KERNEL OPERATORS ON FOCK SPACE

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ABSTRACT. We study on kernel operators (Wick monomials) on symmetric Fock space. We give optimal conditions on kernels so that the corresponding kernel operators are densely defined linear operators on the Fock space. We try to formulate our results in the framework of white noise analysis as much as possible. The most of the results in this paper can be extended to anti-symmetric Fock space.

1. Introduction

Let $\mathcal{F}(L^2)$ be the symmetric Fock space over $L^2(\mathbb{R}^d, dx)$ and let $a(k)$ and $a^*(k)$, $k \in \mathbb{R}^d$, be the annihilation and creation operators (bilinear forms) on $\mathcal{F}(L^2)$ respectively [1, 4, 8]. Consider the following type of integrals

\begin{equation}
T_W = \int_{\mathbb{R}^{(n_1 + n_2)}} w(k_1, \cdots, k_{n_1}, p_1, \cdots, p_{n_2}) \left( \prod_{i=1}^{n_1} a^*(k_i) \right) \left( \prod_{i=1}^{n_2} a(p_i) \right) dk dp,
\end{equation}

where $n_1, n_2 \in \mathbb{N}$. Such integrals, known as Wick monomials, are standard in quantum field theory [2, 8]. There have been many studies on sufficient conditions under which $T_W$ becomes a densely defined (unbounded) operator on $\mathcal{F}(L^2)$ ([3], Proposition 1.2.3 and [8], Theorem X.44).

In white noise analysis [4, 8], the integrals (1.1) have been extensively studied under the name of white noise kernel operators of order $(n_1, n_2)$.
Let
\[(1.2) \quad (\mathcal{S}) \subset (L^2) \subset (\mathcal{S})^*\]
be the Gelfand triple of white noise functions. If the kernel \(w\) belongs to \(\mathcal{S}'(\mathbb{R}^{d(n_1+n_2)})\), \(T_W\) is a continuous linear operator from \((\mathcal{S})\) to \((\mathcal{S})^*\) ([4], Theorem 6.2). In application, it should be important to know conditions on the kernel \(w\) under which \(T_W\) becomes a continuous linear operators from \((\mathcal{S})\) to \((L^2)\). We mention that in quantum probability theory the Wick monomials are defined in terms of stochastic integrals and called Maassen’s kernels [5, 7].

The purpose of this paper is to give minimal conditions on the kernel \(w \in \mathcal{S}'(\mathbb{R}^{d(n_1+n_2)})\) so that the corresponding operator \(T_W\) becomes a densely defined operator on \(\mathcal{F}(L^2)\) (and \((L^2)\)). We formulate our results in the language of white noise analysis. One of our main results can be described as follow: Let \(W\) be a continuous linear map from \(\mathcal{S}(\mathbb{R}^{dn_2})\) to \(L^2(\mathbb{R}^{dn_1})\) defined by a kernel \(w\). Then \(T_W\) formally defined as in (1.1) becomes a continuous linear operator from \((\mathcal{S})\) to \((L^2)\). Thus \(T_W\) defines an unbounded operator on \((L^2)\) with domain \((\mathcal{S})\). See Theorem 3.1 in Section 3.

We organize this paper as follows: In Section 2, we recall the definitions of the symmetric Fock space, and annihilation and creation operators. We then recall the known results ([8], Theorem X.44 and [3], Proposition 1.2.3). In section 3, we first recall Gelfand triple of white noise functions and then consider kernel operators associated to unbounded operators from \(L^2(\mathbb{R}^{dn_2})\) to \(L^2(\mathbb{R}^{dn_1})\) with domain \(\mathcal{S}(\mathbb{R}^{dn_2})\). Finally we consider kernel operators associated to bounded operators from \(L^2(\mathbb{R}^{dn_2})\) to \(L^2(\mathbb{R}^{dn_1})\).

Before closing the introduction, we remark that most of the results in this paper can be extended to kernel operators on anti-symmetric (fermion) Fock space.

