ON THE MODULAR FUNCTION \( j_4 \) OF LEVEL 4

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ABSTRACT. Since the modular curves \( X(N) = \Gamma(N)\backslash \mathfrak{H}^* \) \((N = 1, 2, 3)\) have genus 0, we have field isomorphisms \( K(X(1)) \cong \mathbb{C}(J), K(X(2)) \cong \mathbb{C}(\lambda) \) and \( K(X(3)) \cong \mathbb{C}(j_3) \) where \( J, \lambda \) are the classical modular functions of level 1 and 2, and \( j_3 \) can be represented as the quotient of reduced Eisenstein series. When \( N = 4 \), we see from the genus formula that the curve \( X(4) \) is of genus 0 too. Thus the field \( K(X(4)) \) is a rational function field over \( \mathbb{C} \). We find such a field generator \( j_4(z) = x(z)/y(z) \) \((x(z) = \theta_3(\frac{z}{2}), y(z) = \theta_4(\frac{z}{2})\) Jacobi theta functions\). We also investigate the structures of the spaces \( M_k(\Gamma(4)), S_k(\Gamma(4)) \) and \( M_4(\Gamma(4)), S_4(\Gamma(4)) \) in terms of \( x(z) \) and \( y(z) \). As its application, we apply the above results to quadratic forms.

0. Introduction

Let \( \mathfrak{H} \) be the complex upper half plane. Then \( SL_2(\mathbb{Z}) \) acts on \( \mathfrak{H} \) by \((\begin{array}{cc} a & b \\ c & d \end{array}) \cdot \tau = \frac{a\tau + b}{c\tau + d} \) for \( \tau \in \mathfrak{H} \). Let \( \Gamma(N) \) \((N = 1, 2, 3, \cdots)\) be the principal congruence subgroups of \( SL_2(\mathbb{Z}) \) of level \( N \) and let \( \mathfrak{H}^* \) be the union of \( \mathfrak{H} \) and \( \mathbb{P}^1(\mathbb{Q}) \). The modular curve \( \Gamma(N)\backslash \mathfrak{H}^* \) is a projective closure of smooth affine curve \( \Gamma(N)\backslash \mathfrak{H} \), which we denote by \( X(N) \), with genus \( g_N \). We identify the function field \( K(X(N)) \) on the modular curve \( X(N) \) with the field of modular functions of level \( N \). By the genus formula ([11] Ch. IV §7, or [14] Proposition 1.40), the curves \( X(1), X(2) \) and \( X(3) \) have genus 0. Theoretically, we then have field isomorphisms \( K(X(1)) \cong \mathbb{C}(J), K(X(2)) \cong \mathbb{C}(\lambda) \) and \( K(X(3)) \cong \mathbb{C}(j_3) \) where \( J, \lambda \) are the classical modular functions of level 1 and 2, respectively and \( j_3 \)


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can be represented as the quotient of reduced Eisenstein series ([11] Ch. VII §1.2). Since the curve $X(4)$ is of genus 0 too, the field $K(X(4))$ is a rational function field over $\mathbb{C}$. In this case we shall find such a field generator $j_4$ (§2, Theorem 7) by means of theory of half integral modular forms. For generalities of half integral forms, we refer to [3] and [15].

In §1 we shall show, for later use, the generators and the cusps of the inhomogeneous group $\tilde{\Gamma}(4)$. In §3 we shall investigate the generators of the spaces $M_k(\Gamma(4)), S_k(\Gamma(4)), M_{3/2}(\tilde{\Gamma}(4))$ (the space of half integral modular forms of level 4) and $S_{3/2}(\tilde{\Gamma}(4))$ (the space of half integral cusp forms of level 4) in terms of Jacobi theta functions. Also, we shall prove in Theorem 16 that the normalized field generator $N(j_4)(z)$ is an algebraic integer for $z \in \mathcal{H} \cap \mathbb{Q}(\sqrt{-d}) \ (d > 0)$ (for notations, refer to [1]). In §4 we shall express $j_4$ as the quotient of reduced $\wp$-division values $\wp_{N, \alpha}$ where $\wp$ is the Weierstrass $\wp$-function. And we shall show in Theorem 18 that $\mathbb{Q}(j_4)$ is none other than the field of all the modular functions of level 4 whose Fourier expansions with respect to $q_4 (= e^{2\pi iz/2})$ have rational coefficients.

In §5 we shall apply the result that $K(X(4))$ is equal to $\mathbb{C}(j_4)$ to quadratic forms. Let $Q(n, 1)$ be the set of even unimodular positive definite integral quadratic forms in $n$-variables. For $A[X]$ in $Q(n, 1)$, the theta series $\theta_A(z) = \sum_{X \in \mathbb{Z}^n} e^{\pi i z A[X]} \ (z \in \mathcal{H})$ is a modular form of weight $\frac{n}{2}$. If $n \geq 24$ and $A[X], B[X] \in Q(n, 1)$, then the quotient $\frac{\theta_A(z)}{\theta_B(z)}$ is a modular function of level $N$. We shall extend the results in [5] to the case $N = 4$. In other words, since $\frac{\theta_A(z)}{\theta_B(z)}$ is also a modular function of level 4, we can write it as a rational function of $j_4$ (Theorem 21). In case $n = 24$, we shall be successful in §6 and Appendix B in completely determining the theta series $\theta_A(z)$ as symmetric polynomials over $\mathbb{Q}$ in $\theta_3(\frac{z}{2})$ and $\theta_4(\frac{z}{2})$ where $\theta_3, \theta_4$ are the Jacobi theta functions.

Through this article we adopt the following notations:

- $\mathcal{H}$ - the extended complex upper half plane
- $\Gamma(N) = \{ \gamma \in SL_2(\mathbb{Z}) | \gamma \equiv I \mod N \}$
- $\Gamma_0(N)$ the Hecke subgroup $\{ (a \ b \ c \ d) \in \Gamma(1) | c \equiv 0 \mod N \}$
- $X(N) = \Gamma(N) \backslash \mathcal{H}$
- $X_0(N) = \Gamma_0(N) \backslash \mathcal{H}$
- $\overline{\Gamma}$ the inhomogeneous group of $\Gamma(= \Gamma/ \pm I)$
- $q_n = e^{2\pi iz/h}, \ z \in \mathcal{H}$
- $M_k(\Gamma(N))$ - the space of modular forms of weight $k$ with respect to
the group \( \Gamma(N) \)
\( a \sim b \) means that \( a \) is equivalent to \( b \)
\( z \rightarrow i\infty \) denotes that \( z \) goes to \( i\infty \).

We shall always take the branch of the square root having argument in \( (-\frac{\pi}{2}, \frac{\pi}{2}] \). Thus, \( \sqrt{z} \) is a holomorphic function on the complex plane with the negative real axis \((-\infty, 0] \) removed. For any integer \( k \), we define \( z^{\frac{k}{2}} \) to mean \( (\sqrt{z})^k \).

1. Generators and cusps of \( \overline{\Gamma}(4) \)

Let \( \Gamma_1 \) and \( \Gamma_2 \) be two congruence subgroups of \( \Gamma(1) \) such that \( \Gamma_2 \subseteq \Gamma_1 \).
A subset \( \mathcal{F}_1 \) of the extended upper half plane \( \mathcal{H}^* \) is called a fundamental set for the group \( \overline{\Gamma}_1 \) if it contains exactly one representative of each class of points of \( \mathcal{H}^* \) equivalent under \( \overline{\Gamma}_1 \). A set \( \mathcal{F}_1 \) is called a fundamental region if \( \mathcal{F}_1 \) contains a fundamental set and if \( z \in \mathcal{F}_1, \gamma z \in \mathcal{F}_1 \) and \( \gamma(\neq I) \in \overline{\Gamma}_1 \) imply that \( z \) is a boundary point of \( \mathcal{F}_1 \).

Proposition 1. If \( \overline{\Gamma}_1 = \bigcup_{\nu=1}^{m} \overline{\Gamma}_2 \alpha_{\nu} \) is a coset decomposition of \( \overline{\Gamma}_1 \) and \( \mathcal{F}_1 \) is a fundamental region for \( \overline{\Gamma}_1 \), then \( \mathcal{F}_2 = \bigcup_{\nu=1}^{m} \alpha_{\nu}(\mathcal{F}_1) \) is a fundamental region for \( \overline{\Gamma}_2 \).

Proof. Theorem 2.3.5 [10]. \(\square\)

Theorem 2. Let \( \overline{\Gamma}_1 \) be a congruence subgroup of \( \overline{\Gamma}(1) \) of finite index and \( \mathcal{F} \) be a fundamental region for \( \overline{\Gamma}_1 \). Then the sides of \( \mathcal{F} \) can be grouped into pairs \( \lambda_j, \lambda_j' (j = 1, 2, \ldots, s) \) in such a way that \( \lambda_j \subseteq \mathcal{F} \) and \( \lambda_j' = \gamma_j \lambda_j \) where \( \gamma_j \in \overline{\Gamma}_1 (j = 1, 2, \ldots, s) \). \( \gamma_j \)'s are called boundary substitutions of \( \mathcal{F} \). Furthermore, \( \overline{\Gamma}_1 \) is generated by the boundary substitutions \( \gamma_1, \ldots, \gamma_s \).

Proof. For the first part, one is referred to [10], p. 58. For any \( \gamma \in \overline{\Gamma}_1 \), suppose there exists a sequence of images of \( \mathcal{F} \); \( \mathcal{F}, S_1 \mathcal{F}, S_2 \mathcal{F}, \ldots, S_n \mathcal{F} = \gamma \mathcal{F} \) (\( S_j \in \overline{\Gamma}_1 \)), each adjacent to its successor. Let \( \mathcal{F} \cap S_1 \mathcal{F} \supseteq \lambda_j' \). Since \( \gamma_j \lambda_j = \lambda_j' \) and \( \gamma_j \mathcal{F} \) is another fundamental region, \( \gamma_j \mathcal{F} = S_1 \mathcal{F} \), that is, \( S_1 = \gamma_j \). Then, \( \gamma_j \lambda_{i}, \gamma_j \lambda_{i}^{-1} (i = 1, 2, \ldots, s) \) form the sides of \( S_1 \mathcal{F} \). And \( (\gamma_j \gamma_i \gamma_j^{-1}) \gamma_j \lambda_{i} = \gamma_j \lambda_{i} \), i.e., \( \gamma_j \gamma_i \gamma_j^{-1} (i = 1, \ldots, s) \) are boundary substitutions of \( S_1 \mathcal{F} \). Now, we will use induction on \( n \) to show that \( S_n(=\gamma) \) is generated by \( \gamma_1, \ldots, \gamma_s \) and boundary substitutions are also generated.
by them. The case \( n = 1 \) has been done. Now, denote the sides of \( S_{n-1}\mathcal{F} \) by \( \mu_i, \mu'_i \) \((i = 1, 2, \ldots, s)\). Let \( L_i \mu_i = \mu'_i \) for \( i = 1, \ldots, s \). Then, by induction hypothesis, \( S_{n-1} \) and \( L_i \) \((i = 1, \ldots, s)\) are generated by \( \gamma_1, \ldots, \gamma_s \). If \( S_{n-1}\mathcal{F} \cap S_n\mathcal{F} \supseteq \mu'_j \), then \( L_j \mu_j = \mu'_j \) implies that \( L_j S_{n-1}\mathcal{F} = S_n\mathcal{F} \), i.e., \( S_n = L_j S_{n-1} \). Hence, it is generated by \( \gamma_1, \ldots, \gamma_s \). Also, the set of all points in \( \mathcal{F} \) belonging to the region \( S_n\mathcal{F} \) that can be reached by such sequences is open, and so also is its complement in \( \mathcal{F} \) which must therefore be empty by connectedness of \( \mathcal{F} \). This completes the proof of the theorem.

