:

PROOF OF THEOREM 1. We first prove that

$$\limsup_{h\to 0}\sup_{0<s\le h}\sup_{0\le x\le 1-s}\sup_{0\le y\le 1}\frac{|X(x+s,y)-X(x,y)|}{\sqrt{2\sigma^2(h)\log\{(\frac{1-h}{h}\vee 1)\frac{1}{h}\}}}\le 1 \quad \text{a.s.} \tag{2.1}$$

Let h be in $h_{n+1}\le h\le h_n$, $n\in\mathbb{N}$, where h_n is decreasing and will be specified later on. Note that

$$\sup_{0<s\le h}\sup_{0\le x\le 1-s}\sup_{0\le y\le 1}\frac{|X(x+s,y)-X(x,y)|}{\sqrt{2\sigma^2(h)\log\{(\frac{1-h}{h}\vee 1)\frac{1}{h}\}}}$$

$$\le \sup_{0<s\le h_n}\sup_{0\le x\le 1-s}\sup_{0\le y\le 1}\frac{|X(x+s,y)-X(x,y)|}{\sqrt{2\sigma^2(h_n)\log\{(\frac{1-h_n}{h_n}\vee 1)\frac{1}{h_n}\}}}\frac{\sigma(h_n)}{\sigma(h_{n+1})}. \tag{2.2}$$

For any small $\epsilon>0$, take $T>(1+\epsilon)/(2\epsilon)$, and set $h_n=(n+1)^{-T}$, $n\in\mathbb{N}$. Then we have

$$\lim_{n\to\infty}\frac{\sigma(h_{n+1})}{\sigma(h_n)}=1. \tag{2.3}$$

Indeed, since $\sigma(\cdot)$ is regularly varying,

$$\begin{aligned}1\ge\frac{\sigma(h_{n+1})}{\sigma(h_n)}&=\frac{\sigma\big((1-\frac{1}{n+2})^T h_n\big)}{\sigma(h_n)}\\&=\big(1-\frac{1}{n+2}\big)^{T\gamma}\frac{L\big((1-\frac{1}{n+2})^T h_n\big)}{L(h_n)}\longrightarrow 1\end{aligned}$$

as $n\to\infty$. On the other hand, it follows from Lemma 2 that, for any $\epsilon>0$,

$$\begin{aligned}&P\Big\{\sup_{0<s\le h_n}\sup_{0\le x\le 1-s}\sup_{0\le y\le 1}\frac{|X(x+s,y)-X(x,y)|}{\sqrt{2\sigma^2(h_n)\log\{(\frac{1-h_n}{h_n}\vee 1)\frac{1}{h_n}\}}}>\sqrt{1+\epsilon}\Big\}\\&\le C_\epsilon\Big(\frac{1-s}{h_n}\vee 1\Big)\frac{1}{h_n}\Big\{\Big(\frac{1-h_n}{h_n}\vee 1\Big)\frac{1}{h_n}\Big\}^{-2(1+\epsilon)/(2+\epsilon)}\\&\le C_\epsilon\Big(\frac{1}{h_n}\Big)^{-2\epsilon/(1+\epsilon)}=C_\epsilon(n+1)^{-T(2\epsilon/(1+\epsilon))}.\end{aligned}$$

Since $T\big(2\epsilon/(1+\epsilon)\big) > 1$ for given $\epsilon > 0$, the series

$$\sum_n P\Big\{\sup_{0<s\le h_n}\ \sup_{0\le x\le 1-s}\ \sup_{0\le y\le 1} \frac{|X(x+s,y)-X(x,y)|}{\sqrt{2\sigma^2(h_n)\log\{(\frac{1-h_n}{h_n}\vee 1)\frac{1}{h_n}\}}} > \sqrt{1+\epsilon}\Big\}$$

is convergent. The Borel-Cantelli lemma implies that
(2.4)

$$\limsup_{n\to\infty}\ \sup_{0<s\le h_n}\ \sup_{0\le x\le 1-s}\ \sup_{0\le y\le 1} \frac{|X(x+s,y)-X(x,y)|}{\sqrt{2\sigma^2(h_n)\log\{(\frac{1-h_n}{h_n}\vee 1)\frac{1}{h_n}\}}} \le 1 \quad \text{a.s.}$$

Combining (2.2), (2.3) and (2.4), we obtain (2.1).