2. Preliminary: Fock spaces

In this section, we recall the construction of symmetric Fock spaces, and the theory of creation and annihilation operators on Fock space. We then provide some known facts on kernel operators on Fock space. For more details, we refer the reader to the references [1, 8].
Let $\mathcal{H}$ be a complex Hilbert space. In this paper we are mainly interested in the case of $\mathcal{H} = L^2(\mathbb{R}^d)$. Denote by $\mathcal{H}^{(n)}$ the n-fold tensor product of $\mathcal{H}$: $\mathcal{H}^{(n)} = \mathcal{H} \otimes \mathcal{H} \otimes \cdots \otimes \mathcal{H}$ (n times). Let $S_n$ be the symmetrization operator on $\mathcal{H}^{(n)}$ defined by

\begin{equation}
S_n(\psi_1 \otimes \psi_2 \otimes \cdots \otimes \psi_n) = \frac{1}{n!} \sum_{\pi \in P_n} \psi_{\pi(1)} \otimes \psi_{\pi(2)} \otimes \cdots \otimes \psi_{\pi(n)},
\end{equation}

where $P_n$ is the permutation group on n elements. Put

\begin{equation}
\mathcal{F}^{(n)}(\mathcal{H}) = S_n \mathcal{H}^{(n)}.
\end{equation}

The symmetric Fock space over $\mathcal{H}$ is defined by

\begin{equation}
\mathcal{F}(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{F}^{(n)}(\mathcal{H}),
\end{equation}

where $\mathcal{F}^{(0)}(\mathcal{H}) = \mathbb{C}$. Thus, for $\phi = \{\phi^{(n)}\} \in \mathcal{F}(\mathcal{H})$ and $\psi = \{\psi^{(n)}\} \in \mathcal{F}(\mathcal{H})$, the inner product of $\phi$ and $\psi$ is given by

\begin{equation}
(\phi, \psi)_{\mathcal{F}(\mathcal{H})} = \sum_{n=0}^{\infty} (\phi^{(n)}, \psi^{(n)})_{\mathcal{H}^{(n)}}.
\end{equation}

Let us introduce another Fock space over $\mathcal{H}$, namely white noise Fock space, as follows. Let $\Gamma^{(n)}(\mathcal{H})$ be the Hilbert space of $S_n \mathcal{H}^{(n)}$ equipped with the inner product

\begin{equation}
(\phi^{(n)}, \psi^{(n)})_{\Gamma^{(n)}(\mathcal{H})} = n!(\phi^{(n)}, \psi^{(n)})_{\mathcal{H}^{(n)}}, \quad \phi^{(n)}, \psi^{(n)} \in S_n \mathcal{H}^{(n)}.
\end{equation}

The white noise Fock space over $\mathcal{H}$ is defined by

\begin{equation}
\Gamma(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \Gamma^{(n)}(\mathcal{H}).
\end{equation}

Notice that for $\phi = \{\phi^{(n)}\} \in \Gamma(\mathcal{H})$ and $\psi = \{\psi^{(n)}\} \in \Gamma(\mathcal{H})$ we have

\begin{equation}
(\phi, \psi)_{\Gamma(\mathcal{H})} = \sum_{n=0}^{\infty} (\phi^{(n)}, \psi^{(n)})_{\Gamma^{(n)}(\mathcal{H})} = \sum_{n=0}^{\infty} n!(\phi^{(n)}, \psi^{(n)})_{\mathcal{H}^{(n)}}.
\end{equation}

For the details we refer to [4], Section A.2.

From now on we confine ourselves to the case of $\mathcal{H} = L^2(\mathbb{R}^d; dx)$. If $\mathcal{H} = L^2(\mathbb{R}^d)$, then $\bigotimes_{j=1}^{n} L^2(\mathbb{R}^d) = L^2(\mathbb{R}^{dn})$ and $S(\bigotimes^{n} L^2(\mathbb{R}^d)) = L^2(\mathbb{R}^{dn})$, where $L^2_\sigma$ is the set of functions in $L^2$ which are invariant under permutations of the variables. Let $S(\mathbb{R}^d)$ and $S'((\mathbb{R}^d)$ be the Schwartz space and the space of tempered distributions [8]. For $f \in S(\mathbb{R}^d)$ and
\[ \psi = \{ \psi^{(n)} \} \in \mathcal{F}(L^2) (\Gamma(L^2)), \text{ the annihilation operator } a(f) \text{ and the creation operator } a^*(f) \text{ are defined by} \]

\[ (2.8) \ (a(f)\psi)^n(k_1, \cdots, k_n) = \sqrt{n + 1} \int_{\mathbb{R}^d} f(k)\psi^{(n+1)}(k, k_1, \cdots, k_n)dk, \]

\[ (2.9) \ (a^*(f)\psi)^n(k_1, \cdots, k_n) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} f(k_i)\psi^{(n-1)}(k_1, \cdots, k_i, \cdots, k_n), \]

where \( \hat{k}_i \) means that \( k_i \) is omitted.