Now, we will find the generators of the group \( \Gamma(4) \) by means of Proposition 1 and Theorem 2. It is well known that the fundamental region for \( \Gamma(2) \) is given by the figure ([11], p. 84) where \( T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \ S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \).

![Diagram of fundamental region](image)

On the other hand, \( \Gamma(2) \) has the following right coset decomposition

\[
\Gamma(2) = \Gamma(4) \cup \Gamma(4)T^2 \cup \Gamma(4)\alpha \cup \Gamma(4)\beta
\]

where \( \alpha = ST^{-2}S \) and \( \beta = (ST)^{-1}T^2ST \). Then Proposition 1 gives rise to the following fundamental region for \( \Gamma(4) \).
Note that $-2 \sim 2$, $-1 \sim \frac{1}{3}$, $-\frac{3}{2} \sim \frac{1}{2}$ and $-\frac{5}{3} \sim 1$ in $\Gamma(4) \backslash \mathfrak{H}^*$, which illustrates that there are six $\Gamma(4)$-inequivalent cusps $\infty, 0, 1, -1, -2, \frac{1}{2}$. Now, we will choose appropriate elements from $\Gamma(4)$ which describe the above equivalences. The proof of Lemma 1.41 in [14] provides the idea of explicit construction of them. Based on it, one can have

\[
\begin{align*}
(\frac{1}{4} 0)^1 \cdot 0 &= 0 \\
(\frac{-3}{4} -\frac{4}{11}) \cdot (-1) &= 1/3 \\
(\frac{5}{8} 0) \cdot (-3/2) &= 1/2 \\
(\frac{-4}{7} -\frac{12}{7}) \cdot (-5/3) &= 1 \\
(\frac{1}{0} 0)^1 \cdot (-1) &= 1/3 \\
(\frac{-3}{8} -\frac{4}{11}) \cdot (-3/2) &= 1/2 \\
(\frac{5}{8} 0) \cdot (-5/3) &= 1 \\
(\frac{-4}{7} -\frac{12}{7}) \cdot (-2) &= 2.
\end{align*}
\]

Now, put $\gamma_1 = (\frac{1}{0} 0)^1$, $\gamma_2 = (\frac{1}{4} 0)^1$, $\gamma_3 = (\frac{-3}{4} -\frac{4}{11})$, $\gamma_4 = (\frac{5}{8} 0)$, and $\gamma_5 = (\frac{-4}{7} -\frac{12}{7})$. Then, as described in the above figure, $\gamma_i$ sends boundaries to boundaries for $i = 1, \ldots, 5$ because a linear fractional transformation maps a semicircle to a semicircle.

For the sake of convenience in use, we will express $\gamma_i$'s as a combination of $S$ and $T^2$. Obviously, $\gamma_1 = T^4$. Now, consider the case of $\gamma_2$. $\gamma_2 \infty = \frac{1}{4}$, $S(\gamma_2 \infty) = -4$, $T^4 S \gamma_2 \infty = 0$. By computing $T^4 S \gamma_2$, one gets $T^4 S \gamma_2 = S$. Hence $\gamma_2 = S^{-1} T^{-4} S$.

Next, consider the case of $\gamma_3$. $\gamma_3 \infty = \frac{3}{8}$, $S \gamma_3 \infty = -\frac{3}{8}$, $T^2 S \gamma_3 \infty = -\frac{3}{2}$, $ST^2 S \gamma_3 \infty = \frac{3}{2}$, $T^{-2} ST^2 S \gamma_3 \infty = -\frac{1}{2}$, $ST^{-2} ST^2 S \gamma_3 \infty = 2$. 

\[ T^{-2}S T^{-2}ST^2 S \gamma_3 \infty = 0, \ ST^{-2}S T^{-2}ST^2 S \gamma_3 \infty = \infty. \] By computing \[ ST^{-2}S T^{-2}ST^2 S \gamma_3, \] one gets \[ ST^{-2}S T^{-2}ST^2 S \gamma_3 = T^2. \] Hence, \[ \gamma_3 = S^{-1}T^{-2}S^{-1}T^2S^{-1}T^2 = ST^{-2}ST^2ST^2 \text{ since } S^{-1} = S. \]

By a similar computation, one has \[ \gamma_4 = ST^{-2}ST^{-2}ST^2ST^2 \]
\[ \gamma_5 = T^2S^{-1}T^4ST^2. \]

2. Hauptfunktionen of level 4 as a quotient of Jacobi theta functions

For \( \mu, \nu \in \mathbb{R} \) and \( z \in \mathcal{H} \), put
\[ \Theta_{\mu, \nu}(z) = \sum_{n \in \mathbb{Z}} \exp \left\{ \pi i \left( n + \frac{1}{2} \mu \right)^2 z + \pi i n \nu \right\}. \]

This series uniformly converges for \( \text{Im}(z) \geq \eta > 0 \), and hence defines a holomorphic function on \( \mathcal{H} \).

**Theorem 3.** If \( z \in \mathcal{H} \), then \( \Theta_{\mu, \nu}(z) = e^{-\frac{1}{2} \pi i \nu} \Theta_{\nu, \mu}(-1/z) \).

**Proof.** Theorem 7.1.1 [10]. \( \square \)

We recall the Jacobi theta functions \( \theta_2, \theta_3, \theta_4 \) defined by
\[ \theta_2(z) := \Theta_{1,0}(z) = \sum_{n \in \mathbb{Z}} q_n^{(n+\frac{1}{2})^2} \]
\[ \theta_3(z) := \Theta_{0,0}(z) = \sum_{n \in \mathbb{Z}} q_n^{n^2} \]
\[ \theta_4(z) := \Theta_{0,1}(z) = \sum_{n \in \mathbb{Z}} (-1)^n q_n^{n^2}. \]

Then we have the following transformation formulas.
Theorem 4. For all \( z \in \mathcal{H} \),

\[
\begin{align*}
(i) \quad \theta_2(z + 1) &= e^{\frac{1}{8}\pi i} \theta_2(z) \\
(ii) \quad \theta_2(-1/z) &= (-iz)^{\frac{1}{2}} \theta_4(z) \\
\theta_3(z + 1) &= \theta_4(z) \\
\theta_3(-1/z) &= (-iz)^{\frac{1}{2}} \theta_3(z) \\
\theta_4(z + 1) &= \theta_3(z) \\
\theta_4(-1/z) &= (-iz)^{\frac{1}{2}} \theta_2(z).
\end{align*}
\]

Proof. Theorem 7.1.2 [10].

Let \( x(z) = \theta_3(\frac{z}{2}) \) and \( y(z) = \theta_4(\frac{z}{2}) \). We then readily have the transformation formulas using the above theorem.

Corollary 5. For all \( z \in \mathcal{H} \),

\[
\begin{align*}
(i) \quad \theta_2(z + 4) &= -\theta_2(z) \\
x(z + 2) &= y(z) \\
x(z + 4) &= x(z) \\
y(z + 2) &= x(z), \ y(z + 4) = y(z)
\end{align*}
\]

\[
\begin{align*}
(ii) \quad \theta_2(-2/z) &= (-iz/2)^{\frac{1}{2}} y(z) \\
x(-1/z) &= (-2iz)^{\frac{1}{2}} x(4z) \\
x(-4/z) &= (-iz/2)^{\frac{1}{2}} x(z) \\
y(-1/z) &= (-2iz)^{\frac{1}{2}} \theta_2(2z).
\end{align*}
\]

Theorem 6. \( x(z), y(z) \in M_1(\overline{\Gamma}(4)) \).

Proof. First, we will show the slash operator invariance by making use of the idea from [3], p. 148. For \( \gamma' \in \Gamma_0(4) \) and \( z \in \mathcal{H} \),

\[
(2.1) \quad \Theta(\gamma'z) = j(\gamma', z) \Theta(z)
\]

where \( \Theta(z) = \sum_{n \in \mathbb{Z}} q^n(q = q_1) \) and \( j(\gamma', z) \) is the automorphy factor for \( \Gamma_0(4) \). Then \( x(z) = \Theta(\frac{z}{4}) \) for any \( z \in \mathcal{H} \).

For \( \gamma = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma(4) \), put \( \gamma' = \left( \begin{smallmatrix} a & \frac{b}{4} \\ c & d \end{smallmatrix} \right) \). Note that \( \gamma' \in \Gamma_0(4) \) and \( \gamma' \cdot \frac{z}{4} = \frac{2z}{4} \). Then \( x(\gamma z) = \Theta(\frac{2z}{4}) = \Theta(\gamma' \cdot \frac{z}{4}) \) and, by (2.1),

\[
\Theta \left( \gamma' \cdot \frac{z}{4} \right) = j \left( \gamma', \frac{z}{4} \right) \Theta \left( \frac{4}{z} \right)
\]

\[
= \left( \frac{4c}{d} \right)^{\frac{1}{d}} \sqrt{4c \cdot \frac{z}{4} + d \cdot x(z)}
\]

\[
= \left( \frac{c}{d} \right) \sqrt{cz + d \cdot x(z)} \text{ since } d \equiv 1 \mod 4
\]

\[
= j(\gamma, z)x(z).
\]
This implies

\[ x(\gamma z) = j(\gamma, z)x(z), \]

which means that \( x [\Gamma]_1 = x(z) \) for any \( \gamma \in \Gamma(4) \). For the case \( y(z) \), put \( T = \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right) \) as usual. Then by Corollary 5, \( y(z) = x(z + 2) = x(T^2 z) \).
Since \( \Gamma(4) \) is a normal subgroup of \( \Gamma(1) \), one has \( T^{-2} \Gamma(4) T^2 = \Gamma(4) \). For \( \gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma(4) \), put \( \gamma' = T^2 \gamma T^{-2} = \left( \begin{array}{cc} c & 0 \\ d & 2c \end{array} \right) \in \Gamma(4) \). Then,

\[
y(\gamma z) = x(T^2 \gamma z) = x(T^2(T^{-2} \gamma' T^2)z) = x(\gamma'(T^2 z)) = j(\gamma', T^2 z) x(T^2 z) \text{ by (2.2)} \]
\[
= \left( \frac{c}{d - 2c} \right) \sqrt{c(z + 2) + d - 2c \cdot y(z)} = \left( \frac{c}{d - 2c} \right) \sqrt{cz + d \cdot y(z)}.
\]

To get the identity \( y(\gamma z) = j(\gamma, z)y(z) \), it remains to check that

\[
(2.3) \quad \left( \frac{c}{d - 2c} \right) = \left( \frac{c}{d} \right) \text{ for } \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma(4).
\]

Write \( c = (-1)^{sgn(c)} 2^n \cdot c' \) where \( c' \) is not divisible by 2 and \( c' > 0 \).
Since \( d - 2c \equiv d \mod 8 \), we have \( (\frac{-1}{d-2c}) = (\frac{-1}{d}) \) and \( (\frac{2}{d-2c}) = (\frac{2}{d}) \).
Thus it suffices to show \( (\frac{c'}{d-2c}) = (\frac{c}{d}) \). From the generalized quadratic reciprocity law ([3], p. 153), we recall that \( (\frac{d}{c}) = (-1)^{\frac{c-1}{4} \cdot \frac{d-1}{4}} (\frac{c}{d}) \) if \( c \) or \( d \) is positive. Indeed, since \( d - 2c \equiv 1 \mod 4 \) implies \( (\frac{c}{d-2c}) = (\frac{d-2c}{c}) = (\frac{d}{c}) = (\frac{c}{d}) \),
Next, we check the cusp conditions. We saw in §1 that there are six \( \Gamma(4) \)-inequivalent cusps \( \infty, 0, 1, -1, -2, \frac{1}{2} \).

