PRODUCTS OF WHITE NOISE FUNCTIONALS
AND ASSOCIATED DERIVATIONS

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**Abstract.** Let the Gel'fand triple \((E)_\beta \subset (L^2) \subset (E)^*_\beta\) be the framework of white noise distribution theory constructed by Kondratiev and Stieft. A new class of continuous multiplicative products on \((E)_\beta\) is introduced and associated continuous derivations on \((E)_\beta\) are discussed. Algebraic characterizations of first order differential operators on \((E)_\beta\) are proved. Some applications are also discussed.

1. Introduction

We take, as a framework of white noise distribution theory, a Gel'fand triple

\[(1.1) \quad (E)_\beta \subset (L^2) \subset L^2(E^*, \mu; \mathbb{C}) \subset (E)^*_\beta, \quad 0 \leq \beta < 1,\]

which was constructed in \([18]\), where \(E^*\) is the space of tempered distributions and \(\mu\) is the standard Gaussian measure associated with a Gel'fand triple \(E \subset H \subset E^*\). In particular, if \(\beta = 0\), (1.1) becomes \((E) \subset (L^2) \subset (E)^*\), which was constructed in \([19]\). The white noise distribution theory is an infinite dimensional analogue of the Schwartz distribution theory in which the role of Lebesgue measure on \(\mathbb{R}^n\) is replaced by the Gaussian measure \(\mu\) on \(E^*\). This theory was initiated by Hida \([11]\) and has been considerably developed with applications to

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stochastic analysis, Feynman path integral, infinite dimensional harmonic analysis, quantum probability and Cauchy problems in infinite dimensions and so on, see e.g., [1]–[9], [12]–[15], [22]–[26], [30].

It is well known [13,24] that the Wiener and Wick products are continuous multiplicative operations on \((E)_\beta\) and so \((E)_\beta\) becomes a topological algebra under the Wiener and Wick products. Furthermore, we note that the differential operator \(D_y\) is a first order differential operator as well as a continuous derivation on \((E)_\beta\). Motivated by this point of view, Obata [29] determined all the continuous derivations on \((E)\) with respect to the Wiener products and then showed that they are first order differential operators with variable coefficients. In [1], Chung and Chung showed that the results in [29] can be extended to the Wick product case. Recently, Chung and Chung has introduced a class \(\{\circ_{\gamma}, \gamma \in \mathbb{C}\}\) of multiplicative products on \((E)\) including the Wiener and Wick products and then have showed that with each \(\circ_{\gamma}\) and an \((E)\)-valued distribution \(\Phi\) on \(\mathbb{R}\), we can associate a first order differential operator with variable coefficient \(\Phi\), which is indeed a continuous derivation on \((E)\) with respect to \(\circ_{\gamma}\).

In this paper, we shall see that \(\Xi_{0,m}(\kappa) + N\) is similar to \(N\), i.e., there exists a linear homeomorphism \(G_\kappa \in GL((E)_{\beta(m)})\) such that \(\Xi_{0,m}(\kappa) + N = G_\kappa^{-1}NG_\kappa\) (Corollary 3.7), where \(\Xi_{0,m}(\kappa)\) and \(N\) are the integral kernel operator with kernel distribution \(\kappa \in (E^{\mathbb{C}}_{\kappa}m)^*\) and the number operator, respectively. Next, we define a product \(\circ_\kappa\) associated with \(G_\kappa\), and prove that a first order differential operator with variable coefficient associated with \(\circ_\kappa\) is indeed a continuous derivation on \((E)_\beta\) with respect to \(\circ_\kappa\) (Theorem 5.4), and then \(\Xi_{0,m}(\kappa) + N\) is a continuous derivation on \((E)_\beta\) with respect to the product \(\circ_\kappa\). Finally, as applications, we shall study the eigenvalue problem, Cauchy problem and Poisson type equation associated with \(\Xi_{0,m}(\kappa) + N\).

2. Preliminaries

Let \(H\) be the real Hilbert space of square-integrable functions on \(\mathbb{R}\) with norm \(| \cdot |_0\). Let \(S(\mathbb{R})\) be the Schwarz space consisting of rapidly decreasing \(C^\infty\)-functions. Then we have a Gel'fand triple:

\[
E \equiv S(\mathbb{R}) \subset H \subset S'(\mathbb{R}) \equiv E^* ,
\]
where $E^*$ is the space of the tempered distributions. Note that the Gel'fand triple \((2.1)\) is reconstructed by using a positive self-adjoint operator $A = 1 + t^2 - d^2/dt^2$ on $H$ with Hilbert–Schmidt inverse (see [13], [24], [28]). In fact, $E$ is a nuclear space equipped with the Hilbertian norms $|\xi|_p = |A^p \xi|_0$, $\xi \in E$, $p \in \mathbb{R}$. Let $\mu$ be the standard Gaussian measure on $E^*$ whose characteristic function is given by

$$\int_{E^*} e^{i\langle x, \xi \rangle} \mu(dx) = \exp \left(-\frac{1}{2} |\xi|_0^2 \right), \quad \xi \in E,$$

where $\langle \cdot, \cdot \rangle$ is the canonical bilinear form on $E^* \times E$. The canonical $C$-bilinear form on $(E_C^\otimes n)^* \times (E_C^\otimes n)$ is also denoted by the same symbol $\langle \cdot, \cdot \rangle$. We denote by $(L^2) \equiv L^2(E^*, \mu; \mathbb{C})$ the complex Hilbert space of square integrable functions on $E^*$ with norm $\| \cdot \|_0$. By the Wiener-Itô decomposition theorem, each $\phi \in (L^2)$ admits an expression

$$\phi(x) = \sum_{n=0}^{\infty} \langle : x^\otimes n :, f_n \rangle, \quad x \in E^*, \quad f_n \in H_C^\otimes n,$$

and $\| \phi \|_0^2 = \sum_{n=0}^{\infty} n! |f_n|_0^2$, where $H_C^\otimes n$ is the n-fold symmetric tensor product of the complexification of $H$ and $: x^\otimes n :$ denotes the Wick ordering of $x^\otimes n$.