A vector \( \psi = \{ \psi^{(n)} \} \in \mathcal{F}(L^2) \) for which \( \psi^{(n)} = 0 \) for all but finitely many \( n \) is called a finite particle vector. Denote by \( \mathcal{F}_0 \) the set of finite particle vectors. Put

\[ D_S = \{ \psi : \psi = \{ \psi^{(n)} \} \in \mathcal{F}_0, \psi^{(n)} \in \mathcal{S}(\mathbb{R}^{dn}) \text{ for any } n \}. \]

For each \( p \in \mathbb{R}^d \), we define an operator \( a(p) \) on \( \mathcal{F}(L^2)(\Gamma(L^2)) \) with domain \( D_S \) by

\[ (a(p)\psi)^n(k_1, \cdots, k_n) = \sqrt{n + 1} \psi^{(n+1)}(p, k_1, \cdots, k_n). \]

The adjoint of the operator \( a(p) \) is given formally by

\[ (a^*(p)\psi)^n(k_1, \cdots, k_n) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \delta(p-k_i)\psi^{(n-1)}(k_1, \cdots, k_{i-1}, k_{i+1}, \cdots, k_n), \]

which is a well-defined quadratic form on \( D_S \times D_S \). One can check that, if \( w \) belongs to \( \mathcal{S}'(\mathbb{R}^{d(n_1+n_2)}) \), the integral \( (1.1) \) is defined as a quadratic forms on \( D_S \times D_S \). See [8], Section X.7 for the details.

Let \( A \) be any self-adjoint operator on \( \mathcal{H} \) with domain of essential self-adjointness \( D \). Let \( D_A = \{ \psi \in \mathcal{F}_0 : \psi^{(n)} \in \otimes^n D \text{ for each } n \} \) and define \( d\Gamma(A) \) on \( D_A \cap \mathcal{F}^{(n)}(\mathcal{H}) \) as

\[ (2.10) \ d\Gamma(A) \]

\[ = A \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes A \otimes \cdots \otimes 1 + \cdots + 1 \otimes 1 \otimes \cdots \otimes 1 \otimes A. \]

Then \( d\Gamma(A) \) is essentially self-adjoint on \( D_A [8] \). \( d\Gamma(A) \) is called the second quantization of \( A \). The number operator \( N \) is defined to be \( N = d\Gamma(1) \), which can be written as

\[ (2.11) \ N = \int_{\mathbb{R}^d} a^*(p)a(p) \, dp. \]
For $r \in \mathbb{R}$, let

\begin{equation}
N_r = \int_{\mathbb{R}^d} \mu(p)^r a^*(p)a(p) \, dp,
\end{equation}

where $\mu(p) = (p^2 + 1)^{1/2}$. One can check that $N_r$ is the second quantization of the multiplication operator $\mu(p)$ on $L^2(\mathbb{R}^d)$ [3, 8].

We now collect some more or less well-known results about kernel operators on Fock space. The followings are the results on kernel operators associated to $L^2$-kernels:

**Theorem 2.1** ([8], Theorem X.44). Let $n_1$ and $n_2$ be nonnegative integers and suppose that $w \in L^2(\mathbb{R}^{d(n_1 + n_2)})$. Then there is an operator $T_W$ on $\mathcal{F}_n(L^2(\mathbb{R}^d))$ so that $D_S \subset D(T_W)$ is a core for $T_W$ and

\begin{equation}
T_W = \int_{\mathbb{R}^{d(n_1 + n_2)}} w(k_1, \cdots, k_{n_1}, p_1, \cdots, p_{n_2})(\prod_{i=1}^{n_1} a^*(k_i))(\prod_{i=1}^{n_2} a(p_i))dkdp
\end{equation}

as quadratic forms on $D_S \times D_S$.

(b)

\begin{equation}
T_W^* = \int_{\mathbb{R}^{d(n_1 + n_2)}} w(k_1, \cdots, k_{n_1}, p_1, \cdots, p_{n_2})(\prod_{i=1}^{n_2} a^*(p_i))(\prod_{i=1}^{n_1} a(k_i))dkdp
\end{equation}

as quadratic forms on $D_S \times D_S$.

(c) $(N + 1)^{-n_1/2}T_W(N + 1)^{-n_2/2}$ is a bounded operator on $\mathcal{F}(L^2)$ and the bound

\begin{equation}
\| (N + 1)^{-n_1/2}T_W(N + 1)^{-n_2/2} \| \leq C(n_1, n_2) \| W \|_{L^2}
\end{equation}

holds for some constant $C(n_1, n_2)$.

(d) If $w_n \rightarrow w$ in $L^2(\mathbb{R}^{d(n_1 + n_2)})$, then $T_{W_n} \rightarrow T_W$ strongly on $D_S$.

