STABILITY THEOREMS FOR THE OPERATOR-VALUED FEYNMAN INTEGRAL: THE $\mathcal{L}(L_1(\mathbb{R}), C_0(\mathbb{R}))$ THEORY

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ABSTRACT. In this paper, we prove stability theorems for the operator-valued Feynman integral of certain functionals involving some Borel measures on $(0, t)$ as a bounded linear operator from $L_1(\mathbb{R})$ to $C_0(\mathbb{R})$.

0. Introduction

In 1984, Johnson proved a bounded convergence theorem for the operator-valued function space integral [6]. As far as we know, this is the first stability theorem for the integral as a bounded linear operator on $L_2(\mathbb{R}^n)$ where $n$ is any positive integer. In [9], Johnson and Skoug introduced stability theorems for the integral as an $\mathcal{L}(L_p(\mathbb{R}^N), L_{p'}(\mathbb{R}^N))$ theory, $1 < p \leq 2$, where $N$ is a positive integer such that $N < \frac{2p}{2-p}$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Chang studied stability theorems for the integral as a bounded linear operator from $L_1(\mathbb{R})$ to $C_0(\mathbb{R})$ [2]. In those papers mentioned above, they treated certain functionals which involve only the Lebesgue measure on the interval $(0, t)$.

In [7], Johnson and Lapidus established stability theorems for the integral as an $\mathcal{L}(L_2(\mathbb{R}^N), L_2(\mathbb{R}^N))$ theory for certain functionals involving any Borel measures on $(0, t)$. Chang and Ryu proved theorems insuring stability with respect to potentials and wave functions for the integral as a bounded linear operator on $L_p(\mathbb{R}^N)$ for certain functionals involving some Borel measures on $(0, t)$ [4].

Functionals we consider in this paper are defined in terms of potentials, wave functions and measures. We study the stability of the

Received March 20, 1998.
1991 Mathematics Subject Classification: 28C20.
Key words and phrases: operator-valued Feynman integral, stability theorem, potential, wave function, Borel measure, bounded linear operator.
operator-valued Feynman integral as an $L(L_1(\mathbb{R}), C_0(\mathbb{R}))$ theory for the functionals involving some Borel measures on $(0, t)$ with respect to potentials, wave functions and measures.

1. Preliminaries and notations

Let $\mathbb{R}, \mathbb{C}, \mathbb{C}^+ \text{ and } \tilde{\mathbb{C}}^+$ denote the set of all real numbers, all complex numbers, all complex numbers with positive real part and all nonzero complex numbers with nonnegative real part, respectively. $C_0(\mathbb{R})$ will denote the space of $\mathbb{C}$-valued continuous functions on $\mathbb{R}$ which vanish at $\infty$ with the supremum norm. $L_1(\mathbb{R})$ is the space of Borel measurable, $\mathbb{C}$-valued functions $\psi$ on $\mathbb{R}$ such that $|\psi|$ is integrable with respect to the Lebesgue measure $m$ on $\mathbb{R}$ with the norm $\|\psi\|_1 = \int |\psi| \, dm$. $L(L_1(\mathbb{R}), C_0(\mathbb{R}))$ will denote the space of bounded linear operators from $L_1(\mathbb{R})$ to $C_0(\mathbb{R})$. Let $\tilde{M}(0, t)$ denote the space of complex Borel measures $\eta$ on the interval $(0, t)$ which satisfy the following conditions:

1. If $\mu$ is the continuous part of $\eta$, the Radon-Nikodym derivative $\frac{d\mu}{dm}$ exists and is essentially bounded, where $m$ is the Lebesgue measure on $(0, t)$.

2. $\eta = \sum_{j=1}^{k} w_j \delta_{\tau_j} + \mu$, where $\delta_{\tau_j}$ is the Dirac measure at $\tau_j \in (0, t)$, $0 < \tau_1 < \cdots < \tau_k < t$ and $w_j \in \mathbb{C}$ for $j = 1, 2, \cdots, k$.