(i) \( s = \infty \):

From the definitions of \( x(z) \) and \( y(z) \)

\[
x(z) = \sum_{n \in \mathbb{Z}} q_4^{n^2} = 1 + 2q_4 + 2q_4^2 + 2q_4^3 + \ldots
\]
\[
y(z) = \sum_{n \in \mathbb{Z}} (-1)^n q_4^{n^2} = 1 - 2q_4 + 2q_4^2 - 2q_4^3 + \ldots,
\]

and so \( x(\infty) = y(\infty) = 1 \).

(ii) \( s = 0 \):
Take \( \xi = (S, \sqrt{z}) \) with \( S = \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) \). Observe that \( \xi_\infty = 0 \). Then
\[
x|_{[\xi]_{\frac{1}{2}}} = x(Sz)\sqrt{z}^{-1} \\
= (-2iz)^{\frac{1}{2}}z^{-\frac{1}{2}}x(4z) \quad \text{by Corollary 5} \\
= (-2i)^{\frac{1}{2}}x(4z)
\]
so that we conclude
\[
x(0) = \lim_{z \to i\infty} x|_{[\xi]_{\frac{1}{2}}} = (-2i)^{\frac{1}{2}}.
\]
Similarly
\[
y|_{[\xi]_{\frac{1}{4}}} = y(Sz)\sqrt{z}^{-1} \\
= (-2iz)^{\frac{1}{2}}z^{-\frac{1}{4}}\theta_2(2z) \quad \text{by Corollary 5} \\
= (-2i)^{\frac{1}{2}} \sum_{n \in \mathbb{Z}} q_4^{(n+\frac{1}{2})^2} \\
= (-2i)^{\frac{1}{2}} \sum_{n \in \mathbb{Z}} q_4^{(2n+1)^2} \\
= (-2i)^{\frac{1}{2}}(2q_4 + 2q_4^9 + 2q_4^{25} + \ldots),
\]
hence \( y \) has a zero of order 1 at 0.

(iii) \( s = 1 \):
Take \( \xi = (ST^{-1}S, \sqrt{-z - 1}) \). Then \( \xi_\infty = 1 \) and \( x|_{[\xi]_{\frac{1}{2}}} = x(ST^{-1}Sz) \cdot \sqrt{-z - 1}^{-1} \). It follow from Corollary 5 that \( x(Sz) = (-2iz)^{\frac{1}{2}}x(4z) \), \( x(ST^{-1}z) = (-2iz + 2i)^{\frac{1}{2}}x(4z) \), and \( x(ST^{-1}Sz) = (2i^2z + 2i)^{\frac{1}{2}}(-i)^{\frac{1}{2}}x(z) \) = \((1 + z)^{\frac{1}{2}}x(z)\). Hence, \( x|_{[\xi]_{\frac{1}{4}}} = (1 + z)^{\frac{1}{2}}x(z) \). As \( z \to i\infty \), we have that \( x(1) = -i \). On the other hand, we have \( y|_{[\xi]_{\frac{1}{4}}} = y(ST^{-1}Sz)\sqrt{-z - 1}^{-1} \). Meanwhile, we know again by Corollary 5 that \( y(Sz) = (-2iz)^{\frac{1}{2}}\theta_2(2z) \), \( y(ST^{-1}z) = (-2iz + 2i)^{\frac{1}{2}}\theta_2(2z - 2) = (-2iz + 2i)^{\frac{1}{2}}(-i)\theta_2(2z) \), and \( y(ST^{-1}Sz) = (-i)(1 + z)^{\frac{1}{2}}y(z) \). Thus we come up with \( y|_{[\xi]_{\frac{1}{4}}} = (-i)(1 + z)^{\frac{1}{2}}(-i)^{-1}(1 + z)^{-\frac{1}{2}}y(z) \). As \( z \to i\infty \), \( y(1) = -1 \).

(iv) \( s = -1 \):
Take \( \xi = (STS, \sqrt{z - 1}) \). Then \( \xi_\infty = -1 \) and \( x(STz) = (-2iz - 2i)^{\frac{1}{2}}x(4z) \), \( x(STSz) = (2i^2z^2 - 2i)^{\frac{1}{2}}(-i)^{\frac{1}{2}}x(z) = (1 - z)^{\frac{1}{2}}x(z) \). Therefore, \( x|_{[\xi]_{\frac{1}{4}}} = i^{-1}x(z) \). As \( z \to i\infty \), \( x(1) = -i \). Similarly, \( y(STz) = (-2iz -
2i^{1/2}iθ_2(2z), and \(y(STSz) = \iota(1 - z^{1/2}y(z).\) Hence, \(y|_{s_1} = y(z).\) As \(z \to \iota\infty, y(-1) = 1.\)

(v) \(s = -2: \)

Take \(\xi = (T^{-2}S, \sqrt{2}).\) Then \(\xi\iota\infty = -2\) and \(x(T^{-2}z) = x(z - 2) = y(z),\) \(x(T^{-2}S) = y(-\frac{1}{2}) = (-2i\xi)^{1/2}\theta_2(2z).\) Therefore, \(x|_{s_1} = x(T^{-2}S)z^{-1/2} = (-2i\xi)^{1/2}z^{-1/2}\theta_2(2z) = (-2i)^{1/2}\theta_2(2z) = (-2i)^{1/2}(2q_4 + 2q_6^2 + 2q_8^2 + \ldots).\) It then follows that \(x\) has a zero of order 1 at \(-2.\) In a similar way, \(y(T^{-2}z) = y(z - 2) = x(z)\) and \(y(T^{-2}S) = x(-\frac{1}{2}) = (-2i\xi)^{1/2}x(4z).\) This yields that \(y|_{s_1} = (-2i)^{1/2}x(4z).\) As \(z \to \iota\infty,\) we have \(y(-2) = (-2i)^{1/2}.\)

(vi) \(s = \frac{1}{2}: \)

Take \(\xi = (ST^{-2}S, \sqrt{-2z - 1}).\) Then \(x(ST^{-2}z) = (-2i\xi + 4i)^{1/2}x(4z)\) and \(x(ST^{-2}S) = -(2i\xi + 4i)^{1/2}(1 + 2z)^{1/2}x(z);\) hence \(x|_{s_1} = -i^{-1}x(z).\) As \(z \to \iota\infty, x(\frac{1}{2}) = -i.\) In like manner, \(y(ST^{-2}z) = (-2i\xi + 4i)^{1/2}\theta_2(2z - 4) = (-2i\xi + 4i)^{1/2}(-1)\theta_2(2z),\) \(y(ST^{-2}S) = -(1 + 2z)^{1/2}y(z).\) Therefore \(y|_{s_1} = -i^{-1}y(z).\) As \(z \to \iota\infty,\) we have \(y(\frac{1}{2}) = i.\) \(\square\)

Put
\[
j_4(z) = \frac{x(z)}{y(z)} = 1 + 4q_4 + 8q_4^2 + 16q_4^3 + 32q_4^4 + 56q_4^5 + 96q_4^6 + 160q_4^7 + \cdots.
\]

**Theorem 7.** \(K(X(4)) = \mathbb{C}(j_4)\) and \(j_4\) has the following value at each cusp: \(j_4(\infty) = 1, j_4(0) = \infty\) (a simple pole), \(j_4(1) = i, j_4(-1) = -i, j_4(-2) = 0\) (a simple zero), \(j_4(1/2) = -1.\)

**Proof.** First, we claim that for \(f(z) \in M_4(\overline{\Gamma}(4)), f^2(z) \in M_4(\Gamma(4)).\) In fact, if \(\gamma \in \Gamma(4)\) then we have \(f|_{\gamma} = f(z).\) This is equivalent to \(f(\gamma z) = f(z)j(\gamma, z),\) that is, \(f(\gamma z) = f(z)(\gamma)^{1/2}\gamma^{1/2}\gamma^{1/2}\gamma z + d = f(z)(\gamma)^{1/2}\gamma^{1/2}\gamma^{1/2}\gamma z + d\) since \(d \equiv 1\ mod\ 4.\) Squaring both sides, we have \(f^2(\gamma z) = f^2(z) \cdot (cz + d)\) for any \(\gamma \in \Gamma(4).\) Therefore \(f^2 \in M_4(\Gamma(4)).\) Thus by Theorem 6 \(x^2(z), y^2(z) \in M_1(\Gamma(4)).\) Meanwhile, we saw in the proof of Theorem 6 that each of \(x(z)\) and \(y(z)\) has a simple zero at only one cusp. Observe that for \(f \in M_k(\Gamma(N))\), the sum of zeros is \(\nu_0(f) = \frac{\mu_N k}{12}\) where \(\mu_N = |\overline{\Gamma} : \overline{\Gamma}(N)|.\) It then follows that \(\nu_0(x^2) = \nu_0(y^2) = \frac{\mu_5 k}{12} = 2.\) Since \(x^2\) and \(y^2\) already have a zero of order 2 at cusps, they have no zero in \(S.\) This asserts that \(\deg(j_4)_0 = 1,\) and hence \([K(X(4)) : \mathbb{A}] = 1.\)
\[ \mathbb{C}(j_4) = \text{deg}(j_4)_0 = 1. \] The second part is immediate by definition and Theorem 6.

**Proposition 8.** The cusps of \( \Gamma(4) \) are regular in the sense of half integral weight forms. (for definitions and notations, refer to [3], Ch. IV)

**Proof.** We know that if \( f(z) \in M_{k}(\tilde{\Gamma}(4)) \), then \( f(s) = 0 \) for a \( k \)-irregular cusp \( s \). Since \( x(z) \) and \( y(z) \) belong to \( M_{\frac{k}{2}}(\tilde{\Gamma}(4)) \), if a 1-irregular cusp \( s \) exists then we must have \( x(s) = y(s) = 0 \). We saw, however, in the proof of Theorem 6 that such a cusp does not exist.

**Alternative proof of** Proposition 8. At \( \infty \), we readily see that \( \xi = 1 \), \( h = 1 \) and \( t = 1 \). At 0, take \( \xi = (((0 \hfill 1) \hfill 0) \hfill 1) \), so that \( \xi^{-1} = ((0 \hfill 1) \hfill 0) \), \( -i\sqrt{z} \). We need \( \tilde{\Gamma}(4) \ni \xi ((1 \hfill 0) \hfill 1), t \xi^{-1} = ((-1 \hfill 0) \hfill 1), -it\sqrt{hz - 1} \), which is valid when \( h = 4 \) and \( t = 1 \).

At the cusp 1, take \( \alpha = ((-1 \hfill 0) \hfill -1) \) and \( \xi = ((-1 \hfill 0) \hfill 1) \), \( \sqrt{-z - 1} \) so that \( \xi^{-1} = ((-1 \hfill 0) \hfill 1), \sqrt{z - 1} \). One must choose \( h = 4 \) to obtain \( \alpha((1 \hfill h) \hfill 1, h) = ((-1 \hfill 0) \hfill 1) \). To find \( h \) we compute \( \xi(((1 \hfill \frac{1}{4}) \hfill 1) \), \( t \xi^{-1} = ((-1 \hfill 4\frac{1}{5}) \hfill 1, \sqrt{-4z + 5} \), which implies \( j((\hfill -\frac{3}{4} \hfill -\frac{1}{5}), z) = \frac{1}{t} = \sqrt{-4z + 5} \) provided that \( t = 1 \). Therefore 1 is regular.