(e) $\mathcal{F}_0$ is contained in $D(T_W)$, and on vectors in $\mathcal{F}_0$, $T_W$ is given by the explicit formula

\begin{equation}
(T_W \psi)^{(l-n_2+n_1)} = K(l, n_1, n_2) \times \nabla S \int_{\mathbb{R}^{d(n_2)}} w(k_1, \cdots, k_{n_1}, p_1, \cdots, p_{n_2})\psi(l)(p_1, \cdots, p_{n_2}, k_1, \cdots, k_{n_1+l-n_2})dp
\end{equation}

and

\begin{equation}
(T_W \psi)^{(n)} = 0 \quad \text{if} \quad n < n_1
\end{equation}
where $S$ is the symmetrization operator and $K(l, n_1, n_2) = \left( \frac{1}{((l-n_1)!)(l-n_2)!)} \right)^{1/2}$.

The following is a result on kernel operators associated to more general kernels:

**Theorem 2.2** ([3], Proposition 1.2.3 (a)). Let $w(k, p) \in S'(\mathbb{R}^{dn_1+n_2})$ so that $w$ is a densely defined bilinear form on $S(\mathbb{R}^{dn_1}) \times S(\mathbb{R}^{dn_2})$. Denote by $\|w\|$ the norm of the operator $w$ from $L^2(\mathbb{R}^{dn_2}) \rightarrow L^2(\mathbb{R}^{dn_1})$ given by the kernel $w(k, p)$. Let $T_w$ be the bilinear form given by (1.1). Then there exists a constant $C(n_1, n_2)$ such that the bound

$$\|(N_r + 1)^{-n_1/2} T_w (N_r + 1)^{-n_2/2}\| \leq C(n_1, n_2) \| \left( \prod_{i=1}^{n_1} \mu(k_i)^{-r/2} \right) w \left( \prod_{j=1}^{n_2} \mu(p_j)^{-r/2} \right) \|$$

holds.

3. Kernel operators: Main results

In this section, we reformulate and extend the results in Theorem 2.1 and Theorem 2.2 in the framework of white noise analysis. First let us recall basic facts of white noise analysis. The inner product of $L^2(\mathbb{R}^d; dx)$ is denoted by $(\cdot, \cdot)_2$, the corresponding norm by $\| \cdot \|_2$. Let $(S'(\mathbb{R}^d), \mathcal{B}, \mu)$ be the Gaussian measure on $S'(\mathbb{R}^d)$ with its characteristic function on $S(\mathbb{R}^d)$ given by

$$\int_{S'(\mathbb{R}^d)} \exp(i < x, \xi>) \, d\mu(x) = \exp(-\frac{1}{2}|\xi|^2), \quad \xi \in S(\mathbb{R}^d),$$

where $< \cdot, \cdot >$ is the dual pairing. Let $(L^2) = L^2(S'(\mathbb{R}^d), \mu)$. Recall the definition of $\Gamma(L^2)$ in (2.6). By the chaos decomposition([4, Proposition 2.6]), every $\phi \in (L^2)$ is in one to one correspondence with a sequence $\tilde{\phi} = \{\tilde{\phi}^{(n)}\}$ in $\Gamma(L^2)$, and $\|\phi\|_{L^2} = \|\tilde{\phi}\|_{\Gamma(L^2)}$. Thus one may identify $(L^2)$ with $\Gamma(L^2)$.

Let $H$ be the self-adjoint extension of the differential operator

$$-\Delta + |x|^2 + 1$$

on $L^2(\mathbb{R}^d; dx)$ with domain $S(\mathbb{R}^d)$. Notice that $H \geq 2$. By $\| \cdot \|_{2,p}, \quad p \in \mathbb{R}$, we denote the norm given by $|H^p \cdot |_2$. More generally, for a function $f^{(l)}$
on $\mathbb{R}^{d_l}$, $l \in \mathbb{N}$, we denote by $|f^{(l)}|_{2,p}$, \ $p \in \mathbb{R}$, the norm $|(H^p)^{\Delta l}f^{(l)}|_2$. Let 

$$(S) \subset (L^2) \simeq \Gamma(L^2) \subset (S)^*$$

be the Gelfand triple [4, Chapter 4.A]. Any $\psi \in (S)$ has a chaos decomposition $\psi = \{\psi^{(l)}; l \in \mathbb{N}\}$, where each $\psi^{(l)}$, \ $l \in \mathbb{N}$, belongs to the symmetric Schwartz space $S_s(\mathbb{R}^{d_l})$ [4, Proposition 4.3]. Again we identify $(S)$ with the space of such sequences. Notice that $\psi = \{\psi^{(l)}\}$ belongs to $(S)$ if and only if for any $p \geq 0$

$$||\psi||_{2,p}^2 = \sum_{l=0}^{\infty} l!|\psi^{(l)}|_{2,p}^2 < \infty. \tag{3.1}$$

See [4, Proposition 4.2]. We remark that the inclusions

$$D_S \subset (S) \subset (L^2) \simeq \Gamma(L^2) \subset \mathcal{F}(L^2) \tag{3.2}$$

hold and the embeddings are dense.