Let $\beta$ be a given real number with $0 \leq \beta < 1$. For each $p \geq 0$, define

$$\| \phi \|_{p, \beta} = \sum_{n=0}^{\infty} (n!)^{1+\beta} |f_n|_p^2, \quad \phi \in (L^2),$$

where $\phi$ is given as in (2.2). Let $(E_p)_\beta = \{ \phi \in (L^2) : \| \phi \|_{p, \beta} < \infty \}$ and let $(E)_\beta$ be the projective limit of $\{ (E_p)_\beta : p \geq 0 \}$. Then $(E)_\beta$ is a nuclear space and we have a Gel'fand triple:

$$\tag{2.3} (E)_\beta \subset (L^2) \subset (E)_\beta^*,$$

where $(E)_\beta^*$ is the topological dual space of $(E)_\beta$. The triple (2.3) is called the Kondratiev–Streit space [18]. If $\beta = 0$, then (2.3) is called the Hida–Kubo–Takenaka space and denoted by $(E) \subset (L^2) \subset (E)^*$. An element in $(E)_\beta$ (and in $(E)_\beta^*$) is called a test (and generalized, resp.) white noise functional. We denote by $\langle \cdot, \cdot \rangle$ the canonical $C$-bilinear form on $(E)_\beta^* \times (E)_\beta$. For each $\Phi \in (E)_\beta^*$, there exists a unique sequence $\{ F_n \}_{n=0}^{\infty}$, $F_n \in (E_C^\otimes n)^{sym}$ such that
\[ \langle \Phi, \phi \rangle = \sum_{n=0}^{\infty} n! \langle F_n, f_n \rangle, \quad \phi \in (E)_\beta \]

where \( \phi \) is given as in (2.2). In this case we use a formal expression for \( \Phi \in (E)_\beta^* \):

\[ \Phi(x) = \sum_{n=0}^{\infty} \langle x^{\otimes n}, F_n \rangle, \quad x \in E^*. \]

For each \( \xi \in E_C \), the function \( \phi_\xi \in (E)_\beta \) given by

\[ \phi_\xi(x) = \sum_{n=0}^{\infty} \langle x^{\otimes n}, \xi^{\otimes n} \rangle \frac{1}{n!} = \exp \left( \langle x, \xi \rangle - \frac{1}{2} \langle \xi, \xi \rangle \right), \quad x \in E^* \]

is called an exponential vector. Note that \( \{ \phi_\xi : \xi \in E_C \} \) spans a dense subspace of \( (E)_\beta \). The S–transform of \( \Phi \in (E)_\beta^* \) is a function on \( E_C \) defined by

\[ S\Phi(\xi) = \langle \Phi, \phi_\xi \rangle, \quad \xi \in E_C. \]

In [18], Kondrative and Streit proved a characterization theorem for elements in \( (E)_\beta^* \) and \( (E)_\beta \) by analytic properties of the S–transforms.

By using the characterization theorem, for each \( \Phi, \Psi \in (E)_\beta^* \) the Wick product \( \Phi \circ \Psi \in (E)_\beta^* \) is defined by \( S(\Phi \circ \Psi) = S\Phi \cdot S\Psi \). In facts \( \phi \circ \psi \in (E)_\beta \), \( \phi, \psi \in (E)_\beta \) and the Wick product is a continuous multiplicative operation on \( (E)_\beta \) (see e.g., [24]).

Let \( \mathcal{L}(\mathcal{X}, \mathcal{Y}) \) denote the space of all continuous linear operators from a locally convex space \( \mathcal{X} \) into another locally convex space \( \mathcal{Y} \). Also for notational convenience we write \( \mathcal{L}(\mathcal{X}) = \mathcal{L}(\mathcal{X}, \mathcal{X}) \). For each \( \Xi \in \mathcal{L}((E)_\beta, (E)_\beta^*) \), the \( C \)-valued function \( \hat{\Xi} \) on \( E_C \times E_C \) defined by

\[ \hat{\Xi}(\xi, \eta) = \langle \Xi \phi_\xi, \phi_\eta \rangle, \quad \xi, \eta \in E_C \]

is called the symbol of operator \( \Xi \). The following theorem is a characterization theorem for symbols of operator in \( \mathcal{L}((E)_\beta, (E)_\beta^*) \) and in \( \mathcal{L}((E)_\beta, (E)_\beta^*) \) ([24], [27], [28]).