At the cusp \(-1\), take \( \alpha = ((1 \hfill 0) \hfill 1), \xi = ((-1 \hfill 0) \hfill 1), \sqrt{-z - 1} \) so that \( \xi^{-1} = ((-1 \hfill 0) \hfill 1), \sqrt{-z - 1} \). To get \( \alpha((1 \hfill h) \hfill h) = ((-1 \hfill h) \hfill 1) \), one must take \( h = 4 \). For \( t \), compute \( \xi(((1 \hfill \frac{1}{4}) \hfill 1) \), \( t \xi^{-1} = ((-1 \hfill 4\frac{1}{3}) \hfill 1, \sqrt{-4z - 3} \), which gives \( j((\hfill -\frac{4}{3} \hfill -\frac{1}{3}), z) = \frac{1}{t} = \sqrt{-4z - 3} \) provided that \( t = 1 \). Thus \(-1\) is regular.

At the cusp \(-2\), take \( \alpha = ((-2 \hfill 1) \hfill 1), \xi = ((-2 \hfill 1) \hfill 1), \sqrt{z} \); hence \( \xi^{-1} = ((-1 \hfill 0) \hfill 1, \sqrt{-z - 2}) \). To have \( \alpha((1 \hfill h) \hfill 1) \), \( \alpha^{-1} = ((1 + 2h \hfill 1 - 2h) \hfill 1) \), one is to take \( h = 4 \). For \( t \), compute \( \xi(((1 \hfill \frac{1}{4}) \hfill 1) \), \( t \xi^{-1} = ((-1 \hfill 4\frac{1}{7}) \hfill 1, \sqrt{-4z - 7} \), which implies \( j((\hfill 1 \hfill 4 \hfill 167), z) = \frac{1}{t} = \sqrt{-4z - 7} \) provided that \( t = 1 \). Hence \(-2\) is regular.

Finally at the cusp \( \frac{1}{2} \), take \( \alpha = ((-1 \hfill 1) \hfill 0) \), \( \xi = ((-1 \hfill 1) \hfill 0), \sqrt{-2z - 1} \) so that \( \xi^{-1} = ((-1 \hfill 0) \hfill 1), \sqrt{2z - 1} \). To have \( \alpha((1 \hfill h) \hfill 1) \), \( \alpha^{-1} = ((1 - 2h \hfill 1 + 2h) \hfill 1) \), again one choose \( h = 4 \). To find \( t \), compute \( \xi(((1 \hfill \frac{1}{4}) \hfill 1) \), \( t \xi^{-1} = ((-1 \hfill 4\frac{1}{7}) \hfill 1, \sqrt{-16z + 9} \), which gives \( j((\hfill 1 \hfill 4 \hfill 167), z) = \frac{1}{t} = \sqrt{-16z + 9} \) when \( t = 1 \), which amounts to say that \( \frac{1}{2} \) is regular.
3. Structures of $M_k(\Gamma(4))$ and $S_k(\Gamma(4))$

We recall from [10] and [14] the following facts:

**FACT 1.** For $k \geq 2$ and $\Gamma'$ a congruence subgroup of $\Gamma(1)$, we have

\[
dim M_k(\Gamma') = \begin{cases} 
  g + \sigma_\infty(\Gamma') - 1 & \text{if } k = 2 \\
  (k - 1)(g - 1) + \frac{k}{2} \cdot \sigma_\infty(\Gamma') + \sum_{i=1}^{r} \left[ \frac{k(e_i - 1)}{2e_i} \right] & \text{if } k \text{ even} \\
  (k - 1)(g - 1) + \frac{u_k}{2} + \frac{u'(k-1)}{2} + \sum_{i=1}^{r} \left[ \frac{k(e_i - 1)}{2e_i} \right] & \text{if } k \text{ odd, } -1 \notin \Gamma' 
\end{cases}
\]

where $g$ is the genus of $\Gamma' \setminus \Gamma^*$, $\sigma_\infty(\Gamma')$ the number of $\Gamma'$-inequivalent cusps, $e_1, \ldots, e_r$ the orders of inequivalent elliptic elements of $\Gamma'$ and $u$ (resp. $u'$) the number of inequivalent regular (resp. irregular) cusps of $\Gamma'$.

\[
dim S_k(\Gamma') = \begin{cases} 
  \dim M_k(\Gamma') - \sigma_\infty(\Gamma') & \text{if } k > 2 \\
  g & \text{if } k = 2 \\
  0 & \text{otherwise.}
\end{cases}
\]

For $k = 1$ and $\Gamma' = \Gamma(N)$,

\[
dim M_1(\Gamma(N)) = \frac{\mu_N}{2N} \text{ with } \mu_N = [\overline{\Gamma}(1) : \overline{\Gamma}(N)], \text{ if } u \geq 2g - 2
\]

\[
dim S_1(\Gamma(N)) = 0 \text{ for } 3 \leq N \leq 11.
\]

**FACT 2.** Let $X(z) = \theta_4^4(z)$, $Y(z) = \theta_4^4(z)$ and $\lambda(z) = \frac{X(z)}{Y(z)}$. Then $X, Y \in M_2(\Gamma(2))$ and $K(X(2)) = C(\lambda)$.

**THEOREM 9.** (i) For $k \geq 1$, $\dim M_{2k}(\Gamma(2)) = k + 1$ and $\dim S_{2k}(\Gamma(2)) = k - 2$ if $k \geq 3$.

(ii) $M_{2k}(\Gamma(2))$ is spanned over $C$ by $k + 1$ functions $X^k, X^{k-1}Y, \ldots, Y^k$.

(iii) $S_{2k}(\Gamma(2))$ is spanned by $k - 2$ functions $\Delta_2 X^{k-3}, \Delta_2 X^{k-2}Y, \ldots, \Delta_2 Y^{k-3}$ where $\Delta_2 = XY(X - Y) \in S_6(\Gamma(2))$ and $k \geq 3$.

**Proof.** If $\Gamma' = \Gamma(2)$, we have $g = 0, \sigma_\infty = 3, e_i = 0$ for all $i$, $u = 3$, and $\mu_2 = 6$. Then (i) follows from Fact 1. Now, consider (iii). Note that $\lambda(\infty) = 0, \lambda(1) = \infty$, and $\lambda(0) = 1$ imply that $\Delta_2$ is a cusp form. For any $f \in M_6(\Gamma(2))$, the number of zeros of $f$ is

\[
\nu_0(f) = \frac{\mu_2 \cdot 6}{12} = \frac{6 \cdot 6}{12} = 3.
\]
Since $\Delta_2$ is a cusp form, $\nu_0(\Delta_2) \geq 3$. But, it follows by (3.1) that $\nu_0(\Delta_2) = 3$. Also, all zeros of $\Delta_2$ appear at the cusps, which means that $\Delta_2(z) \neq 0$ on $\mathfrak{H}$. Observe that each function stated in (iii) is in $S_{2k}(\Gamma(2))$ and the cardinality is the same as dim $S_{2k}(\Gamma(2))$. Therefore it is necessary to check their independency to justify (iii). Suppose that

$$\sum_{i=0}^{k-3} c_i \Delta_2 X^{k-3-i} Y^i = 0$$

for $c_i \in \mathbb{C}$.

Since $\Delta_2 Y^{k-3}$ never vanishes in $\mathfrak{H}$, dividing the above by $\Delta_2 Y^{k-3}$, we have

$$\sum_{i=0}^{k-3} c_i \lambda^{k-3-i} = 0.$$

Since $\lambda$ is transcendental over $\mathbb{C}$, $c_i = 0$ for all $i$. (ii) can be proved in a similar fashion. \hfill \Box

**Theorem 10.**

(i) $\dim M_k(\Gamma(4)) = 2k + 1$ for $k \geq 1$, $\dim S_k(\Gamma(4)) = 2k - 5$ for $k \geq 3$.

(ii) $M_k(\Gamma(4))$ is spanned over $\mathbb{C}$ by the functions $x^{2k}, x^{2k-1} y, \ldots, y^{2k}$.

(iii) Let $\Delta_4 = x y(x^4 - y^4)$. Then $\Delta_4 \in S_3(\Gamma(4))$ and for $k \geq 3$, $S_k(\Gamma(4))$ is spanned by $\Delta_4 x^{2k-6}, \Delta_4 x^{2k-7} y, \ldots, \Delta_4 y^{2k-6}$.

**Proof.** If $\Gamma' = \Gamma(4)$, we have $g = 0$, $\sigma_\infty = 6$, $e_i = 0$ for all $i$, $u = 6$, and $\mu_4 = 24$. Then (i) is immediate by Fact 1. We consider (ii) because (ii) can be handled in a similar way. By Theorem 6, the functions mentioned in (iii) belong to $M_k(\Gamma(4))$. Since $y(0) = 0$ and $x(-2) = 0$, $\Delta_4(0) = \Delta_4(-2) = 0$. And

$$(3.2) \quad \frac{\Delta_4}{y^6} = j_4(j_4^4 - 1).$$

If $s \neq 0$, $-2$ then $j_4(s)$ is a 4-th root of unity. Also, for $s \neq 0$, $y(s) \neq 0$. Hence, by (3.2), $\Delta_4$ is a cusp form. For any $f \in M_3(\Gamma(4))$, the number of zeros is

$$(3.3) \quad \nu_0(f) = \frac{\mu_4 \cdot 3}{12} = 6.$$ 

Since $\Delta_4$ is a cusp form, $\nu_0(\Delta_4) \geq 6$. But, by (3.3), $\nu_0(\Delta_4) = 6$ so that $\Delta_4$ never vanishes on $\mathfrak{H}$. Now, all functions in (iii) are in $S_k(\Gamma(4))$. It
remains to check that they are linearly independent because the cardinality is equal to the dimension of $S_k(\Gamma(4))$. Suppose that

$$
\sum_{i=0}^{2k-6} c_i \Delta_4 x^{2k-6-i} y^i = 0 \text{ for } c_i \in \mathbb{C}.
$$

Since $\Delta_4 y^{2k-6}$ never vanishes on $\mathfrak{H}$, dividing the above by $\Delta_4 y^{2k-6}$, we have

$$(3.4) \sum_{i=0}^{2k-6} c_i j_4^{2k-6-i} = 0.$$  

Here we have to show that $j_4$ is transcendental over $\mathbb{C}$. Choose any $c \in \mathbb{C}$ and consider $j_4 - c$. Since $j_4 - c$ is a nonconstant modular function, it has at least one zero. This implies that the image of $j_4$ is all of $\mathbb{C}$. But if we had an algebraic equation satisfied by $j_4$, then the image of $j_4$ would be mapped into the set of solutions of the algebraic equation which is at most a finite set. This is impossible. Therefore $c_i = 0$ for all $i$ in $(3.4)$.

\[\square\]

**Remark.** For any $\frac{k}{2} \in \mathbb{N}$, $M_{\frac{k}{2}}(\tilde{\Gamma}(4)) = M_{\frac{k}{2}}(\Gamma(4))$. Indeed, for $\gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma(4)$,

$$j(\gamma, z) = \left( \frac{c}{d} \right) \sqrt{cz+d} \text{ since } d \equiv 1 \text{ mod } 4.$$  

Since $k$ is even, $j(\gamma, z)^k = (cz+d)^{\frac{k}{2}}$, that is, $M_{\frac{k}{2}}(\tilde{\Gamma}(4))$ has the same automorphy factor as that of $M_{\frac{k}{2}}(\Gamma(4))$.

Before going further we will show the algebraic independency of $x(z)$ and $y(z)$. To this end, we need the following lemma.

**Lemma 11.** If $f_k + f_{k-1} + \cdots + f_0 = 0$ where $k \in \mathbb{N}$ and $f_i \in M_{\frac{k}{2}}(\tilde{\Gamma}(4))$ for $i = 0, \ldots, k$, then $f_i = 0$ for all $i = 0, \ldots, k$.