Let us return to the discussion of kernel operators. Denote the set of continuous linear map from a topological space $X$ to a topological space $Y$ by $\mathcal{L}(X,Y)$. Recall that $W \in \mathcal{L}(L^2(\mathbb{R}^{dn_2}), L^2(\mathbb{R}^{dn_1}))$ is Hilbert-Schmidt operator if and only if there is a kernel function $w \in L^2(\mathbb{R}^{(n_1+n_2)})$ with

$$(Wf)(k) = \int_{\mathbb{R}^{dn_2}} w(k,p)f(p)dp, \quad f \in L^2(\mathbb{R}^{dn_2}). \tag{3.3}$$

Moreover the Hilbert-Schmidt norm of $W$ equals to the $L^2$-norm of the kernel function $w$ [8, Theorem VI.23]. Therefore one can see that Theorem 2.1 is the result for kernel operators associated with Hilbert-Schmidt operators from $L^2(\mathbb{R}^{dn_2})$ to $L^2(\mathbb{R}^{dn_1})$.

In order to describe the main idea in this paper, consider the operators $N_\tau$ defined in (2.12). $N_\tau$ can be written formally by

$$N_\tau = \int_{\mathbb{R}^{2d}} (k^2 + 1)^{\tau/4} \delta(k-p)(p^2 + 1)^{\tau/4} a^*(k)a(p) \, dkdp.$$ 

Thus, in a sense $N_\tau$, \ $\tau \geq 0$, can be viewed as a kernel operator associated with the kernel $w(k,p) = (k^2 + 1)^{\tau/4} \delta(k-p)(p^2 + 1)^{\tau/4}$, which is not in $L^2(\mathbb{R}^{2d})$. However the kernel defines a continuous linear map from $S(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$ via the expression (3.3).

Let $W$ be a continuous linear map from $S(\mathbb{R}^{dn_2})$ to $L^2(\mathbb{R}^{dn_1})$, i.e., $W \in \mathcal{L}(S(\mathbb{R}^{dn_2}), L^2(\mathbb{R}^{dn_1}))$. Then there exist constants $p > 0$ and $C > 0.$
such that

\[(3.4) \quad \|Wf\|_{L^2} \leq C\| (H^p)^{\otimes n_2} f \|_{2} \quad f \in S(\mathbb{R}^{d_{n_2}}).\]

See [8, Chapter V]. Let \( W \in \mathcal{L}(S(\mathbb{R}^{d_{n_2}}), L^2(\mathbb{R}^{d_{n_1}})) \) be given. For any \( l \geq n_2, S(\mathbb{R}^{d_l}) \simeq S(\mathbb{R}^{d_{n_2}}) \otimes S(\mathbb{R}^{d(l-n_2)}), \) and so \( W \otimes 1^{(l-n_2)} \) is a continuous linear map from \( S(\mathbb{R}^{d_l}) \) to \( L^2(\mathbb{R}^{d(l-n_2)+n_1}) \), where \( 1^{(l-n_2)} \) is the identity map on \( L^2(\mathbb{R}^{d(l-n_2)}) \).

The following is our main result:

**Theorem 3.1.** Let \( n_1 \) and \( n_2 \) be non-negative integers and suppose that a continuous linear map \( W \) from \( S(\mathbb{R}^{d_{n_2}}) \) to \( L^2(\mathbb{R}^{d_{n_1}}) \) is given. For any \( \psi = \{ \psi^{(l)} \} \in (S), \) let \( T_W \psi \) be defined by

\[(3.5) \quad (T_W \psi)^{(l-n_2+n_1)} = K(l, n_1, n_2) S(W \otimes 1^{(l-n_2)}) \psi^{(l)}, \quad l \geq n_2, \]

\[(T_W \psi)^{(n)} = 0, \quad n < n_1,\]

where \( S = S_{l-n_2+n_1} \) is the symmetrization operator and \( K(l, n_1, n_2) \) is the constant given in Theorem 2.1 (e). Then the following results hold:

(a) \( T_W \) becomes a continuous linear map from \( (S) \) to \( (L^2) \). Thus \( T_W \) is an unbounded linear operator on \( \Gamma(L^2) \) (and so \( \mathcal{F}(L^2) \)) with domain \((S)\).