**Theorem 2.1.** A \( \mathcal{C} \)-valued function \( \Theta : E_C \times E_C \to \mathbb{C} \) is a symbol of an \( \Xi \in \mathcal{L}((E)_\beta, (E)_\beta^*) \) if and only if \( \Theta \) satisfies the following conditions:

(O1) For each \( \xi, \xi', \eta, \eta' \in E_C \), the function

\[ (z, w) \mapsto \Theta(z\xi + \xi', w\eta + \eta') \]
is an entire function on \( \mathbb{C} \times \mathbb{C} \);

\((O2)\) There exist constants \( K \geq 0, C \geq 0 \) and \( p \geq 0 \) such that

\[
|\Theta(\xi, \eta)| \leq K \exp C \left( |\xi|^{\frac{2}{p+q}} + |\eta|^{\frac{2}{p}} \right), \quad \xi, \eta \in E_C.
\]

Moreover, \( \Theta \) is the symbol of an operator in \( \mathcal{L}((E)_{\beta}) \) if and only if it satisfies \((O1)\) and

\((O2')\) For any \( p \geq 0, \epsilon > 0 \), there exist \( q \geq 0 \) and \( K \geq 0 \) such that

\[
|\Theta(\xi, \eta)| \leq K \exp \epsilon \left( |\xi|^{\frac{2}{p+q}} + |\eta|^{\frac{2}{p}} \right), \quad \xi, \eta \in E_C.
\]

For each \( y \in E^*_C \) there exists a unique operator \( D_y \in \mathcal{L}((E)_{\beta}) \) such that \( D_y \phi_\xi = \langle y, \xi \rangle \phi_\xi, \xi \in E_C \) which is called the annihilation operator. The adjoint operator \( D_y^* \in \mathcal{L}((E)_{\beta}^*) \) of \( D_y \) is called the creation operator. By Theorem 2.1, we can show that for each \( \kappa \in (E_C^\otimes (l+m))^* \), there exists a unique operator \( \Xi_{l,m}(\kappa) \in \mathcal{L}((E)_{\beta}, (E)_{\beta}^*) \) such that

\[
\Xi_{l,m}(\kappa)(\xi, \eta) = \langle \Xi_{l,m}(\kappa) \phi_\xi, \phi_\eta \rangle = \langle \kappa, \eta^\otimes l \otimes \xi^\otimes m \rangle e^\langle \xi, \eta \rangle, \quad \xi, \eta \in E_C.
\]

This operator \( \Xi_{l,m}(\kappa) \) is called the integral kernel operator with kernel distribution \( \kappa \). In particular, \( \Delta_G = \Xi_{0,2}(\tau) \) and \( N = \Xi_{1,1}(\tau) \) are called the Gross Laplacian and number operator, respectively, where \( \tau \) is the trace defined by \( \langle \tau, \xi \otimes \eta \rangle, \xi, \eta \in E_C \). It is well-known that for each \( \kappa \in (E_C^\otimes (l+m))^* \), \( \Xi_{l,m}(\kappa) \in \mathcal{L}((E)_{\beta}) \) if and only if \( \kappa \in (E_C^\otimes l) \otimes (E_C^\otimes m)^* \).

3. Transformation group on white noise functionals

In this section we obtain a two-parameter transformation group \( G \) on \( (E)_{\beta} \). Moreover, we shall prove that there exists an element \( G_\kappa \in G \) such that \( G_\kappa (\Xi_{0,m}(\kappa) + N)G_\kappa^{-1} = N \). For notational convenience, we define a function \( \beta : \mathbb{N} \cup \{0\} \to [0, 1) \) by

\[
\beta(m) = \begin{cases} 
0, & m = 0, 1 \\
1 - \frac{2}{m}, & m = 2, 3, \ldots 
\end{cases}
\]
Lemma 3.1. Let $\kappa \in (E_C^\otimes m)^*$ and $a \in C$. Then there exists a unique operator $G_{\kappa,a} \in \mathcal{L}((E)_{(m)})$ such that

$$G_{\kappa,a}\phi_\xi = \exp\{\langle \kappa, \xi^\otimes m \rangle \} \phi_a \xi, \quad \xi \in E_C.$$ 

Proof. Let $\kappa \in (E_C^\otimes m)^*$ and $a \in C$. For any $\xi, \eta \in E_C$, we put

$$\Theta(\xi, \eta) = \exp\{\langle \kappa, \xi^\otimes m \rangle + a(\xi, \eta) \}.$$ 

Then the function $\Theta$ satisfies conditions (O1) and (O2') in Theorem 2.1 with $\beta = \beta(m)$. Hence by Theorem 2.1, there exists a unique operator $G_{\kappa,a} \in \mathcal{L}((E)_{(m)})$ such that

$$\widehat{G}_{\kappa,a}(\xi, \eta) = \langle G_{\kappa,a}\phi_\xi, \phi_\eta \rangle = \Theta(\xi, \eta).$$

This completes the proof.

Let $C$ and $C^* = C - \{0\}$ be the additive and multiplicative groups of complex numbers, respectively. Let $GL(X)$ denote the group of all linear homeomorphisms from a locally convex space $X$ onto itself.

Theorem 3.2. Let $\kappa \in (E_C^\otimes m)^*$ be fixed. And let $\mathcal{G} = \{G_{a,b} ; a \in C, b \in C^*\}$. Then $\mathcal{G}$ is a subgroup of $GL((E)_{(m)})$.

Proof. By Lemma 3.1, we have $G_{0_\kappa,1}\phi_\xi = \phi_\xi$ for any $\xi \in E_C$, and

$$G_{a',b'}(a_{b,b}^\xi) = \exp\{\langle a + a'^b, \kappa, \xi^\otimes m \rangle \} \phi_{b,b} = G_{a-a'^b,\kappa, b'}^\xi,$$

for any $a, a' \in C$ and $b, b' \in C^*$ and $\xi \in E_C$. But $\{\phi_\xi : \xi \in E_C\}$ spans a dense subspace of $(E)_{(m)}$ and for any $a \in C, b \in C^*$, $G_{a,b}$ is continuous. Hence it follows that for any $\phi \in (E)_{(m)}$,

$$G_{0_\kappa,1}\phi = \phi \quad \text{and} \quad G_{a',b'}(G_{a,b}\phi) = G_{a-a'^b,\kappa, b'}(\phi).$$

Then $G_{(-a'^b)^\kappa, b'^{-1}}$ is the inverse of $G_{a,b}$ in $\mathcal{G}$.