**Proof.** Fix an arbitrary point $z \in \mathfrak{H}$. Put $\gamma_i = \left( \begin{array}{cc} 1 & 0 \\ 4i & 1 \end{array} \right)$ for $i = 0, \ldots, k + 1$. Then $j(\gamma_i, z) = \left( \frac{d}{4i} \right) \sqrt{4iz+1} = \sqrt{4iz+1}$ are distinct. By the assumption,

$$f_k(\gamma_i z) + f_{k-1}(\gamma_i z) + \cdots + f_0(\gamma_i z) = 0.$$

Since $f_i \in M_{\frac{k}{2}}(\tilde{\Gamma}(4))$ for $i = 0, \ldots, k$, we have

$$j(\gamma_i, z)^k f_k(z) + j(\gamma_i, z)^{k-1} f_{k-1}(z) + \cdots + f_0(z) = 0.$$

for $i = 0, \ldots, k + 1$. This gives rise to the following linear system
\[
\begin{pmatrix}
    j(\gamma_1, z)^k & \cdots & j(\gamma_i, z) & 1 & f_k(z) \\
    j(\gamma_2, z)^k & \cdots & j(\gamma_2, z) & 1 & f_{k-1}(z) \\
    \vdots & \ddots & \vdots & \vdots & \vdots \\
    j(\gamma_{k+1}, z)^k & \cdots & j(\gamma_{k+1}, z) & 1 & f_0(z)
\end{pmatrix} = 0.
\]
Note that the determinant of the above system is the well-known Vandermonde determinant, which is nonzero because $j(\gamma, z)$'s are all distinct. Hence, $f_i(z) = 0$ for each $i$. Since $z$ is arbitrary, $f_i = 0$ for any $i$. □

Now, suppose that there exists a polynomial $F \in \mathbb{C}[X_1, X_2]$ which is satisfied by $x(z)$ and $y(z)$. By Theorem 6 and Lemma 11, we may assume that $F$ is homogeneous. Let $\deg F = n$. Then,
\[
\frac{F(x, y)}{y^n} = \sum_{k=0}^{n} a_k j_4^k = 0
\]
for $a_k \in \mathbb{C}$. Since $j_4$ is transcendental over $\mathbb{C}$, it follows that $a_k = 0$ for any $k$; hence $F = 0$. This guarantees the algebraic independency of $x$ and $y$.

**Theorem 12.**
\[
X(z) = \theta_2(z)^4 = \frac{1}{4}(x^4 - 2x^2y^2 + y^4) \\
Y(z) = \theta_3(z)^4 = \frac{1}{4}(x^4 + 2x^2y^2 + y^4).
\]

**Proof.** Note that $\infty$ is equivalent to $\frac{1}{2}$, $1 \sim -1$ and $0 \sim -2$ in the curve $\Gamma(2) \setminus \mathcal{D}^*$. Thus $\theta_2^4(\infty) = 0$ implies $\theta_2^4(\frac{1}{2}) = 0$. Also, $\theta_3^4(1) = 0$ implies $\theta_3^4(-1) = 0$. Considering the values of $x$ and $y$ at the cusps, we obtain
\[
(x^4 - 2x^2y^2 + y^4)(\infty) = 0 \quad (x^4 - 2x^2y^2 + y^4)(\frac{1}{2}) = 0 \quad (x^4 - 2x^2y^2 + y^4)(1) = 0 \quad (x^4 - 2x^2y^2 + y^4)(-1) = 0.
\]
Let us recall that for $f \in M_2(\Gamma(4))$, the number of zeros is
\[
\nu_0(f) = \frac{\mu_4 \cdot 2}{12} = 4.
\]
Since $\theta_2^4$ (resp. $\theta_3^4$) has a zero of order 1 at $\infty$ (resp. at 1) in $q_2$ expansion, it has a zero of order 2 in $q_4$ expansion. Meanwhile, (3.5)
shows that $\nu_0(\theta_2^4) = \nu_0(\theta_4^4) = 4$. Hence it turns out that they have no other zeros except those mentioned above. On the other hand, it follows from the equality $(x^4 \pm 2x^2y^2 + y^4) = (x^2 \pm y^2)^2$ that they have zeros of even order. Again by (3.5), they have no other zeros except those.

Therefore $\frac{\theta_2^4}{x^4 - 2x^2y^2 + y^4}$ has no zeros and no poles, which claims that the quotient is a constant. We use the transformation formula for $\theta_2$ in Theorem 4 and Theorem 6 to get that $\theta_2^4(0) = -1$ and $(x^4 - 2x^2y^2 + y^4)(0) = ((-2i)^\frac{1}{2})^4 = -4$. Hence, the constant should be $\frac{1}{4}$. Likewise, we can show the other case.

**Theorem 13** (Extended Version of Theorem 10). (i) For $k \geq 1$, \text{dim} \ M_{\frac{k}{2}}(\tilde{\Gamma}(4)) = k + 1$ and $M_{\frac{k}{2}}(\tilde{\Gamma}(4))$ is spanned by $x^k, x^{k-1}y, \ldots, y^k$, that is, it is the space of all polynomials in $\mathbb{C}[x, y]$ having pure weight $\frac{k}{2}$.

(ii) For $k \geq 6$, \text{dim} \ S_{\frac{k}{2}}(\tilde{\Gamma}(4)) = k - 5$ and $S_{\frac{k}{2}}(\tilde{\Gamma}(4))$ is generated by $\Delta_4x^{k-6}, \Delta_4x^{k-7}y, \ldots, \Delta_4y^{k-6}$ with $\Delta_4$ as in Theorem 10.

**Proof.** For (i), it is enough to consider the case $\frac{k}{2} \notin \mathbb{N}$. Note that $x^k, x^{k-1}y, \ldots, y^k$ are linearly independent and belong to $M_{\frac{k}{2}}(\tilde{\Gamma}(4))$ due to Theorem 6. Let $\alpha \in M_{\frac{k}{2}}(\tilde{\Gamma}(4))$. Then, $\alpha \cdot x \in M_{k+2}(\tilde{\Gamma}(4))$. Since $\frac{k+1}{2} \in \mathbb{N}$, by Theorem 10, we obtain

$$\alpha \cdot x = c_0x^{k+1} + c_1x^ky + \cdots + c_{k+1}y^{k+1}$$

for $c_i \in \mathbb{C}$. Now, evaluate the above at the cusp $s = -2$. Then $x(-2) = 0$ and $y(-2) \neq 0$ give $c_{k+1} = 0$. Since $x(z) \neq 0$ on $S$, we can divide the both sides in (3.6) by $x$. Then $\alpha \in \mathbb{C}x^k + \cdots + \mathbb{C}y^k$, from which (i) follows. (ii) can be similarly proved. The only nontrivial part is that $\Delta_4x^{k-6}, \Delta_4x^{k-7}y, \ldots, \Delta_4y^{k-6}$ span $S_{\frac{k}{2}}(\tilde{\Gamma}(4))$. Let $\beta \in S_{\frac{k}{2}}(\tilde{\Gamma}(4))$. Then $\beta \cdot x \in M_{k+2}(\tilde{\Gamma}(4))$. Since $\frac{k+1}{2}$ is an integer, it turns out that

$$\beta \cdot x = c_0\Delta_4x^{k-5} + \cdots + c_{k-5}\Delta_4y^{k-5}$$

for $c_i \in \mathbb{C}$. Comparing the order of zero at $-2$, we see that all terms except $c_{k-5}\Delta_4y^{2k-6}$ have the orders greater than or equal to 2. But the term $c_{k-5}\Delta_4y^{k-5}$ has the order 1 at $-2$, which forces us to have $c_{k-5} = 0$. Dividing the both sides of (3.7) by $x$, we come up with $\beta \in \mathbb{C}\Delta_4x^{k-6} + \cdots + \mathbb{C}\Delta_4y^{k-6}$. \hfill $\Box$
Example. Define
\[ \Theta(z) := \sum_{n \in \mathbb{Z}} q^{n^2} \quad (z \in \mathcal{H}). \]

Then \( \Theta \in M_{\frac{1}{2}}(\tilde{\Gamma}_0(4)) \) ([3], p. 184). Hence, \( \Theta \in M_{\frac{1}{2}}(\tilde{\Gamma}(4)) \) and, by Theorem 13, it can be written as a linear combination of \( x \) and \( y \)
\begin{equation}
\Theta = ax + by
\end{equation}
for some \( a, b \in \mathbb{C} \). Observe that \( \Theta(\infty) = 1 \) and \( \Theta(\frac{1}{2}) = 0 \) because \( \frac{1}{2} \) is a 1-irregular cusp of \( \Gamma_0(4) \). Evaluating (3.8) at the cusps \( \infty \) and \( \frac{1}{2} \), we get \( a = b = \frac{1}{2} \). Therefore the result is
\[ \Theta = \frac{1}{2}x + \frac{1}{2}y. \]

Before closing this section we try to find the relations between \( j_4 \) and the classical modular functions \( J \) and \( \lambda \).

Theorem 14. (i) We have
\[ \lambda = \frac{j_4^4 - 2j_4^2 + 1}{j_4^4 + 2j_4^2 + 1} \]
and the irreducible polynomial of \( j_4 \) is \( Z^4 + 2\frac{\lambda + 1}{\lambda - 1} Z^2 + 1 \in \mathbb{C}(\lambda)[Z] \) over \( \mathbb{C}(\lambda)(= K(X(2))) \).

(ii) Let \( J \) be the classical modular function of level 1 with \( J(i) = 1 \). Then one has
\[ J = \frac{1}{108} \frac{(j_4^8 + 14j_4^4 + 1)^3}{(j_5^8 - j_4^8)} \]
and the irreducible polynomial of \( j_4 \) over \( \mathbb{C}(J) \) is \( (Z^8 + 14Z^4 + 1)^3 - 108J(Z^8 - Z)^6 \in \mathbb{C}(J)[Z] \).

Proof. In (i), the equality of \( \lambda \) follows from Theorem 12. Observe that
\[ [K(X(4)) : K(X(2))] = [\tilde{\Gamma}(2) : \tilde{\Gamma}(4)] = 4. \]
Hence, \( \deg(\text{Irr}(j_4, \mathbb{C}(\lambda))) = 4 \). Clearly, \( j_4 \) satisfies \( Z^4 + 2\frac{\lambda + 1}{\lambda - 1} Z^2 + 1 \). Thus, the two polynomials are the same. Since
\[ J = \frac{4 (\lambda^2 - \lambda + 1)^3}{27 \lambda^2(\lambda - 1)^2} \]
([10], p. 228), plugging (i) into the above we come up with the equality in (ii). Now, \( K(X(1)) = \mathbb{C}(J), K(X(4)) = \mathbb{C}(j_4) \) and
\[ [K(X(4)) : K(X(1))] = [\tilde{\Gamma}(1) : \tilde{\Gamma}(4)] = 24. \]
And $\deg(\text{Irr}(j_4, \mathbb{C}(J))) = 24$. By the same reason as in (i), the second part of (ii) follows. 

For $z \in \mathbb{Q}(\sqrt{-d}) \cap \mathbb{R}$ ($d > 0$), it is well-known that $j(z) (= 1728J(z))$ is an algebraic integer ([6], [14]). For algebraic proofs, see [2], [7], [13] and [16]. Therefore it is natural to ask whether $j_4(z)$ is so or not. Although we have Theorem 14 at hand, the answer for the above question seems to be negative because the modular function $J(z)$ has no Fourier expansion of the form $q^{-1}(1 + \sum_{n \geq 1} a_n q^n)$. To support the above claim, let us find a counter example as follows. Observe that

$$(3.9) \quad \theta_2(2z) = \frac{1}{2} \left( \theta_3 \left( \frac{z}{2} \right) - \theta_4 \left( \frac{z}{2} \right) \right).$$

$$(3.10) \quad \theta_3(2z) = \frac{1}{2} \left( \theta_3 \left( \frac{z}{2} \right) + \theta_4 \left( \frac{z}{2} \right) \right).$$

**Lemma 15.** (i) For $x \in \mathbb{R}_+$, $j_4(x^i) > 0$.
(ii) For $z \in \mathbb{R}_+$, $j_4(2z)^2 = \frac{1}{2}(j_4(z) + j_4(z)^{-1})$.
(iii) $j_4 \left( \frac{i}{2^n} \right) = \frac{j_4(2^n i)}{j_4(2^{n+1} i)} + 1$ for $n \in \mathbb{N} \cup \{0\}$.
(iv) $j_4(2z)^4 = \frac{1}{1-j_4(z)}$.