(b) If the adjoint \( W^* \) of \( W \) defines a continuous linear map from \( S(\mathbb{R}^{d_{n_1}}) \) to \( L^2(\mathbb{R}^{d_{n_2}}) \), then \( T_W^* \) is also continuous map from \( (S) \) to \( (L^2) \). In this case \( T_W^* = (T_W)^* \) on \((S)\), and so \( T_W \) is closable.

(c) Let \( p \) be any positive constant satisfying (3.4). Then \( T_W(d\Gamma(H^{2p}))^{-n_2/2} \) \( (N+1)^{-n_1/2} \) is a bounded operator on \( \mathcal{F}(L^2) \), and the bound

\[
\| T_W(d\Gamma(H^{2p}))^{-n_2/2}(N+1)^{-n_1/2} \|_{\mathcal{F}(L^2) \rightarrow \mathcal{F}(L^2)} \leq C(n_1, n_2) \| W(H^{-p})^{\otimes n_2} \|_{\mathcal{L}(L^2(\mathbb{R}^{d_{n_2}}), L^2(\mathbb{R}^{d_{n_1}}))} \]

holds for some constant \( C(n_1, n_2) \). The same result as above holds on \( (L^2) \) if one replaces \( T_W(d\Gamma(H^{2p}))^{-n_2/2}(N+1)^{-n_1/2} \) by \( T_W(d\Gamma(H^{2p}))^{-n_2/2}(N+1)^{-n_1/2}(N+1)^{-n_1/2} \) for \( n_2, (n_1-n_2)/2 \).

**Remark 3.2.** (a) If \( W \in \mathcal{L}(S(\mathbb{R}^{d_{n_2}}), L^2(\mathbb{R}^{d_{n_1}})) \) is defined by the kernel \( w(k, p) \) via the expression (3.3), then \( T_W \) can be expressed as the integral (1.1) as a bilinear form \( (S) \times (S) \).

(b) Theorem 3.1(c) is the result analogous to Theorem 2.2 [3, Proposition 1.2.3 (a)].
Proof of Theorem 3.1. (a) For any $\psi = \{\psi^{(l)}\} \in (S)$, it follows from (3.5) and (3.4) that

$$
| (T_W \psi)^{(l-n_2+n_1)} |_2 \leq K(l, n_1, n_2) (W \otimes 1^{(l-n_2)}) \psi^{(l)} |_2 \\
\leq K(l, n_1, n_2) \| W (H^{-p})^{\otimes n_2} \| \| ((H^p)^{\otimes n_2} \otimes 1^{(l-n_2)}) \psi^{(l)} |_2.
$$

(3.6)

Notice that $K(l, n_1, n_2)^2 \leq l^{n_2} (l + n_1)^{n_1}$. Since $H \geq 2$, we have that

$$
|((H^p)^{\otimes n_2} \otimes 1^{(l-n_2)}) \psi^{(l)} |_2 \leq |(H^p)^{\otimes \tilde{l}} \psi^{(l)} |_2.
$$

Thus it follows from (3.6) that

$$
\| T_W \psi \|_{(L^2)^l} \leq C \sum_{l \geq n_2} (l - n_2 + n_1)! l^{n_2} (l + n_1)^{n_1} |\psi^{(l)} |_{2,p}^2 \\
\leq C \| \psi \|_{2,\tilde{p}}^2
$$

for some $\tilde{p} \geq p$. This proves the part (a).

(b) For given $l \geq n_2$, let $f \in L^2(\mathbb{R}^d)$ and $g \in S(\mathbb{R}^{d(l-n_2+n_1)})$. Using the fact $K(l, n_1, n_2) = K(l - n_2 + n_1, n_2, n_1)$ and $S^2 = S$, we have

$$
K(l, n_1, n_2) (S(W \otimes 1^{(l-n_2)}) S f, g)_2 \\
= K(l - n_2 + n_1, n_2, n_1) (S f, S(W^* \otimes 1^{(l-n_2)}) S g)_2.
$$

(3.7)

Thus, it follows from the above relation that $(T_W)^* = T_{W^*}$ on $(S)$. This proves the part (b).