Let $F_{\kappa,a} \in \mathcal{L}((E)^*_{(m)})$ be the adjoint operator of $G_{\kappa,a}$. Then by using similar arguments as in [5], we obtain explicit expressions $F_{\kappa,a}\Phi$ and $G_{\kappa,a}\Phi$ for $\Phi \in (E)^*_{(m)}$ and $\phi \in (E)_{(m)}$. 

Let $\mathcal{F}_{\kappa,a} \in \mathcal{L}((E)^*_{(m)})$ be the adjoint operator of $G_{\kappa,a}$.
THEOREM 3.3. Let $\Phi(x) = \sum_{n=0}^{\infty} \langle : x^{\otimes n} : , F_n \rangle \in (E)^{\beta(m)}_\kappa$, $F_n \in (E^{\otimes n})_{\text{sym}}^*$. Then we have, for any positive integer $m$,

$$F_{n,a} \Phi(x) = \sum_{n=0}^{\infty} \langle : x^{\otimes n} : , \sum_{k=0}^{\lfloor \frac{n}{m} \rfloor} \frac{1}{k!} a^{n-mk} F_{n-mk} \otimes K^{\otimes k} \rangle.$$

THEOREM 3.4. For $\phi(x) = \sum_{n=0}^{\infty} \langle : x^{\otimes n} : , f_n \rangle \in (E)^{\beta(m)}_\kappa$, $f_n \in E^{\otimes n}_C$, we have, for any positive integer $m$,

$$G_{n,a} \phi(x) = \sum_{n=0}^{\infty} \langle : x^{\otimes n} : , \sum_{k=0}^{\infty} \frac{(n+mk)!}{n!k!} a^n \kappa^{\otimes k} \otimes m_k f_{n+mk} \rangle.$$

PROPOSITION 3.5. The operator $G_{n,a}$ is expressed by

$$G_{n,a} = e^{(\log a)N} \circ e^{\Xi_{0,m}(\kappa)}.$$

Proof. Let $\phi(x) = \sum_{n=0}^{\infty} \langle : x^{\otimes n} : , f_n \rangle$. Then by Theorem 3.4, it holds that $G_{n,a} \phi = e^{(\log a)N} \phi$ and $G_{n,a} \phi = e^{\Xi_{0,m}(\kappa)} \phi$. Clearly, $G_{0,a} \circ G_{n,a} = G_{n,a}$. Hence we complete the proof. □

PROPOSITION 3.6. For any $\alpha_1, \alpha_2 \in C$, we have

$$G_{n,a}(\alpha_1 \Xi_{0,m}(\kappa) + \alpha_2 N) = ((\alpha_1 + \alpha_2 m)e^{-m \log a} \Xi_{0,m}(\kappa) + \alpha_2 N)G_{n,a}.$$

Proof. Note that for any $\kappa \in (E^{\otimes m})^*$,

$$[N, \Xi_{0,m}(\kappa)] = -m \Xi_{0,m}(\kappa).$$

Therefore, we have

$$e^{\Xi_{0,m}(\kappa)} N = (N + m \Xi_{0,m}(\kappa)) e^{\Xi_{0,m}(\kappa)}$$

and

$$e^{(\log a)N} \Xi_{0,m}(\kappa) = e^{-m \log a} \Xi_{0,m}(\kappa) e^{(\log a)N}.$$

Hence by Proposition 3.5, we obtain that

$$G_{n,a}(\alpha_1 \Xi_{0,m}(\kappa) + \alpha_2 N) = ((\alpha_1 + \alpha_2 m)e^{-m \log a} \Xi_{0,m}(\kappa) + \alpha_2 N)G_{n,a}.$$

Thus we complete the proof. □

COROLLARY 3.7. Let $\kappa \in (E^{\otimes m})^*$ and let $G_{\kappa} = G_{\frac{1}{m} \kappa,1}$. Then we have

$$G_{\kappa}(\Xi_{0,m}(\kappa) + N) = N G_{\kappa}.$$

Proof. The proof is straightforward from Proposition 3.6. □
4. Products on white noise functionals

In this section, we introduce a new class of continuous multiplicative products \( \circ_\kappa \) on \(( E )_\beta \) indexed by \( \kappa \in ( E )_\beta^{\otimes m} \).

Let \( \mathcal{H} \) be a linear homeomorphism on \(( E )_\beta \). Then we can define a continuous multiplicative operation \( \circ_{\mathcal{H}} \) on \(( E )_\beta \) associated with \( \mathcal{H} \) by

\[
\phi \circ_{\mathcal{H}} \psi = \mathcal{H}^{-1}( \mathcal{H} \phi \circ \mathcal{H} \psi ), \quad \phi, \psi \in ( E )_\beta,
\]

where \( \circ \) is the Wick product.

**PROPOSITION 4.1.** Let \( \Xi \in \mathcal{L}( ( E )_\beta ) \). Then \( \Xi \) is a derivation with respect to the multiplication \( \circ_{\mathcal{H}} \), i.e., \( [\Xi, \phi \circ_{\mathcal{H}} \psi] = \Xi \phi \circ_{\mathcal{H}} \psi + \phi \circ_{\mathcal{H}} \Xi \psi \), \( \phi, \psi \in ( E )_\beta \) if and only if \( \mathcal{H} \Xi = \Xi \mathcal{H}^{-1} \) is a derivation with respect to the Wick product.