**Proof.** It follows from the definition that $\theta_3 \left( \frac{z}{2} \right) = \sum_{n \in \mathbb{Z}} e^{\pi i \frac{q^2}{4} n^2} = \sum_{n \in \mathbb{Z}} e^{-\pi i n^2} > 0$. And by Theorem 4 (ii) and (3.9), $\theta_4 \left( \frac{z}{2} \right) = \theta_4 \left( -\frac{z}{2} \right) = (-i\frac{z}{2})^{1/2} \theta_2 \left( \frac{2z}{x} \right) = \sqrt{2} \cdot \frac{1}{2} \left( \theta_3 \left( \frac{1}{2z} \right) - \theta_4 \left( \frac{1}{2z} \right) \right) > 0$. This implies (i). For the second, we readily get that

$$j_4(2z)^2 = \frac{\theta_3(z)^2}{\theta_4(z)^2} = \frac{\theta_3 \left( \frac{z}{2} \right)^2 + \theta_4 \left( \frac{z}{2} \right)^2}{2 \theta_3 \left( \frac{z}{2} \right) \theta_4 \left( \frac{z}{2} \right)}$$

by [10], Theorem 7.1.8

$$= \frac{1}{2} (j_4(z) + j_4(z)^{-1}).$$

Finally, for $n \in \mathbb{N} \cup \{0\}$

$$j_4 \left( \frac{i}{2^n} \right) = \frac{\theta_3 \left( \frac{i}{2^{n+1}} \right)}{\theta_4 \left( \frac{i}{2^{n+1}} \right)} = \frac{\theta_3(2^{n+1} i)}{\theta_2(2^{n+1} i)}$$

by Theorem 4 (ii)

$$= \frac{\theta_3(2^{n-1} i) + \theta_4(2^{n-1} i)}{\theta_3(2^{n-1} i) - \theta_4(2^{n-1} i)}$$

by (3.9) and (3.10)

$$= \frac{j_4(2^n i) + 1}{j_4(2^n i) - 1}.$$
Also, \( j_4(2z)^4 = \frac{\theta_3(z)^4}{\theta_4(z)^4} = \frac{\theta_3(z)^4}{\theta_3(z)^4 - \theta_2(z)^4} = \frac{1}{1 - \lambda(z)}. \) This completes the lemma.

In Lemma 15 (iii), let us take \( n = 0. \) Then we come up with \( j_4(i) = 1 \pm \sqrt{2}. \) By Lemma 15 (i), \( j_4(i) > 0 \) and so

\[(3.11) \quad \quad j_4(i) = 1 + \sqrt{2}. \]

Applying again Lemma 15 (i) and (ii) we obtain that \( j_4(2i) = \sqrt{2} \) and \( j_4(4i) = \sqrt{\frac{3 + \sqrt{2}}{2} - 1}. \) We claim at this stage that \( j_4(4i) \) cannot be an algebraic integer. Suppose that \( j_4(4i) \) belongs to the ring \( \mathcal{O} \) of algebraic integers. Then

\[\frac{\sqrt{2} + 1}{\sqrt{32}} \in \mathcal{O}, \text{ which implies } \frac{1}{\sqrt[8]{32}} \in \mathcal{O} \text{ because } \sqrt{2} + 1 \in \mathcal{O}^x.\]

We conclude from the above that

\[\left(\frac{1}{\sqrt[8]{32}}\right)^8 = \frac{1}{32} \in \mathcal{O},\]

which is a contradiction. Therefore \( j_4(4i) \) is not an algebraic integer. In order to overcome this obstacle we borrow the notion of normalized series from Conway-Norton’s paper ([1]).

**Theorem 16.** Let \( N(j_4)(z) = \frac{4}{j_4(z) - 1} + 2 = \frac{1}{q_4} + 0 + 2q_4^3 - q_4^7 - 2q_4^{11} + 3q_4^{15} + 2q_4^{19} + \cdots \) be the normalized generator of \( K(X(4)) \). Then for \( \tau \in \mathbb{Q}(-d) \cap \mathcal{O}, \) \( N(j_4)(\tau) \) is an algebraic integer.

**Proof.** Let \( j \) be the modular function whose Fourier expansion with respect to \( q \) is \( \frac{1}{q} + 744 + 196884q + \cdots \). Then \( j(\tau) \) is an algebraic integer for such \( \tau \). Note that \( J = \frac{j}{1728} \). By Theorem 14, we see that \( J = \frac{1}{108} \left( \frac{j_4^8 + 14j_4^4 + 1}{j_4^8 - j_4^4} \right)^3. \) Hence, substituting \( \frac{4}{N-2} + 1 \) for \( j_4 \), we obtain that

\[j = 2^4 \cdot \frac{(j_4^8 + 14j_4^4 + 1)^3}{(j_4^8 - j_4^4)^4} \]

\[= \frac{(N^8 + 224N^4 + 256)^3}{(N - 2)^4 N^4 (N^3 + 2N^2 + 4N + 8)^2} \text{ where } N = N(j_4)(\tau).\]
This implies that \( N(j_4)(\tau) \) is integral over \( \mathbb{Z}[j(\tau)] \) and hence \( N(j_4)(\tau) \) is integral over \( \mathbb{Z} \).

\[ \square \]

4. Examples

Theorem 13 implies that any \( f \) in \( M_4(\Gamma(4)) \) is a homogeneous polynomial in \( x \) and \( y \) whose degree is \( k \). Furthermore the polynomial expression is unique due to the algebraic independency of \( x \) and \( y \). In this section we will describe the modular function \( j_4 \) in terms of \( \varphi \)-division values and Fricke functions. First, we recall the definition of the \( N \)-th division values of \( \varphi \):

\[ \varphi_{N, \bar{a}}(\tau) := \varphi \left( \frac{a_1 \tau + a_2}{N}; L_\tau \right) \]

where \( \bar{a} = \left( \begin{array}{c} a_1 \\ a_2 \end{array} \right) \), \( L_\tau = \mathbb{Z} \tau + \mathbb{Z} \) and \( \varphi \) is the Weierstrass \( \varphi \)-function. Since \( \varphi(u; L) = \varphi(v; L) \) if and only if \( u \equiv \pm v \pmod{L} \), we see that

\[ \varphi_{N, \bar{a}} = \varphi_{N, \bar{b}} \iff \bar{a} \equiv \bar{b} \pmod{N\mathbb{Z}^2}. \]  

(4.1)

Now, define the reduced \( \varphi \)-division value \( \varphi_{N, \bar{a}}^* \) by

\[ \varphi_{N, \bar{a}}^* := \sum_{t \pmod{N}} \sum_{d \equiv 1 \pmod{N} \atop d > 0} \mu(d) \frac{\varphi_{N, \bar{a}}}{d^2} \]

where \( \bar{a} \) runs \( \pmod{N} \) with \( (a_1, a_2) = 1 \) and \( \mu \) is the Möbius function. Then we have \( \varphi_{N, \bar{a}}^* \in M_4(\Gamma(N)) \) and at the cusps of \( \Gamma(N) \)

\[ \varphi_{N, \bar{a}}^* \left( -\frac{d}{c} \right) = \begin{cases} N^2 - \frac{N^2}{\sigma_{\infty}(N)} & \text{if } -\frac{d}{c} \text{ is } \Gamma(N)\text{-equivalent to } -\frac{a_2}{a_1}, \\ -\frac{N^2}{\sigma_{\infty}(N)} & \text{otherwise.} \end{cases} \]  

(4.3)

For the standard facts mentioned above, we refer to [11], p. 171. By theorem 13, \( \varphi_{4, \bar{a}}^* \) is a homogeneous polynomial in \( x \) and \( y \) of degree 4. Thus we can write \( \varphi_{4, \bar{a}}^* \) as follows:

\[ \varphi_{4, \bar{a}}^* = c_0 x^4 + c_1 x^3 y + c_2 x^2 y^2 + c_3 x y^3 + c_4 y^4 \]
for \( c_i \in \mathbb{C} \). Using (4.3) and the values of \( x \) and \( y \) at the cusps of \( \Gamma(4) \) (see Theorem 6), we can determine the coefficients \( c_i \). In fact,

\[
\begin{align*}
  c_0 &= -\frac{1}{4} \varphi_{4, a}^*(0) \\
  c_1 &= \frac{1}{4} \varphi_{4, a}^*(\infty) - \frac{1}{4} \varphi_{4, a}^* \left( \frac{1}{2} \right) - \frac{1}{4} i \varphi_{4, a}^*(-1) + \frac{1}{4} i \varphi_{4, a}^*(1) \\
  c_2 &= \frac{1}{2} \varphi_{4, a}^*(\infty) + \frac{1}{2} \varphi_{4, a}^* \left( \frac{1}{2} \right) + \frac{1}{4} \varphi_{4, a}^* (0) + \frac{1}{4} i \varphi_{4, a}^*(-2) \\
  c_3 &= \frac{1}{4} \varphi_{4, a}^*(\infty) - \frac{1}{4} \varphi_{4, a}^* \left( \frac{1}{2} \right) + \frac{1}{4} i \varphi_{4, a}^*(-1) - \frac{1}{4} i \varphi_{4, a}^*(1) \\
  c_4 &= -\frac{1}{4} \varphi_{4, a}^*(-2)
\end{align*}
\]

with \( i = \sqrt{-1} \). Recall that there are 6 distinct reduced \( \varphi \)-division values which correspond to the cusps of \( \Gamma(4) \). They are as follows:

\[
\begin{align*}
  s &= \infty, \quad \varphi_{4, \left( \frac{0}{1} \right)}^* = \frac{2}{3} x^4 + 4x^3 y + 4x^2 y^2 + 4xy^3 + \frac{2}{3} y^4 \\
  s &= 0, \quad \varphi_{4, \left( \frac{1}{0} \right)}^* = -\frac{10}{3} x^4 + \frac{2}{3} y^4 \\
  s &= 1, \quad \varphi_{4, \left( \frac{-1}{1} \right)}^* = \frac{2}{3} x^4 - 4ix^3 y - 4x^2 y^2 - 4ixy^3 + \frac{2}{3} y^4 \\
  s &= -1, \quad \varphi_{4, \left( \frac{1}{1} \right)}^* = \frac{2}{3} x^4 - 4ix^3 y - 4x^2 y^2 + 4ixy^3 + \frac{2}{3} y^4 \\
  s &= -2, \quad \varphi_{4, \left( \frac{1}{2} \right)}^* = \frac{2}{3} x^4 - \frac{10}{3} y^4 \\
  s &= \frac{1}{2}, \quad \varphi_{4, \left( \frac{-1}{2} \right)}^* = \frac{2}{3} x^4 - 4x^3 y + 4x^2 y^2 - 4xy^3 + \frac{2}{3} y^4.
\end{align*}
\]

Using the above result, we get

\[
\frac{\varphi_{4, \left( \frac{0}{1} \right)}^* - \varphi_{4, \left( \frac{1}{0} \right)}^*}{\varphi_{4, \left( \frac{1}{2} \right)}^* - \varphi_{4, \left( \frac{0}{1} \right)}^*} = \frac{-4x^4 - 4x^3 y - 4x^2 y^2 - 4xy^3}{-4x^3 y - 4x^2 y^2 - 4xy^3 - 4y^4}
\]

\[
= \frac{-4x(x^3 + x^2 y + xy^2 + y^3)}{-4y(x^3 + x^2 y + xy^2 + y^3)}
= \frac{x}{y} = j_4.
\]

In this way, one can have a field generator of \( K(X(4)) \) in terms of reduced \( \varphi \)-division values.
REMARK. (Generation of $j_4$ with Fricke functions) Recalling the definition of Fricke function $f_{a_1,a_2}$ where $(a_1,a_2) \in \mathbb{Z}^2$ and both $a_1$ and $a_2$ are not multiple of $N$ ([6], [14]), then,

$$f_{a_1,a_2} = -2^7 \cdot 3^5 \frac{G_4 G_6}{\Delta} \varphi_{N,\tilde{a}}$$

with $\tilde{a} = (\frac{a_1}{a_2})$.