Next we prove that

$$\liminf_{h\to 0}\ \sup_{0\le x\le 1-h}\ \sup_{0\le y\le 1} \frac{|X(x+h,y)-X(x,y)|}{\sqrt{2\sigma^2(h)\log\{(\frac{1-h}{h}\vee 1)\frac{1}{h}\}}} \ge 1 \quad \text{a.s.} \tag{2.5}$$

It is obvious that there exists $\beta > 0$ big enough such that

$$\Big(\frac{1-h}{h}\vee 1\Big)\frac{1}{h} \ge \Big(\log\frac{1}{h}\Big)^{\beta}$$

for $h > 0$ sufficiently small. For such h, one can choose large integer N such that

$$\Big(\log\frac{1}{h}\Big)^{\beta} > N^2.$$

Define nonnegative integers $\nabla_1(h)$ and $\nabla_2(h)$ by

$$\nabla_1(h) = \Big[\frac{1-h}{Nh}\Big], \qquad \nabla_2(h) = \Big[\frac{1}{Nh}\Big]$$

for $h > 0$ sufficiently small. Then there exists a constant $c > 0$ such that

$$\nabla_1(h)\nabla_2(h) \ge c\Big(\log\frac{1}{h}\Big)^{\beta} \tag{2.6}$$

provided $h > 0$ is small enough. For $i = 0, 1, 2, \cdots, \nabla_1(h)$ and $j =$

$0, 1, 2, \cdots, \nabla_2(h)$, we also define

$$Z(i,j) = X(Nhi + h, Nhj) - X(Nhi, Nhj).$$

Then $Z(i,j)/\sigma(h)$ are standard normal random variables. It follows that, for any $0 < \epsilon' < \epsilon < 1$ and small h,

(2.7)

$$\begin{aligned} P\Big\{ \sup_{0\le x\le 1-h} \sup_{0\le y\le 1} \frac{X(x+h,y) - X(x,y)}{\sigma(h)} \\ < \sqrt{2(1-\epsilon)\log\{(\frac{1-h}{h} \vee 1)\frac{1}{h}\}}\Big\} \\ \le P\Big\{ \max_{0\le i\le \nabla_1(h)} \max_{0\le j\le \nabla_2(h)} \frac{Z(i,j)}{\sigma(h)} < \sqrt{2(1-\epsilon')\log(\nabla_1(h)\nabla_2(h))}\Big\}. \end{aligned}$$

Let $r(i,i',j,j') = correlation(Z(i,j), Z(i',j'))$, $i \ne i'$, $j \ne j'$, and set $l = |i - i'| \ge 1$, $m = |j - j'| \ge 1$. Using the relation $ab = \frac{1}{2}(a^2 + b^2 - (a-b)^2)$, it follows from the condition (ii) that

$$\begin{aligned} &|covariance(Z(i,j), Z(i',j'))| = |E\{Z(i,j)Z(i',j')\}| \\ &= \big|E\{X((Ni+1)h, Njh)X((Ni'+1)h, Nj'h) \\ &\qquad - X((Ni+1)h, Njh)X(Ni'h, Nj'h) \\ &\qquad - X(Nih, Njh)X((Ni'+1)h, Nj'h) \\ &\qquad + X(Nih, Njh)X(Ni'h, Nj'h)\}\big| \\ &= \frac{1}{2}\Big|\sigma^2\Big(\sqrt{(Nl+1)^2h^2 + (Nm)^2h^2}\Big) - \sigma^2\Big(\sqrt{(Nl)^2h^2 + (Nm)^2h^2}\Big) \\ &\quad - \Big(\sigma^2\Big(\sqrt{(Nl)^2h^2 + (Nm)^2h^2}\Big) - \sigma^2\Big(\sqrt{(Nl-1)^2h^2 + (Nm)^2h^2}\Big)\Big)\Big| \\ &= \frac{1}{2}\left|\int_{\sqrt{(Nl)^2h^2+(Nm)^2h^2}}^{\sqrt{(Nl+1)^2h^2+(Nm)^2h^2}} d\sigma^2(x) - \int_{\sqrt{(Nl-1)^2h^2+(Nm)^2h^2}}^{\sqrt{(Nl)^2h^2+(Nm)^2h^2}} d\sigma^2(x)\right| \\ &\le c\frac{\sigma^2\Big(\sqrt{(Nl+1)^2h^2 + (Nm)^2h^2}\Big)h^2}{(Nl-1)^2h^2 + (Nm)^2h^2}, \end{aligned}$$