(c) Since each $\psi^{(l)}$ belongs to $S(\mathbb{R}^d)$, we have

$$
|((H^p)^{\otimes n_2} \otimes 1^{(l-n_2)}) \psi^{(l)} |_2^2 = (\psi^{(l)}, ((H^{2p})^{\otimes n_2} \otimes 1^{(l-n_2)}) \psi^{(l)})_2 \\
= (\psi^{(l)}, S((H^{2p})^{\otimes n_2} \otimes 1^{(l-n_2)}) \psi^{(l)})_2.
$$

(3.8)

For any $j \in \{1, 2, \ldots, l\}$, denote by $H_j$ the operator $H$ acting on $L^2(\mathbb{R}^d, dx_j)$, i.e., $L^2(\mathbb{R}^d)$-space with $x_j$-variable. For any $A = \{i_1, i_2, \ldots, i_{n_2}\} \subset \{1, 2, \ldots, l\}$, $i_j < i_k$ if $j < k$, we write for a sake of brevity that

$$(H^{2p})^{\otimes A} = 1 \otimes \cdots \otimes H_i^{2p} \otimes 1 \otimes \cdots \otimes H_j^{2p} \otimes 1 \otimes \cdots \otimes H_k^{2p} \otimes 1 \otimes \cdots \otimes 1.$$

Then it can be checked that

$$
S((H^{2p})^{\otimes n_2} \otimes 1^{(l-n_2)}) S = \frac{(l - n_2)! n_2!}{l!} \sum_{A: |A| = n_2} (H^{2p})^{\otimes A} \\
\leq \frac{(l - n_2)!}{l!} (d\Gamma(H^{2p}))^{n_2},
$$

(3.9)
where the sum is taken over the subsets of \(\{1, 2, \cdots, l\}\) with \(n_2\)-elements. By substituting (3.9) into (3.8), it follows from (3.6) that

\[
\|(T_W \psi)(l-n_2+n_1)\|_2 \leq C \left( \frac{(l-n_2+n_1)!}{(l-n_2)!} \right)^{1/2} \|W(H^{-p})^{\otimes n_2}\| \|d\Gamma(H^{2p})^{n_2/2}\| \|(l-n_2+n_1)\|_2 \\|T_W \psi\|_2 \cdot |l-n_2+n_1|_2
\]

Thus the first part of (c) follows from the above estimate. And

\[
\|T_W \psi\|^2_{\mathcal{F}(L^2)} = \sum_{l \geq n_2} (l-n_2+n_1)! \|(T_W \psi)(l-n_2+n_1)\|_2^2
\]

The second part of (c) follows from (3.10) and the above equality. This completes the proof of the theorem. \(\square\)

Next, let us consider optimal conditions on kernels so that the corresponding kernel operators are densely defined operators on Fock spaces \(\mathcal{F}(L^2(\mathbb{R}^d))\) and \(\Gamma(L^2(\mathbb{R}^d))\). For any dense subset \(D\) in \(L^2(\mathbb{R}^d)\), denote by \(D_s^{(n)}\) the \(n\)-fold symmetric algebraic tensor product of \(D\). For a given dense subset \(D \subset L^2(\mathbb{R}^d)\), let

\[
\mathcal{F}_0(D) = \{f = (f^{(n)}) \in \mathcal{F}_0 : f^{(n)} \in D_s^{(n)}, n \in \mathbb{N}\}.
\]

Denote by \(\mathcal{E}(D)\) the algebra generated by the exponential vectors \(e(f), f \in D\), where for \(f \in D\)

\[
e(f) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{\otimes n}.
\]

Then \(\mathcal{F}_0(D)\) and \(\mathcal{E}(D)\) are dense in \(\mathcal{F}(L^2(\mathbb{R}^d))\) and also in \(\Gamma(L^2(\mathbb{R}^d))\).

**Theorem 3.3.** Let \(n_1\) and \(n_2\) be nonnegative integers. For a dense subset \(D \subset L^2(\mathbb{R}^d)\), let \(D^{(n_2)}\) be the \(n_2\)-fold algebraic tensor product of \(D\). Suppose \(W\) be an unbounded operator from \(L^2(\mathbb{R}^{dn_2})\) to \(L^2(\mathbb{R}^{nd_i})\) with domain \(D^{(n_2)}\). For any \(\psi = \{\psi^{(l)}\} \in \mathcal{F}_0(D)\), define \(T_W \psi\) by the expression (3.5). Then the following results hold:

(a) \(T_W\) is well-defined on \(\mathcal{F}_0(D)\). Thus \(T_W\) becomes an unbounded operator on \(\mathcal{F}(L^2(\mathbb{R}^d))\) and also on \(\Gamma(L^2(\mathbb{R}^d))\) with domain \(\mathcal{F}_0(D)\).

(b) \(T_W\) is well-defined on \(\mathcal{E}(D)\). Thus \(T_W\) becomes an unbounded operator on \(\mathcal{F}(L^2(\mathbb{R}^d))\) and also on \(\Gamma(L^2(\mathbb{R}^d))\) with domain \(\mathcal{E}(D)\).