From the equality $j_4 = \frac{\varphi_4(\frac{1}{2})^* - \varphi_4(\frac{d}{q})^*}{\varphi_4(\frac{d}{2})^* - \varphi_4(\frac{d}{q})^*}$ and (4.2), it follows that

$$j_4 = \frac{\sum_{t \equiv 0, dt \equiv 1 \mod 4} \mu(d) \mu(d)}{\sum_{t \equiv 0, dt \equiv 1 \mod 4} \mu(d) \mu(d)}(f_{t,0} - f_{0,t}).$$

In the above, consider the summation $\sum_{d>0, dt \equiv 1 \mod 4} \mu(d) \mu(d)$. Note that when $t = 0, 2$ there is no $d$ satisfying the congruence equation $dt \equiv 1 \mod 4$. Now put $a = \sum_{d>0, dt \equiv 3 \mod 4} \mu(d) \mu(d)$ and $b = \sum_{d>0, dt \equiv 3 \mod 4} \mu(d) \mu(d)$. Then in (4.4),

$$j_4 = \frac{a (f_{1,0} - f_{0,1}) + b (f_{3,0} - f_{0,3})}{a (f_{1,2} - f_{0,1}) + b (f_{3,0} - f_{0,3})} = \frac{a (f_{1,0} - f_{0,1}) + b (f_{1,0} - f_{0,1})}{a (f_{1,2} - f_{0,1}) + b (f_{1,2} - f_{0,1})} \quad \text{by (4.1)}$$

$$= \frac{f_{1,0} - f_{0,1}}{f_{1,2} - f_{0,1}}.$$

**Lemma 17.** For $n$ even, let $f \in M_2(\Gamma(4))$. If $f$ has a Fourier expansion with rational coefficients, then it can be written as a homogeneous polynomial over $\mathbb{Q}$ in $x$ and $y$ whose degree is $n$.

**Proof.** By Theorem 10,

$$f = \sum_{j=0}^{n} a_j x^{n-j} y^j, \quad a_j \in \mathbb{C}.$$

We must show that each $a_j$ lies in $\mathbb{Q}$. Considering Fourier expansions of $f$ and $x^{n-j} y^j$ gives

$$f = \sum_{i=0}^{\infty} b_i q_4^i, \quad b_i \in \mathbb{Q}$$

$$x^{n-j} y^j = \sum_{i=0}^{\infty} c_{ij} q_4^i, \quad c_{ij} \in \mathbb{Q}.$$
Plugging (4.5) into the above and comparing the coefficients of \( q_4 \)-expansion, we get the following linear system:

\[
(4.6) \quad (c_{ij})_{i \geq 0, \ 0 \leq j \leq n} \begin{pmatrix} a_0 \\ \vdots \\ a_n \end{pmatrix} = (b_i)_{i \geq 0}.
\]

Note that the \( j \)-th column of the matrix \((c_{ij})\) corresponds to the Fourier coefficients of \( x^{n-j}y^j \). Since \( x^n, x^{n-1}y, \ldots, y^n \) are linearly independent over \( \mathbb{C} \), the matrix \((c_{ij})\) has rank \( n + 1 \). This allows us to choose \( n + 1 \) rows from \((c_{ij})\) which are linearly independent. Without loss of generality, we may assume that the matrix \((c_{ij})_{0 \leq i,j \leq n}\) is invertible. Now, instead of (4.6), consider the following system:

\[
(c_{ij})_{0 \leq i,j \leq n} \begin{pmatrix} a_0 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} b_0 \\ \vdots \\ b_n \end{pmatrix}.
\]

Multiplying through by the inverse of the matrix \((c_{ij})_{0 \leq i,j \leq n}\), we have \( a_0, \ldots, a_n \in \mathbb{Q} \) as desired.

Theorem 18. \( \mathbb{Q}(j_4) \) coincides with the field \( \mathcal{F}_4 \) of all the modular functions of level 4 whose Fourier expansions with respect to \( q_4 \) have rational coefficients.

Proof. By [14], Proposition 6.9 we know that \( \mathcal{F}_4 = \mathbb{Q}(J(z), J(4z), f_{1,0}(z)) \). Since \( x \) and \( y \) have rational Fourier coefficients, so also has \( j_4 \). Hence, \( \mathbb{Q}(j_4) \) is contained in \( \mathcal{F}_4 \). For the reverse inclusion we need to show that \( J(z), J(4z), f_{1,0}(z) \in \mathbb{Q}(j_4) \). From Theorem 14 (ii) we see that \( J(z) \in \mathbb{Q}(j_4^{4}(z)) \), and so \( J(z) \in \mathbb{Q}(j_4) \). Next, observe that \( J(4z) \in \mathbb{Q}(j_4^{4}(4z)) = \mathbb{Q}(x^{4}(4z)) \). To claim \( J(4z) \in \mathbb{Q}(j_4) \), it is enough to show that \( x^{4}(4z) \) and \( y^{4}(4z) \) are homogeneous polynomials over \( \mathbb{Q} \) in \( x \) and \( y \) of degree 4. By an example in §3, we obtain \( \frac{1}{2}(x + y) = \theta_3(2z) \). Simple calculation leads us to \( \frac{1}{2}(x - y) = \theta_2(2z) \). Therefore, \( x^{4}(4z) = \theta_3(2z)^4 = \frac{1}{2}(x + y)^4 \) and \( y^{4}(4z) = \theta_4(2z)^4 = \theta_3(2z) \theta_2(2z) = \frac{1}{2}(x + y)^4 - \frac{1}{2}(x - y)^4 \). Finally, we consider the Fricke function \( f_{1,0}(z) \). Recall that \( f_{1,0} = -27 \cdot 3^5 \frac{G_6}{\Delta} \). As is shown in the proof of [14], Proposition 6.9 or [11], pp. 169-170, \( \pi^{-2} \theta_{4}(\frac{x}{y}) \) has rational Fourier coefficients. On the other hand \( \pi^{-4}G_4, \pi^{-6}G_6 \) and \( \pi^{-12}\Delta \) have the same
property. Furthermore, they can be viewed as modular forms of level 4. Thus, by Lemma 17, they can be written as homogeneous polynomials over $\mathbb{Q}$ in $x$ and $y$. This implies that $f_{1,0}(z) \in \mathbb{Q}(j_4)$.

5. Application to quadratic forms

Lemma 19. (i) Let $f \in M_{2k}(\Gamma(1))$. Then $f$ is a symmetric homogeneous polynomial over $\mathbb{C}$ in $x^4(z)$ and $y^4(z)$ whose degree is $k$.

(ii) Let $g \in M_{2k}(\Gamma(2))$. Then $g$ is a symmetric homogeneous polynomial in $x^2(z)$ and $y^2(z)$ whose degree is $2k$.

Proof. By Theorem 9 and 10,

$$f(z) = p_1(X(z), Y(z)) = p_2(x(z), y(z))$$

where $p_1$ and $p_2$ are homogeneous polynomials in two variables with deg $p_1 = k$ and deg $p_2 = 4k$. We claim that $p_1$ and $p_2$ are symmetric. In fact,

$$p_1(X, Y) = f|_{ST}$$ since $f \in M_{2k}(\Gamma(1))$

$$= p_1(X|_{ST}, Y|_{ST}) = p_1(Y, X)$$ by Theorem 4.

Also,

$$p_2(x, y) = f|_{T}$$ since $f \in M_{2k}(\Gamma(1))$

$$= p_2(x(z + 2), y(z + 2)) = p_2(y, x)$$ by Corollary 5.

Recall from Theorem 12 that $X = \frac{1}{4}(x^2 - y^2)^2$ and $Y = \frac{1}{4}(x^2 + y^2)^2$. Substituting $x$ for $x$ and $-y$ for $y$ we see that $X$ and $Y$ are unchanged. This implies by (5.1) that $p_2(x, -y) = p_2(x, y)$, that is, $p_2$ involves terms whose degree of $y$ is even. Also, substituting $x$ for $x$ and $iy$ for $y$, $X$ and $Y$ interchange with each other. Since $p_1$ is symmetric, by (5.1) we have $p_2(x, iy) = p_2(x, y)$, i.e., $p_2$ has terms whose degree of $y$ is a multiple of 4.

In the case of (ii), $p_2$ is symmetric and the equality $p_2(x, -y) = p_2(x, y)$ still holds. The assertion follows from these facts.

For $p(x) \in \mathbb{C}[x]$, we call $p(x)$ symmetric if $p(x) = x^k p(\frac{1}{x})$ with $k = \deg p(x)$.

Corollary 20. (i) Let $f_1, f_2 \in M_{2k}(\Gamma(1))$. Then,

$$\frac{f_1(z)}{f_2(z)} = \frac{p(j_4^4(z))}{q(j_4^4(z))}$$
where $p$ and $q$ are symmetric polynomials in one variable whose degrees are less than or equal to $k$.

(ii) Let $g_1, g_2 \in M_{2k}(\Gamma(2))$. Then,

$$\frac{g_1(z)}{g_2(z)} = \frac{p(j_4^2(z))}{q(j_4^2(z))}$$

where $p$ and $q$ are symmetric polynomials of degree less than or equal to $2k$.

**Proof.** Obvious. \qed

Now, we will consider the theta series associated to quadratic forms. Let $Q(n, 1)$ be the set of even unimodular positive definite integral quadratic forms in $n$-variables. Then $n \equiv 0 \mod 8$ ([12], ch.V). For $A[X]$ in $Q(n, 1)$, the theta series defined by

$$\theta_A(z) = \sum_{X \in \mathbb{Z}^n} e^{\pi i A[X]} (z \in \mathcal{H})$$

is a modular form of weight $\frac{n}{2}$ and level 1. In cases $n = 8$ and 16, the quotients $\frac{\theta_A}{\theta_B}$ are 1 for $A[X], B[X] \in Q(n, 1)$. If $n \geq 24$, then we have the following theorem.

**THEOREM 21.** For any two quadratic forms $A[X], B[X] \in Q(n, 1)$,

$$\frac{\theta_A(z)}{\theta_B(z)} = \frac{p(j_4(z))}{q(j_4(z))}$$

where $p$ and $q$ are symmetric polynomials over $\mathbb{Q}$ in $j_4$ of degree $n$.