Let $r \in (2, \infty]$ and $\eta \in \tilde{M}(0, t)$. Let $L_{1;r;\eta}([0, t] \times \mathbb{R}) \equiv L_{1;r;\eta}$ be the space of all Borel measurable $\mathbb{C}$-valued functions $\theta$ on $[0, t] \times \mathbb{R}$ such that

\begin{equation}
\| \theta \|_{1;r;\eta} = \left\{ \int_{(0, t)} \| \theta(s, \cdot) \|_1^r \, d|\eta|(s) \right\}^{\frac{1}{r}}
\end{equation}

is finite. If $\theta$ is in $L_{1;r;\eta}$ and $\eta = \mu + \nu$ is the Lebesgue decomposition, it is not difficult to show that $\theta \in L_{1;r;\mu} \cap L_{1;r;\nu}$. Let $\eta \in \tilde{M}(0, t)$. A Borel measurable $\mathbb{C}$-valued function $\theta$ on $[0, t] \times \mathbb{R}$ is said to belong to $L_{\infty;1;\eta}$ if

\begin{equation}
\| \theta \|_{\infty;1;\eta} = \int_{(0, t)} \| \theta(s, \cdot) \|_{\infty} \, d|\eta|(s)
\end{equation}
is finite. For $\lambda \in \mathbb{C}^+$, $\psi \in L_1(\mathbb{R})$ and a positive real number $s$, let

\begin{equation}
(C_{\lambda/s}\psi)(\xi) = \left(\frac{\lambda}{2\pi s}\right)^{1/2} \int_{\mathbb{R}} \psi(u) \exp\left(-\frac{\lambda(u - \xi)^2}{2s}\right) \, dm(u).
\end{equation}

Then $C_{\lambda/s}$ is in $\mathcal{L}(L_1(\mathbb{R}), C_0(\mathbb{R}))$ and $\|C_{\lambda/s}\| \leq (|\lambda|/2\pi s)^{1/2}$ [8]. And as a function of $\lambda$, $C_{\lambda/s}$ is analytic in $\mathbb{C}^+$ and is weakly continuous in $\mathbb{C}^+$ [8]. Let $\theta$ be in $L_1(\mathbb{R})$ and let $M_\theta$ be the operator of multiplication from $C_0(\mathbb{R})$ to $L_1(\mathbb{R})$ given by $M_\theta \psi = \psi \theta$. Then $M_\theta$ is in $\mathcal{L}(C_0(\mathbb{R}), L_1(\mathbb{R}))$ and $\|M_\theta\| \leq \|\theta\|_1$ [3]. It will be convenient to let $\theta(s)$ denote $M_{\theta(s,r)}$ for $\theta$ in $L_{1;r;\eta}$.

Let $C[0,t]$ be the space of continuous functions on $[0,t]$ and the Wiener space, $C_0[0,t]$, will consist of those $x$ in $C[0,t]$ such that $x(0) = 0$. Integration over $C_0[0,t]$ will always be with respect to the Wiener measure $m_w$.

Let $F$ be a functional from $C[0,t]$ to $\mathbb{C}$. Given $\lambda > 0, \psi \in L_1(\mathbb{R})$ and $\xi \in \mathbb{R}$, let

\begin{equation}
(I_\lambda(F)\psi)(\xi) = \int_{C_0[0,t]} F(\lambda^{-\frac{1}{2}}x + \xi) \psi(\lambda^{-\frac{1}{2}}x(t) + \xi) \, dm_w(x).
\end{equation}