(c) Suppose that \(W^{*}\) is well-defined on \(D^{(n_1)}\), i.e., \(D^{(n_1)} \subset D(W^{*})\). Then \(T_W^{*} \subset (T_W)^{*}\). Thus \(T_W\) is closable.
Proof. (a) Notice that for \( \psi^{(l)} = S(f_1 f_2 \cdots f_l), \ l \geq n_2 \)

\[
S(W \otimes 1^{(l-n_2)})\psi^{(l)} = \sum_{\pi \in S(l)} W(f_{\pi(1)} \cdots f_{\pi(n_2)}) f_{\pi(n_2+1)} \cdots f_{\pi(l)},
\]

where \( S(l) \) is the permutation group of \( \{1, \cdots l\} \). Since each \( W(f_{\pi(1)} \cdots f_{\pi(n_2)}) \) belongs to \( L^2(\mathbb{R}^{dn_2}) \), we see that \( S(W \otimes 1^{(l-n_2)})\psi^{(l)} \in L^2(\mathbb{R}^{d(l-n_2+n_1)}) \).

Thus (a) follows from (3.5).

(b) A direct computation shows that for \( g \in D \)

\[
T_W(e(g)) = \sum_{l \geq n_2} \frac{K(l, n_1, n_2)}{l!} S(W(g^{\otimes n_2}) \cdot g^{\otimes (l-n_2)}),
\]

and

\[
\|T_W(e(g))\|_{\Gamma(L^2(\mathbb{R}^d))}^2 \leq \sum_{l \geq n_2} (l - n_2 + n_1)! \frac{K(l, n_1, n_2)^2}{(l)!^2} |Wg^{\otimes n_2}|^2 \|g\|_{L^2}^{2(l-n_2)}
< \infty.
\]

Notice that \( \|\psi\|_{\mathcal{F}(L^2)} \leq \|\psi\|_{\Gamma(L^2)} \). This proves the part (b).

(c) This follows the method similar to that used in the proof of the part (b) of Theorem 3.1. \( \square \)

In the rest of the paper, we consider the class of kernel operators associated with bounded operators from \( L^2(\mathbb{R}^{dn_2}) \) to \( L^2(\mathbb{R}^{dn_1}) \). In this case we have the results analogous to Theorem 2.1.

**Theorem 3.4.** Let \( n_1 \) and \( n_2 \) be non-negative integers and let \( W \in \mathcal{L}(L^2(\mathbb{R}^{dn_2}), L^2(\mathbb{R}^{dn_1})) \). Let \( T_W \) be an unbounded operator on \( \mathcal{F}(L^2) \) defined as in (3.5). Then the following results hold:

(a) \( T_W \) is well-defined on \( \mathcal{F}_0 \), and leaves \( \mathcal{F}_0 \) invariant, i.e., \( T_W \mathcal{F}_0 \subset \mathcal{F}_0 \).

(b) \((T_W)^* = T_W^* \) on \( \mathcal{F}_0 \), and so \( T_W \) is closable.

(c) If \( W_n \to W \) in \( \mathcal{L}(L^2(\mathbb{R}^{dn_2}), L^2(\mathbb{R}^{dn_1})) \), then \( T_{W_n} \to T_W \) strongly on \( \mathcal{F}_0 \).

(d) \( T_W(N+1)^{-(n_1+n_2)/2} \) is a bounded operator on \( \mathcal{F}(L^2) \), and there exists a constant \( C(n_1, n_2) \) such that the bound

\[
\|T_W(N+1)^{-(n_1+n_2)/2}\| \leq C(n_1, n_2)\|W\|
\]

holds.

Proof. (a) Since \( S(W \otimes 1^{(l-n_2)})\psi^{(l)} \in \mathcal{F}^{(l-n_2+n_1)}(L^2) \), the part (a) follows from (3.5).
(b) For any $f \in L^2(\mathbb{R}^d)$ and $g \in L^2(\mathbb{R}^{d(l-n_2+n_1)})$, the equality (3.7) holds. This implies the part (b).

(c) It follows from (3.5) that for any $\psi = \{\psi^{(i)}\} \in \mathcal{F}_0$

$$|(T_{W_n}\psi)^{(l-n_2+n_1)} - (T_W\psi)^{(l-n_2+n_1)}|_2 \leq K(l, n_1, n_2)\|W_n - W\| |\psi^{(i)}|_2.$$

Thus (c) follows from the above bound.

(d) If one replace $H$ by $1$ and then replace $d\Gamma(H^{2p})$ by $(N + 1)$ in Theorem 3.1 (c), the part (d) follows from Theorem 3.1 (c) as a corollary. \hfill \square

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References


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