**Proof.** The proof is clear from the definition of \( \circ_{\mathcal{H}} \).  \( \square \)

From now on we only consider the case \( \mathcal{H} = G_\kappa \) for a fixed kernel \( \kappa \in ( E )_\beta^{\otimes m} \) with integer \( m \geq 2 \), where \( G_\kappa \) is given in Corollary 3.7. We put \( \circ_\kappa = \circ_{G_\kappa} \). That is, \( \circ_\kappa \) is a continuous multiplicative operation on \(( E )_{\beta(m)} \) defined by

\[
(4.1) \quad \phi \circ_\kappa \psi = G_\kappa^{-1}( G_\kappa \phi \circ G_\kappa \psi ), \quad \phi, \psi \in ( E )_{\beta(m)}.
\]

**PROPOSITION 4.2.** Let \( \kappa \in ( E )_\beta^{\otimes m} \). Then there exists a unique operator \( T_\kappa \in \mathcal{L}( ( E )_{\beta(m)}, ( E )^*_{\beta(m)} ) \) such that the following hold:

\[
(4.2) \quad \phi_\xi \circ_\kappa \phi_\eta = \widehat{T_\kappa}(\xi, \eta)\phi_{\xi + \eta}, \quad \xi, \eta \in E_\kappa,
\]

and

\[
(4.3) \quad \langle \phi \circ_\kappa \phi_\xi, \phi_\eta \rangle = e^{(\xi, \eta)} \langle T_\kappa \phi_\xi \circ \phi_\eta, \phi \rangle, \quad \phi \in ( E )_{\beta(m)}, \xi, \eta \in E_\kappa.
\]

**Proof.** By using the equation (4.1) we can easily see that

\[
\phi_\xi \circ_\kappa \phi_\eta = \exp \left\{ \frac{1}{m}(\kappa, (\xi + \eta)^{\otimes m} - \xi^{\otimes m} - \eta^{\otimes m}) \right\} \phi_{\xi + \eta}, \quad \xi, \eta \in E_\kappa.
\]
Note that the function \( \Theta_\kappa(\xi, \eta) = \exp \left\{ \frac{1}{m} \langle \kappa, (\xi + \eta)^\otimes m - \xi^\otimes m - \eta^\otimes m \rangle \right\} \) satisfies (O1) and (O2) in Theorem 2.1. So, there exists a unique operator \( T_\kappa \in \mathcal{L}(E)_{\beta(m)}, (E)_{\beta(m)}^* \) such that (4.2) holds. Now for any \( \xi, \eta, \zeta \in E_C \) we observe that

\[
\langle \langle \phi_\zeta \circ_\kappa \phi_\xi, \phi_\eta \rangle \rangle = \langle \langle T_\kappa \phi_\zeta, \phi_\xi \rangle \rangle \langle \langle \phi_\zeta + \xi, \phi_\eta \rangle \rangle \\
= e^{\langle \xi, \eta \rangle} \langle \langle T_\kappa^* \phi_\xi, \phi_\zeta \rangle \rangle \langle \langle \phi_\zeta, \phi_\eta \rangle \rangle \\
= e^{\langle \xi, \eta \rangle} \langle \langle T_\kappa^* \phi_\xi \circ \phi_\eta, \phi_\zeta \rangle \rangle.
\]

Since \( \Theta_\kappa \) is symmetric, we have \( T_\kappa^* = T_\kappa \), and hence (4.3) holds for all \( \phi = \phi_\zeta, \zeta \in E_C \). Thus the proof follows from the fact that \{\phi_\zeta ; \zeta \in E_C\} spans a dense linear subspace of \( (E)_{\beta(m)} \).

**Example 4.3.** Let \( \kappa \in (E_C^\otimes 2)^* \) and \( \widetilde{\kappa} \in \mathcal{L}(E_C, E_C^*) \) be given by

\[
\langle \langle \widetilde{\kappa} \xi, \eta \rangle \rangle = \langle \kappa, \xi \otimes \eta \rangle, \quad \xi, \eta \in E_C.
\]

Then we have

\[
\phi_\xi \circ_\kappa \phi_\eta = e^{\langle \kappa, \xi \otimes \eta \rangle} \phi_{\xi + \eta} = e^{\frac{1}{2} \langle \langle \widetilde{\kappa} + \widetilde{\kappa}^* \rangle \xi + \eta, \phi \rangle} \phi_{\xi + \eta}, \quad \xi, \eta \in E_C.
\]

Hence we obtain that \( T_\kappa = \Gamma((\widetilde{\kappa} + \widetilde{\kappa}^*)/2) \), where \( \Gamma(A) \) is the second quantization operator of \( A \in \mathcal{L}(E_C, E_C^*) \). In this case, the equation (4.3) becomes

\[
\langle \langle \phi \circ_\kappa \phi_\xi, \phi_\eta \rangle \rangle = e^{\langle \xi, \eta \rangle} \langle \langle \frac{1}{2} \langle \langle \widetilde{\kappa} + \widetilde{\kappa}^* \rangle \xi + \eta, \phi \rangle, \phi \rangle \rangle, \quad \phi \in (E)_{\beta(m)}, \xi, \eta \in E_C.
\]