If $I_\lambda(F)\psi$ is in $C_0(\mathbb{R})$ as a function of $\xi$ and if the correspondence $\psi \to I_\lambda(F)\psi$ gives an element of $\mathcal{L} \equiv \mathcal{L}(L_1(\mathbb{R}), C_0(\mathbb{R}))$, we say that the operator-valued function space integral $I_\lambda(F)$ exists. Next suppose that there exists $\lambda_0 (0 < \lambda_0 < \infty)$ such that $I_\lambda(F)$ exists for all $\lambda$ in $(0, \lambda_0)$ and further suppose that there exists an $\mathcal{L}$-valued function which is analytic in $\mathbb{C}^+_{\lambda_0} \equiv \{\lambda \in \mathbb{C} \mid Re\lambda > 0, |\lambda| < \lambda_0\}$ and agree with $I_\lambda(F)$ on $(0, \lambda_0)$. Then this $\mathcal{L}$-valued function is denoted by $I^{an}_\lambda(F)$ and is called the operator-valued analytic Wiener integral of $F$ associated with $\lambda$. Finally, let $q$ be in $\mathbb{R}$ with $0 < |q| < \lambda_0$. Suppose there exists an operator $J^{an}_q(F)$ in $\mathcal{L}$ such that for every $\psi$ in $L_1(\mathbb{R})$, $J^{an}_q(F)\psi$ is the weak limit of $I^{an}_\lambda(F)\psi$ as $\lambda \to -iq$ through $\mathbb{C}^+_{\lambda_0}$. Then $J^{an}_q(F)$ is called the operator-valued Feynman integral of $F$ associated with $q$. 
As we continue, we will need to write
\[ [w_1 \theta(\tau_1, x(\tau_1)) + \cdots + w_m \theta(\tau_m, x(\tau_m)) + \theta(s, x(s))]^n \]
as a product of monomials. However, we will need more refined breakdown of the sum. It will be convenient to introduce a prime notation on sum like
\[ \sum' \quad \text{this sum is to be over integers} \]
\[ q_0, q_1, \cdots, q_{m-k}, \text{ where } q_0 \geq 0, q_1 \geq 1, \cdots, q_{m-k} \geq 1 \text{ and } q_0 + \cdots + q_{m-k} = n. \]
Using this notation, we have the following equality \[^3\]
\begin{equation}
\left[ \sum_{j=1}^{m} w_j \theta(\tau_j, x(\tau_j)) + \theta(s, x(s)) \right]^n
= \sum_{k=0}^{m} \sum_{1 \leq z_1 < \cdots < z_{m-k} \leq m} \sum' \frac{n!}{q_0! q_1! \cdots q_{m-k}!} \left[ w_{z_1} \theta(\tau_{z_1}, x(\tau_{z_1})) \right]^{q_1} \cdots \left[ w_{z_{m-k}} \theta(\tau_{z_{m-k}}, x(\tau_{z_{m-k}})) \right]^{q_{m-k}} \theta(s, x(s))^{q_0}.
\end{equation}

Let \( \eta \in \tilde{M}(0, t) \) and \( \theta \in L_{1_{\tilde{r}; \eta}}. \) Set
\begin{equation}
F_n(x) := \left( \int_{(0,t)} \theta(s, x(s)) \, d\eta(s) \right)^n, \quad x \in C[0, t], \quad n = 0, 1, 2, \cdots.
\end{equation}

Here, if \( n = 0, \) from the definition, we have \( I_\lambda(F_0) = C_{\lambda/t}. \)

We use the following two theorems from \([1,3]\) in the sequel.

**Theorem 1.1.** Let \( \eta = \sum_{j=1}^{m} w_j \delta_{\tau_j} + \mu \) where \( \delta_{\tau_j} \) is the Dirac measure at \( \tau_j \in (0, t), 0 < \tau_1 < \cdots < \tau_m < t \) and \( w_j \in \mathbb{C} \) for \( j = 1, 2, \cdots, m. \) Suppose that \( \theta(\tau_j, \cdot), j = 1, 2, \cdots, m, \) are essentially bounded. Then the operators \( I_\lambda^n(F_n) \) and \( J_q^{an}(F_n) \) exist for all \( \lambda \in \mathbb{C}^+ \) and all real
\( q \neq 0 \), respectively. Further for \( \lambda \in \mathbb{C}^+, \psi \in L_1(\mathbb{R}) \) and \( \xi \in \mathbb{R} \),

\[
(1.7) \quad (I^n_\lambda (F_n) \psi)(\xi) = \sum_{k=0}^{m} \sum_{1 \leq z_1 < \cdots < z_{m-k} \leq m} \sum_{q_0+q_1+\cdots+q_{m-k}=n} \frac{n! w_{z_1}^{q_1} \cdots w_{z_{m-k}}^{q_{m-k}}}{q_1! \cdots q_{m-k}!} \int_{\Delta^{z_1, \cdots, z_{m-k}}_{q_0, j_1, \cdots, j_{m-k+1}}} \cdots \circ L_{m-k}(\psi)(\xi) \ d \times_{i=1}^{q_0} \mu(s_i),
\]