where $c > 0$ is a constant. Hence we have, for any small h,

$$\begin{aligned}|r(i,j,i',j')| &\le c\frac{\sigma^2\left(h\sqrt{(Nl+1)^2+(Nm)^2}\right)}{\sigma^2(h)\left((Nl-1)^2+(Nm)^2\right)}\\ &\le c\frac{(Nl+1)^2+(Nm)^2}{(Nl-1)^2+(Nm)^2}\left((Nl+1)^2+(Nm)^2\right)^{-(1-\gamma)}\\ &\le c\left((Nl-1)^2+(Nm)^2\right)^{-(1-\gamma)} < \left(l^2+m^2\right)^{-(1-\gamma)}\\ &\le (2lm)^{-(1-\gamma)} < (lm)^{-\nu},\end{aligned}$$

where N is sufficiently large and $0 < \nu = 1-\gamma$. To estimate the upper bound for the last inequality of (2.7), let us now apply Lemma 3 for

$$Y_{l_il_j} = \frac{Z(i,j)}{\sigma(h)}, \quad i = 0,1,2,\cdots,\nabla_1(h)\ ; \quad j = 0,1,2,\cdots,\nabla_2(h),$$

$$\left|r_{ij}^{i'j'}\right| = |r(i,j,i',j')| < (lm)^{-\nu},\ l = |i-i'| \ge 1,\ m = |j-j'| \ge 1,$$

$$u = u(h) = \{(2-\eta)\log(\nabla_1(h)\nabla_2(h))\}^{1/2} \quad \text{and} \quad \eta = 2\epsilon' < \frac{(1-\delta)\nu}{1+\nu+\delta}.$$

Then the last inequality of (2.7) is less than or equal to

$$\{\Phi(u(h))\}^{(\nabla_1(h)+1)(\nabla_2(h)+1)} + c\,(\nabla_1(h)\nabla_2(h))^{-\delta_0}$$

for some $\delta_0 > 0$. Thus we have

$$\begin{aligned}&P\Big\{\sup_{0\le x\le 1-h}\sup_{0\le y\le 1}\frac{X(x+h,y)-X(x,y)}{\sqrt{2\sigma^2(h)\log\{(\frac{1-h}{h}\vee 1)\frac{1}{h}\}}} < \sqrt{1-\epsilon}\Big\}\\ &\le \exp\big(-c(\nabla_1(h)\nabla_2(h))^{\epsilon}\big) + c\big(\nabla_1(h)\nabla_2(h)\big)^{-\delta_0} \qquad (2.8)\\ &\le c\,(\nabla_1(h)\nabla_2(h))^{-\delta_0},\end{aligned}$$

where c is a constant, which may different in lines. Let $h_n = \exp(-n^a)$, $0 < a < 1$, $n \in \mathbb{N}$. Note that the β in (2.6) can be taken sufficiently large so that $\beta > 1/(a\delta_0)$. Thus (2.8) implies that

$$\begin{aligned}&P\Big\{\sup_{0\le x\le 1-h_n}\sup_{0\le y\le 1}\frac{X(x+h_n,y)-X(x,y)}{\sqrt{2\sigma^2(h_n)\log\{(\frac{1-h_n}{h_n}\vee 1)\frac{1}{h_n}\}}} < \sqrt{1-\epsilon}\Big\}\\ &\le c\Big(\log\frac{1}{h_n}\Big)^{-\beta\delta_0} = cn^{-a\beta\delta_0}\end{aligned}$$

and

$$\sum_n P\Big\{\sup_{0\le x\le 1-h_n}\sup_{0\le y\le 1}\frac{X(x+h_n,y)-X(x,y)}{\sqrt{2\sigma^2(h_n)\log\{\big(\frac{1-h_n}{h_n}\vee 1\big)\frac{1}{h_n}\}}}<\sqrt{1-\epsilon}\Big\}<\infty.$$

By the Borel-Cantelli lemma, we obtain

$$\liminf_{n\to\infty}\sup_{0\le x\le 1-h_n}\sup_{0\le y\le 1}\frac{X(x+h_n,y)-X(x,y)}{\sqrt{2\sigma^2(h_n)\log\{\big(\frac{1-h_n}{h_n}\vee 1\big)\frac{1}{h_n}\}}}\ge\sqrt{1-\epsilon}\qquad \text{a.s.}$$

Let h be in $h_{n+1}\le h<h_n$. It follows from Lemma 4 that the result (2.5) holds. The inequalities (2.1) and (2.5) complete the proof of Theorem 1. □

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