Moreover, if \( \kappa \in (E_C^\otimes 2)^*_{sym} \), then we have

\[
\langle \langle \phi \circ_\kappa \phi_\xi, \phi_\eta \rangle \rangle = e^{\langle \xi, \eta \rangle} \langle \langle \phi_{\langle \langle \kappa, \xi \rangle + \eta, \phi \rangle}, \phi \rangle \rangle, \quad \phi \in (E)_{\beta(m)}, \xi, \eta \in E_C.
\]

In particular, \( \circ_{\gamma} \) becomes the \( \gamma \)-product \( \circ_{\gamma} \) studied in [2]. Furthermore, \( \circ_0 \) is the Wick product and \( \circ_{\tau} \) is the Wiener product (see [2], [24]).
5. First order differential operators

In [2], Chung and Chung discussed the first order $\gamma$-differential operator $\Xi \in \mathcal{L}(\langle E \rangle)$ with coefficient $\Phi \in E_C^* \otimes \langle E \rangle$, where $\Xi$ is given by

$$\Xi = \int_{\mathbb{R}} \Phi(t) \circ \partial_t dt$$

as a formal integral expression. We now introduce a first order differential operator associated with the $\circ_\kappa$-product.

**Proposition 5.1.** Let $\kappa \in (E_C^\otimes)^*$ and let $\Phi \in E_C^* \otimes \langle E \rangle_{\beta(m)}$. Then there exists a unique $\Xi \in \mathcal{L}(\langle E \rangle_{\beta(m)})$ such that

$$\Xi(\xi, \eta) = \langle\langle \Phi, \xi \rangle \circ_\kappa \phi_\xi, \phi_\eta \rangle, \quad \xi, \eta \in E_C,$$

where $\langle \Phi, \xi \rangle \in \langle E \rangle_{\beta(m)}$ is given by

$$\langle\langle \Phi, \xi \rangle, \phi \rangle = \langle\langle \Phi, \xi \otimes \phi \rangle, \phi \in \langle E \rangle_{\beta(m)}.$$

**Proof.** The proof is immediately from that the function

$$\Theta(\xi, \eta) = \langle\langle \Phi, \xi \rangle \circ_\kappa \phi_\xi, \phi_\eta \rangle, \quad \xi, \eta \in E_C$$

satisfies (O1) and (O2') in Theorem 2.1 with $\beta = \beta(m)$. \qed

**Definition 5.2.** Let $\kappa \in (E_C^\otimes)^*$ and $\Phi \in E_C^* \otimes \langle E \rangle_{\beta(m)}$. The operator $\Xi$ defined in Proposition 5.1 is called a first order $\kappa$-differential operator with coefficient $\Phi$ and denoted by

$$\Xi = \int_{\mathbb{R}} \Phi(t) \circ_\kappa \partial_t dt$$

as a formal integral expression.

**Theorem 5.3.** Let $\kappa \in (E_C^\otimes)^*$ and $\Phi \in E_C^* \otimes \langle E \rangle_{\beta(m)}$. Then for $\Xi \in \mathcal{L}(\langle E \rangle_{\beta(m)})$ the following statements are equivalent:

(i) $\Xi$ is a first order $\kappa$-differential operator with coefficient $\Phi$.

(ii) For any $\xi \in E_C$ and $n \geq 0$, we have

$$\Xi(\langle \cdot, \cdot^\otimes n \rangle) = n(\langle \cdot, \cdot^\otimes (n-1), \cdot^\otimes (n-1) \rangle \circ_\kappa \langle \Phi, \xi \rangle).$$

(iii) For any $\xi \in E_C$ and $n \geq 0$, $\Xi(\langle \cdot, \xi^\otimes n \rangle) = n(\langle \cdot, \xi \rangle^\otimes (n-1) \circ_\kappa \langle \Phi, \xi \rangle)$. 

Proof. (i) ⇒ (ii) Since $\Xi$ is a first order $\kappa$-differential operator with coefficient $\Phi$, the symbol of $\Xi$ is given by

$$
\hat{\Xi}(\xi, \eta) = \langle \Xi \phi_\xi, \phi_\eta \rangle = \langle \langle \Phi, \xi \rangle \circ_\kappa \phi_\xi, \phi_\eta \rangle, \quad \xi, \eta \in E_\mathcal{C}.
$$

Hence we obtain that for any $\xi \in E_\mathcal{C}$, $\Xi \phi_\xi = \langle \Phi, \xi \rangle \circ_\kappa \phi_\xi$. Therefore for any $\xi \in E_\mathcal{C}$, $\phi \in (E)$ and $z \in \mathbb{C}$, we have

$$
\sum_{n=0}^{\infty} \frac{1}{n!} \langle \Xi(\langle \cdot \otimes^n \cdot, \xi \otimes^n \rangle), \phi \rangle z^n
$$

$$
= \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \langle \langle \Phi, \xi \rangle \circ_\kappa \langle \cdot \otimes^{(n-1)} \cdot, \xi \otimes^{(n-1)} \rangle, \phi \rangle z^n.
$$

Thus the proof follows.

(ii) ⇒ (i) The proof is obvious.