where

\[
(1.8) \quad \Delta^{z_1, \cdots, z_{m-k}}_{q_0, j_1, \cdots, j_{m-k+1}} = \{(s_1, \cdots, s_{q_0}) \in (0, t)^{q_0} \mid 0 < s_1 < \cdots < s_{j_1} < \tau_{z_1} < s_{j_1+1} < \cdots < s_{j_1+\cdots+j_{m-k}} < \tau_{z_{m-k}} < s_{j_1+\cdots+j_{m-k+1}} < \cdots < s_{q_0} < t \}
\]

and for \((s_1, \cdots, s_{q_0}) \in \Delta^{z_1, \cdots, z_{m-k}}_{q_0, j_1, \cdots, j_{m-k+1}}\) and \(\alpha \in \{0, 1, \cdots, m - k\}\)

\[
(1.9) \quad L_\alpha = \theta(\tau_{z_\alpha})^{q_\alpha} \circ C_{\lambda/(s_{j_1+\cdots+j_{\alpha+1}}-\tau_{z_\alpha})} \circ \theta(s_{j_1+\cdots+j_{\alpha+1}}) \circ \cdots \circ \theta(s_{j_1+\cdots+j_{\alpha+1}}) \circ C_{\lambda/(\tau_{z_{\alpha+1}}-s_{j_1+\cdots+j_{\alpha+1}})}.
\]

(It is convenient to let \(\theta(\tau)^q\) denote the operator of multiplication by \([\theta(\tau, \cdot)]^q\), that is, \(\theta(\tau)^q = M_{[\theta(\tau, \cdot)]^q}\). We use the conventions \(\tau_0 = 0, \tau_{m+1} = t\) and \(\theta(\tau_0)^{q_0} = 1\), where \(1\) is the inclusion map.)

For all real \( q \neq 0 \), \((J^n_q (F_n) \psi)(\xi)\) is given by the right hand side of (1.7) with \(\lambda = -iq\). Finally we have for \(\lambda \in \mathbb{C}^+\),

\[
(1.10) \quad \|I^n_\lambda (F_n)\| \leq B_n(|\lambda|),
\]
(1.11) 
\[ B_n(|\lambda|) := (n!)^{\frac{1}{p}} \left[ \left( \frac{|\lambda|}{2\pi} \right)^{\frac{m+1}{p}} \vee \left( \frac{|\lambda|}{2\pi} \right)^{\frac{1}{q}} \right] \left[ \min_{1 \leq j \leq m} (\tau_j - \tau_{j-1}) \right]^{-\frac{m+1}{p}} \left[ \frac{\Gamma(l(1 - \frac{r'}{2}))}{\Gamma(1 - \frac{r'}{2})} \right]^{-\frac{m+1}{r'}} \left[ \frac{\Gamma(l(1 - \frac{r}{2}))}{\Gamma(1 - \frac{r}{2})} \right]^{\frac{m+1}{r}} \left[ \sum_{j=1}^{m} w_j \left( \|\theta(\tau_j, \cdot)\|_{\infty} \vee \|\theta(\tau_j, \cdot)\|_{1} \right) \right] \right]
\[ + \left( \sum_{j=1}^{m} (\tau_j - \tau_{j-1})^{1 - \frac{r'}{2}} \right)^{\frac{1}{r'}} \left( \frac{|\lambda|}{2\pi} \right)^{\frac{1}{q}} \|\theta\|_{1r ; \mu} \left\| \int_0^t \frac{d\mu}{dm} \right\|_{\infty} \left( \frac{1 - \frac{r'}{2}}{r'} \right)^{\frac{1}{r'}} \right]^{n}, \]

where \( l \) is a positive integer such that \( \Gamma(l(1 - \frac{r'}{2})) \) is the minimum value of \( \{ \Gamma(i(1 - \frac{r'}{2})) | i \in \mathbb{N} \} \), \( \Gamma \) is the gamma function, \( \frac{1}{r} + \frac{1}{r'} = 1 \) and the notation \( a \vee b \) means the maximum value of \( a \) and \( b \). The inequality (1.10) also holds for \( J_q^m(F_n) \) with \( |\lambda| \) replaced by \( |q| \).