**Proof.** From Lemma 17 and Lemma 19 we see that $\theta_A$ and $\theta_B$ are symmetric homogeneous polynomials over $\mathbb{Q}$ in $x(z)$ and $y(z)$ whose degree is $n$. In both cases the coefficients of the term $x^n$ do not vanish because $\theta_A(0) = \theta_B(0) = 1, x(0) \neq 0$ and $y(0) = 0$ by Appendix A. Now the result follows. \qed

**6. Examples**

In case $n = 24$, we are able to completely determine the polynomials discussed in Theorem 21.
Lemma 22. Let $E_6$ be the Eisenstein series of weight 6 of level 1 with $E_6(\infty) = 1$ and $F = (2\pi)^{-12}\Delta$ where $\Delta$ is the modular discriminant. Then we have

$$E_6 = -\frac{1}{64}x^{12} + \frac{33}{64}x^8y^4 + \frac{33}{64}x^4y^8 - \frac{1}{64}y^{12}$$

$$F = \frac{1}{2^{16}}x^4y^4(x^4 - y^4)^4.$$ 

Proof. By Lemma 17 and Lemma 19, $E_6$ can be written as

$$E_6 = ax^{12} + bx^8y^4 + bx^4y^8 + ay^{12}$$

for some $a, b \in \mathbb{Q}$. Evaluating both sides at some cusps of $\Gamma(4)$, we will determine $a$ and $b$. First, at $s = 0, 1 = E_6(0) = a \cdot x(0)^{12} = a \cdot (-2i)^{12} = a \cdot (\frac{-1}{64})$; hence $a = -\frac{1}{64}$. Next, at $s = \infty, 1 = E_6(\infty) = a + b + b + a = 2 \cdot (-\frac{1}{64}) + 2b$ and hence $b = \frac{33}{64}$. Now, consider the case of $F$. As is well known ([10], p. 222), we have the following equality:

$$F = \frac{1}{2^8}\theta_2^8\theta_3^8\theta_4^8$$

$$= \frac{1}{2^8}X^2Y^2(Y - X)^2 \text{ by the relation } \theta_3^4 = \theta_2^4 + \theta_4^4 \text{ and Fact 2}$$

$$= \frac{1}{2^8}\frac{1}{4^4}(x^4 - y^4)^4(x^2y^2)^2 \text{ by Theorem 12}$$

$$= \frac{1}{2^{16}}x^4y^4(x^4 - y^4)^4.$$ 

This completes the lemma. 

\[\square\]

Proposition 23. For $A \in \mathbb{Q}(24, 1)$,

$$\theta_A(z) = a^2x^{24} + (2ab + \frac{g_A}{2^{14}})x^{20}y^4 + (b^2 + 2ab - \frac{g_A}{2^{14}})x^{16}y^8$$

$$+ (2a^2 + 2b^2 + \frac{3g_A}{2^{15}})x^{12}y^{12}$$

$$+ (b^2 + 2ab - \frac{g_A}{2^{14}})x^8y^{16} + (2ab + \frac{g_A}{2^{16}})x^4y^{20} + a^2y^{24}$$

where $a = -\frac{1}{64}, b = \frac{33}{64}$ and $g_A = c_A + \frac{762048}{691} = r_A(1) + 1008 \in \mathbb{Z}$ depending on Niemeier's classification ([8]).

Proof. Since $E_{12}$ and $F$ span $M_{12}(\Gamma(1))$, we can express

(6.1) $$\theta_A = E_{12} + c_AF = E_6^2 + g_AF.$$
By comparing $q$-expansion we get $g_A = c_A + \frac{762048}{691}$. Now, plugging the results in Lemma 22 into (6.1), we obtain the assertion. \hfill \Box

Appendix A

For 6 cusps of $\Gamma(4)$, we have the following table:

<table>
<thead>
<tr>
<th></th>
<th>$\infty$</th>
<th>0</th>
<th>1</th>
<th>-1</th>
<th>-2</th>
<th>$\frac{1}{2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X$</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>$Y$</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>0</td>
<td>1</td>
<td>$\infty$</td>
<td>$\infty$</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>$x$</td>
<td>1</td>
<td>$\sqrt{-2i}$</td>
<td>-i</td>
<td>-i</td>
<td>0</td>
<td>-i</td>
</tr>
<tr>
<td>$y$</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>$\sqrt{-2i}$</td>
<td>i</td>
</tr>
<tr>
<td>$j_4$</td>
<td>1</td>
<td>$\infty$</td>
<td>i</td>
<td>-i</td>
<td>0</td>
<td>-1</td>
</tr>
</tbody>
</table>

Appendix B

From Proposition 23, the formula (9) in [4] and following Niemeier's notation,

$$
\theta_{3 \times E_8}(z) = \theta_{E_8} \oplus D_{4}(z) = \frac{1}{4096} x^{24} + \frac{21}{2048} x^{20} y^4 + \frac{591}{1024} x^{16} y^8 + \frac{707}{512} x^{12} y^{12} + \frac{591}{1024} x^8 y^{16} + \frac{21}{2048} x^4 y^{20} + \frac{1}{4096} y^{24} 
$$

$$
\theta_{E_7} \oplus E_7 \oplus D_{10}(z) = \theta_{E_7} \oplus A_{17}(z) = \frac{1}{4096} x^{24} + \frac{3}{512} x^{20} y^4 + \frac{272}{1024} x^{16} y^8 + \frac{85}{128} x^{12} y^{12} + \frac{663}{1024} x^8 y^{16} + \frac{3}{512} x^4 y^{20} + \frac{1}{4096} y^{24} 
$$

$$
\theta_{D_{24}}(z) = \frac{1}{4096} x^{24} + \frac{33}{2048} x^{20} y^4 + \frac{495}{4096} x^{16} y^8 + \frac{748}{1024} x^{12} y^{12} + \frac{495}{1024} x^8 y^{16} + \frac{33}{2048} x^4 y^{20} + \frac{1}{4096} y^{24} 
$$

$$
\theta_{D_{12} \oplus D_{12}}(z) = \frac{1}{4096} x^{24} + \frac{15}{2048} x^{20} y^4 + \frac{639}{4096} x^{16} y^8 + \frac{689}{1024} x^{12} y^{12} + \frac{639}{1024} x^8 y^{16} + \frac{15}{2048} x^4 y^{20} + \frac{1}{4096} y^{24} 
$$

$$
\theta_{3 \times D_8}(z) = \frac{1}{4096} x^{24} + \frac{9}{2048} x^{20} y^4 + \frac{687}{4096} x^{16} y^8 + \frac{671}{1024} x^{12} y^{12} + \frac{687}{1024} x^8 y^{16} + \frac{9}{2048} x^4 y^{20} + \frac{1}{4096} y^{24} 
$$

$$
\theta_{D_8 \oplus A_{13}}(z) = \frac{1}{4096} x^{24} + \frac{21}{2048} x^{20} y^4 + \frac{675}{4096} x^{16} y^8 + \frac{1351}{2048} x^{12} y^{12} + \frac{675}{1024} x^8 y^{16} + \frac{21}{4096} x^4 y^{20} + \frac{1}{4096} y^{24} 
$$

$$
\theta_{1 \times E_6}(z) = \theta_{E_6} \oplus D_7 \oplus A_{17}(z) = \frac{1}{4096} x^{24} + \frac{15}{2048} x^{20} y^4 + \frac{699}{4096} x^{16} y^8 + \frac{1333}{2048} x^{12} y^{12} + \frac{699}{1024} x^8 y^{16} + \frac{15}{4096} x^4 y^{20} + \frac{1}{4096} y^{24} 
$$
\[ \theta_{4 \times D_6}(z) = \theta_{D_6} \bigoplus A_8 \bigoplus A_8(z) = \frac{1}{4096} x^{24} + \frac{3}{1024} x^{20} y^4 + \frac{711}{4096} x^{16} y^8 + \frac{331}{512} x^{12} y^{12} + \frac{711}{4096} x^8 y^{16} + \frac{3}{1024} x^4 y^{20} + \frac{1}{4096} y^{24} \]

\[ \theta_{D_6} \bigoplus D_6 \bigoplus A_4 \bigoplus A_1(z) = \frac{1}{4096} x^{24} + \frac{9}{4096} x^{20} y^4 + \frac{723}{4096} x^{16} y^8 + \frac{1315}{2048} x^{12} y^{12} + \frac{723}{4096} x^8 y^{16} + \frac{9}{4096} x^4 y^{20} + \frac{1}{4096} y^{24} \]

\[ \theta_{3 \times A_8}(z) = \frac{1}{4096} x^{24} + \frac{21}{8192} x^{20} y^4 + \frac{717}{4096} x^{16} y^8 + \frac{2639}{4096} x^{12} y^{12} + \frac{717}{4096} x^8 y^{16} + \frac{21}{8192} x^4 y^{20} + \frac{1}{4096} y^{24} \]

\[ \theta_{A_4}(z) = \frac{1}{4096} x^{24} + \frac{69}{8192} x^{20} y^4 + \frac{621}{4096} x^{16} y^8 + \frac{2733}{4096} x^{12} y^{12} + \frac{621}{4096} x^8 y^{16} + \frac{69}{8192} x^4 y^{20} + \frac{1}{4096} y^{24} \]

\[ \theta_{A_1} \bigoplus A_1(z) = \frac{1}{4096} x^{24} + \frac{33}{8192} x^{20} y^4 + \frac{693}{4096} x^{16} y^8 + \frac{2675}{4096} x^{12} y^{12} + \frac{693}{4096} x^8 y^{16} + \frac{33}{8192} x^4 y^{20} + \frac{1}{4096} y^{24} \]

\[ \theta_{6 \times D_1}(z) = \theta_{D_1} \bigoplus (4 \times A_3)(z) = \frac{1}{4096} x^{24} + \frac{3}{2048} x^{20} y^4 + \frac{735}{4096} x^{16} y^8 + \frac{653}{1024} x^{12} y^{12} + \frac{735}{4096} x^8 y^{16} + \frac{3}{2048} x^4 y^{20} + \frac{1}{4096} y^{24} \]

\[ \theta_{6 \times A_6}(z) = \frac{1}{4096} x^{24} + \frac{15}{8192} x^{20} y^4 + \frac{729}{4096} x^{16} y^8 + \frac{2621}{4096} x^{12} y^{12} + \frac{729}{4096} x^8 y^{16} + \frac{15}{8192} x^4 y^{20} + \frac{1}{4096} y^{24} \]

\[ \theta_{9 \times A_4}(z) = \frac{1}{4096} x^{24} + \frac{9}{8192} x^{20} y^4 + \frac{741}{4096} x^{16} y^8 + \frac{2603}{4096} x^{12} y^{12} + \frac{741}{4096} x^8 y^{16} + \frac{9}{8192} x^4 y^{20} + \frac{1}{4096} y^{24} \]

\[ \theta_{9 \times A_2}(z) = \frac{1}{4096} x^{24} + \frac{3}{4096} x^{20} y^4 + \frac{747}{4096} x^{16} y^8 + \frac{1297}{4096} x^{12} y^{12} + \frac{747}{4096} x^8 y^{16} + \frac{3}{4096} x^4 y^{20} + \frac{1}{4096} y^{24} \]

\[ \theta_{12 \times A_4}(z) = \frac{1}{4096} x^{24} + \frac{3}{8192} x^{20} y^4 + \frac{753}{4096} x^{16} y^8 + \frac{2585}{4096} x^{12} y^{12} + \frac{753}{4096} x^8 y^{16} + \frac{3}{8192} x^4 y^{20} + \frac{1}{4096} y^{24} \]

\[ \theta_{24 \times A_1}(z) = \frac{1}{4096} x^{24} + \frac{759}{4096} x^{16} y^8 + \frac{121}{256} x^{12} y^{12} + \frac{759}{4096} x^8 y^{16} + \frac{1}{4096} y^{24} \]

\[ \theta_{C_0}(z) = \frac{1}{4096} x^{24} + \frac{3}{4096} x^{20} y^4 + \frac{771}{4096} x^{16} y^8 + \frac{1279}{2048} x^{12} y^{12} + \frac{771}{4096} x^8 y^{16} - \frac{3}{4096} x^4 y^{20} + \frac{1}{4096} y^{24} \]

References


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