(ii) ⇔ (iii) Note that for any $\xi \in E_\mathcal{C}$, $G_\kappa^{-1}(\langle \cdot, \xi \rangle) = \langle \cdot, \xi \rangle$. Hence by the definition of $\circ_\kappa$, we obtain that for any $\xi \in E_\mathcal{C}$

$$
\sum_{n=0}^{\infty} \frac{1}{n!} \langle \cdot, \xi \rangle \circ_\kappa^n = G_\kappa^{-1} \phi_\xi = e^{\frac{1}{m} \langle \kappa, \xi \otimes^m \rangle} \phi_\xi.
$$

Therefore (ii) implies that for any $\xi \in E_\mathcal{C}$

$$
\sum_{n=0}^{\infty} \frac{1}{n!} \Xi(\langle \cdot, \xi \rangle \circ_\kappa^n) = e^{\frac{1}{m} \langle \kappa, \xi \otimes^m \rangle} \sum_{n=0}^{\infty} \frac{1}{n!} \Xi(\langle \cdot \otimes^n \cdot, \xi \otimes^n \rangle)
$$

$$
= \sum_{n=0}^{\infty} \frac{1}{n!} \langle \cdot, \xi \rangle \circ_\kappa^n \circ_\kappa \langle \Phi, \xi \rangle.
$$

Hence (ii) implies that for any $\xi \in E_\mathcal{C}$ and $z \in \mathbb{C}$

$$
\sum_{n=0}^{\infty} \frac{1}{n!} \Xi(\langle \cdot \otimes^n \cdot, \xi \otimes^n \rangle) z^n = \sum_{n=0}^{\infty} \frac{1}{n!} \langle \cdot \otimes^n \cdot, \xi \otimes^n \rangle \circ_\kappa \langle \Phi, \xi \rangle z^{n+1}.
$$

Similarly (iii) implies that for any $\xi \in E_\mathcal{C}$ and $z \in \mathbb{C}$

$$
\sum_{n=0}^{\infty} \frac{1}{n!} \Xi(\langle \cdot \otimes^n \cdot, \xi \otimes^n \rangle) z^n = \sum_{n=0}^{\infty} \frac{1}{n!} \langle \cdot \otimes^n \cdot, \xi \otimes^n \rangle \circ_\kappa \langle \Phi, \xi \rangle z^{n+1}.
$$

Thus we complete the proof. \qed
Theorem 5.4. Let \( \kappa \in (E_C^\otimes m)^* \) and \( \Xi \in \mathcal{L}((E)_{\beta(m)}) \). Then \( \Xi \) is a derivation with respect to \( \odot_\kappa \) if and only if \( \Xi \) is a first order \( \kappa \)-differential operator with some coefficient \( \Phi \in E_C^* \otimes (E)_{\beta(m)} \).

Proof. Let \( \Xi \) be a first order \( \kappa \)-differential operator with coefficient \( \Phi \in E_C^* \otimes (E)_{\beta(m)} \). Then we note that \( G_\kappa(\cdot, \xi) = G_\kappa^{-1}(\cdot, \xi) = \langle \cdot, \xi \rangle \) for all \( \xi \in E_C \). Then the operator \( \Xi' = G_\kappa \Xi G_\kappa^{-1} \) is a first order Wick differential operator with coefficient \( \Phi' \), where \( \Phi' \in E_C^* \otimes (E)_{\beta(m)} \) is given by \( \langle \Phi', \xi \rangle = G_\kappa \langle \Phi, \xi \rangle \) for each \( \xi \in E_C \). In fact, we have

\[
\Xi'(\langle \cdot, \xi \rangle^{\otimes n}) = G_\kappa \Xi G_\kappa^{-1}(\langle \cdot, \xi \rangle^{\otimes n})
= G_\kappa \Xi(\langle \cdot, \xi \rangle^{\otimes n})
= G_\kappa(n\langle \cdot, \xi \rangle^{\otimes (n-1)} \odot_\kappa \langle \Phi, \xi \rangle)
= n\langle \cdot, \xi \rangle^{\otimes (n-1)} \odot_\kappa \langle \Phi', \xi \rangle.
\]

So, by Theorem 4.5 in [1], \( \Xi' \) is a derivation with respect to \( \odot \) and hence by Proposition 4.1, \( \Xi \) is a derivation with respect to \( \odot_\kappa \).

Conversely, let \( \Xi \) be a derivation with respect to \( \odot_\kappa \). Define a map \( \widetilde{\Phi} : E_C \rightarrow (E)_{\beta(m)} \) by \( \widetilde{\Phi}(\xi) = \Xi(\langle \cdot, \xi \rangle), \xi \in E_C \). Then \( \Phi \in \mathcal{L}(E_C, (E)_{\beta(m)}) \). Hence there exists a unique \( \Phi \in E_C^* \otimes (E)_{\beta(m)} \) such that

\[
\langle \Phi, \xi \rangle = \Xi(\langle \cdot, \xi \rangle), \quad \xi \in E_C.
\]

Since \( \Xi \) is a derivation with respect to \( \odot_\kappa \), for any \( \xi \in E_C \) and \( n \geq 0 \) we have

\[
\Xi(\langle \cdot, \xi \rangle^{\otimes n}) = n\langle \cdot, \xi \rangle^{\otimes (n-1)} \odot_\kappa \Xi(\langle \cdot, \xi \rangle) = n\langle \cdot, \xi \rangle^{\otimes (n-1)} \odot_\kappa \langle \Phi, \xi \rangle.
\]

Thus by Proposition 5.3, \( \Xi \) is a first order \( \kappa \)-differential operator with coefficient \( \Phi \).

\[\square\]

Example 5.5. For each \( y \in E_C^* \) the differential operator \( D_y \) is a first order \( \kappa \)-differential operator with coefficient \( y \otimes 1 \).