Let \( \lambda_0 > 0 \) be given and \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) be an analytic function in \( \mathbb{C}_{\lambda_0}^+ \) such that \( \sum_{n=0}^{\infty} |a_n| B_n(|\lambda|) < \infty \) for all \( \lambda \) in \( \mathbb{C}_{\lambda_0}^+ \).

Let

(1.12) \[ F(y) = f \left( \int_{(0,t)} \theta(s, y(s))d\eta(s) \right) \quad \text{for} \ y \ \text{in} \ C[0,t]. \]

**Theorem 1.2.** Let \( \eta = \sum_{j=1}^{m} w_j \delta_{\tau_j} + \mu \) where \( \delta_{\tau_j} \) is the Dirac measure at \( \tau_j \in (0, t), 0 < \tau_1 < \cdots < \tau_m < t \) and \( w_j \in \mathbb{C} \) for \( j = 1, 2, \cdots, m \). Suppose that \( \theta(\tau_j, \cdot), j = 1, 2, \cdots, m, \) are essentially bounded. Then for \( \lambda \in (0, \lambda_0) \) and \( \xi \in \mathbb{R}, \) \( \sum_{n=0}^{\infty} a_n F_n(\lambda^{-\frac{1}{2}} x + \xi) \) converges absolutely for a.e. \( x \in C_0[0,t] \). Also the operators \( I_\lambda^m(F) \) and \( J_q^m(F) \) exist for all
\[ \lambda \in \mathbb{C}_{\lambda_0}^+ \text{ and for all } q \text{ with } 0 < |q| < \lambda_0, \text{ respectively. Further for } \lambda \in \mathbb{C}_{\lambda_0}^+ \]

\[ I_{\lambda}^{an}(F) = \sum_{n=0}^{\infty} a_n I_{\lambda}^{an}(F_n) \]

and

\[ J_{q}^{an}(F) = \sum_{n=0}^{\infty} a_n J_{q}^{an}(F_n), \]

where \( F_n \) is the functional defined in (1.6). Moreover, for \( \lambda \in \mathbb{C}_{\lambda_0}^+ \), the series in (1.13) and (1.14) satisfy

\[ \|I_{\lambda}^{an}(F')\| \leq \sum_{n=0}^{\infty} |a_n| B_n(|\lambda|) \]

and

\[ \|J_{q}^{an}(F')\| \leq \sum_{n=0}^{\infty} |a_n| B_n(|q|) \]

and both of them converge in the operator norm.

**2. Stability theorems**

Firstly, we establish the stability for the operator-valued Feynman integral of functionals involving some Borel measures on \((0,t)\) with respect to potentials.

**Theorem 2.1.** Let \( \eta \) be in \( \tilde{M}(0,t) \) with \( \eta = \mu + \sum_{p=1}^{k} w_p \delta_{r_p} \). Let \( H \in L_{1,r;\eta} \) and \( H(\tau_p,\cdot) \) be essentially bounded for each \( p = 1, 2, \ldots, k \). Let \( \theta^{(N)}, N = 1, 2, \ldots, \) be Borel measurable functions on \([0,t] \times \mathbb{R}\) such
that for $\eta \times m$-a.e.

$$\theta^{(N)} \to \theta \quad \text{as } N \to \infty$$

and

$$|\theta^{(N)}| \leq |H| \quad \text{for } N = 1, 2, \ldots.$$  

Then $\theta$ and $\theta^{(N)}$ belong to $L_{1;r,\eta}$. Let $F_n^{(N)}$ be defined in (1.6) with $\theta$ replaced by $\theta^{(N)}$. Then for all real $q > 0$, $J_q^{an}(F_n)$ and $J_q^{an}(F_n^{(N)})$ exist for each $N \in \mathbb{N}$ and as $N \to \infty$,