Example 5.6. Let \( \kappa \in (E_C^\otimes m)^* \). Then by Corollary 3.7 and Proposition 4.1, \( \Xi_{0,m}(\kappa) + N \) is a derivation with respect to \( \odot_\kappa \). Moreover, by Theorem 5.4, \( \Xi_{0,m}(\kappa) + N \) is a first order \( \kappa \)-differential operator with coefficient \( \Phi_0 \), where \( \langle \Phi_0, \xi \rangle = (\Xi_{0,m}(\kappa) + N)(\langle \cdot, \xi \rangle) = \langle \cdot, \xi \rangle \). In particular, \( \gamma \Delta_G + N \) is the first order \( \gamma_t \)-differential operator with coefficient \( \Phi_0 \) (see [2]).
6. Applications

In this section, we shall discuss the eigenvalue problem, Cauchy problem and Poisson type equation associated with $\Xi_{0,m}(\kappa) + N$, $\kappa \in (E_C^{\otimes m})^*$. We now consider the eigenvalue problem associated with $\Xi_{0,m}(\kappa)+N$, i.e., we consider

$$ (\Xi_{0,m}(\kappa) + N)\psi = \lambda \psi, $$

where $\psi \in (E)_{\beta(m)}$ and $\lambda \in \mathbb{C}$ are unknown.

By using the fact that $\Xi_{0,m}(\kappa) + N = G_{\kappa}^{-1}NG_{\kappa}$, we easily have the following proposition:

**PROPOSITION 6.1.**

(i) $\lambda$ is an eigenvalue of $\Xi_{0,m}(\kappa) + N$ if and only if $\lambda$ is an eigenvalue of $N$.

(ii) $\phi$ is an eigenfunction of $\Xi_{0,m}(\kappa) + N$ if and only if $G_{\kappa}\phi$ is an eigenfunction of $N$.

(iii) The set of all eigenvalues of $\Xi_{0,m}(\kappa) + N$ is $\{0,1,2,\cdots\}$.

Now, we consider the following Cauchy problem:

$$ \frac{du_t}{dt} = -(\Xi_{0,m}(\kappa) + N)u_t, \quad u_0 = \phi \in (E)_{\beta(m)}. $$

Note that $\tilde{u}_t = G_{0,e^{-t}}$ is the one-parameter subgroup of $GL((E)_{\beta(m)})$ with infinitesimal generator $-N$ (see [5], [17], [24]). Hence we can easily check that $u_t = G_{\kappa}^{-1}G_{0,e^{-t}}G_{\kappa}$ is the one-parameter subgroup of $GL((E)_{\beta(m)})$ with infinitesimal generator $-(\Xi_{0,m}(\kappa) + N)$. Thus we have the following theorem.

**THEOREM 6.2.** Let $\phi \in (E)_{\beta(m)}$. Then $u_t = G_{\kappa}^{-1}G_{0,e^{-t}}G_{\kappa}\phi \in (E)_{\beta(m)}$ is a unique solution of the equation (6.2).

Finally, we consider the following Poisson type equation:

$$ (\Xi_{0,m}(\kappa) + N + \lambda I)u = \phi, $$

where $\phi \in (E)_{\beta(m)}$ and $\lambda > 0$. 
The $\lambda$-potential ($\lambda > 0$) of test functional $\phi \in (E)_{\beta(m)}$ is defined by

$$H_\lambda \phi = \int_0^\infty e^{-\lambda t} G_{0,e^{-t}} \phi dt,$$

where the integral is a white noise integral (see [17], [24]). For the case $\lambda = 0$, define the normalized potential of $\phi \in (E)_{\beta(m)}$ by

$$G\phi = \int_0^\infty G_{0,e^{-t}}(\phi - E(\phi)) dt,$$

where $E(\phi)$ is the expectation of $\phi$.

**Theorem 6.3** [17]. Let $\phi \in (E)_{\beta(m)}$. Then we have

$$NG\phi = \phi - E(\phi) \quad \text{and} \quad (N + \lambda I) H_\lambda \phi = \phi.$$

**Theorem 6.4.** Let $\kappa \in (E^\infty_m)^*$ and $\phi \in (E)_{\beta(m)}$. Then $u = G_\kappa^{-1} H_\lambda G_\kappa \phi \in (E)_{\beta(m)}$ is a solution of the equation (6.3).

**Proof.** Let $\phi \in (E)_{\beta(m)}$. Then by Theorem 6.3, $v = H_\lambda G_\kappa \phi$ is a solution of the equation $(N + \lambda I)v = G_\kappa \phi$. Hence we obtain that

$$(G_\kappa^{-1}(N + \lambda I)G_\kappa)G_\kappa^{-1}v = \phi.$$ 

Thus by Corollary 3.7, we have

$$(\Xi_{0,m}(\kappa) + N + \lambda I)G_\kappa^{-1}v = \phi,$$

That is, $u = G_\kappa^{-1} H_\lambda G_\kappa \phi$ satisfies the equation (6.3). \hfill \square

**Theorem 6.5.** Let $\phi \in (E)_{\beta(m)}$. Then we have

$$(\Xi_{0,m}(\kappa) + N)G_\kappa^{-1}G_\kappa \phi = \phi - E(G_\kappa \phi).$$

**Proof.** Let $\phi \in (E)_{\beta(m)}$. Then by Theorem 6.3, we have

$$NGG_\kappa \phi = G_\kappa \phi - E(G_\kappa \phi).$$

Hence we have

$$(G_\kappa^{-1}NG_\kappa)G_\kappa^{-1}G_\kappa \phi = \phi - G_\kappa^{-1}E(G_\kappa \phi) = \phi - E(G_\kappa \phi).$$

Thus by Corollary 3.7, we complete the proof. \hfill \square
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