$$J_q^{an}(F_n^{(N)}) \to J_q^{an}(F_n) \quad \text{in the operator norm.}$$

**Proof.** By (2.1), $\|\theta^{(N)}\|_{1;r,\eta} \leq \|H\|_{1;r,\eta}$ for $N = 1, 2, \ldots$ and so $\theta^{(N)}$ and $\theta$ are in $L_{1;r,\eta}$. And $\theta(\tau_p, \cdot)$, $p = 1, 2, \ldots, k$, are essentially bounded. Hence by Theorem 1.1, for each $N \in \mathbb{N}$, $J_q^{an}(F_n)$ and $J_q^{an}(F_n^{(N)})$, $N = 1, 2, \ldots$, exist for all real $q > 0$. For each $\psi \in L_1(\mathbb{R})$ and $q > 0$,

$$\|J_q^{an}(F_n^{(N)})\psi - J_q^{an}(F_n)\psi\|_\infty$$

$$\leq \|\psi\|_1 \sum_{k=0}^{m} \sum_{1 \leq z_1 < \cdots < z_{m-k} \leq m} \sum' \frac{m! \prod_{j_1 \cdots j_{m-k}=q_0} |z_{j_1} \cdots z_{j_{m-k}}|^{q_m-k}}{q_1! \cdots q_{m-k}!}$$

$$\left(\frac{|q|}{2\pi}\right)^{\frac{q_0+m-k+1}{2}} \sum_{j_1 + \cdots + j_{m-k+1}=q_0}$$

$$\int_{\Delta_{q_0, j_1, \cdots j_{m-k+1}}} [s_1 \cdots (\tau_{z_1} - s_{j_1})(s_{j_1+1} - \tau_{z_1}) \cdots (t - s_{q_0})]^{-\frac{1}{2}}$$

$$\int_{R^{q_0+m-k}} \left[ \prod_{i=1}^{q_0} (\theta^{(N)}(s_i, v_i)) \prod_{j=1}^{m-k} (\theta^{(N)}(\tau_{z_j}, v_{j}'))^{q_j} \right.$$

$$- \prod_{i=1}^{q_0} \theta(s_i, v_i) \prod_{j=1}^{m-k} (\theta(\tau_{z_j}, v_{j}'))^{q_j} \bigg| d \times v_i \bigg| d \times v_{j} \bigg| d \times |\mu|(s_i) \bigg].$$
Since
\[
\| J_q^n(F^{(N)}_n) \psi - J_q^n(F_n) \psi \|_\infty \\
\leq \| \psi \|_1 (\| J_q^n(F^{(N)}_n) \| + \| J_q^n(F_n) \|) < \infty,
\]
by applying the Fubini theorem we obtain the above inequality.

(2.2)
\[
\| J_q^n(F^{(N)}_n) - J_q^n(F_n) \| \\
\leq \sum_{k=0}^{m} \sum_{1 \leq z_1 < \ldots < z_{m-k} \leq m} \sum_{q_0+q_1+\ldots+q_{m-k}=n}^{n-1} \frac{n! \prod_{i=1}^{m-k} w_{z_i}^{q_i} \prod_{i=m-k+1}^{m} w_{z_i}^{q_i}}{q_1! \cdots q_{m-k}!} \\
\left( \frac{|q|}{2\pi} \right)^{q_0+m-k+1} \sum_{j_1+\ldots+j_{m-k+1}=q_0} \int_{\Delta_{\mu_j}^{z_1,\ldots,z_{m-k}}_{\mu_{j_1},\ldots,\mu_{j_{m-k+1}}} \mu_0} L(N) d \times \prod_{i=1}^{q_0} \mu_i(s_i),
\]
where

(2.3)
\[
L(N) = L(N; q_0; s_1, \ldots, s_{q_0}) \\
= \left[ s_1 \cdots (\tau_{z_1} - s_{j_1}) (s_{j_1+1} - \tau_{z_1}) \cdots (t - s_{q_0}) \right]^{-\frac{1}{2}} \\
\int_{\mathbb{R}^{q_0+m-k}} \prod_{i=1}^{q_0} g^{(N)}(s_i, v_i) \prod_{j=1}^{m-k} g^{(N)}(\tau_{z_j}, v'_j)^{q_j} \\
- \prod_{i=1}^{q_0} \theta(s_i, v_i) \prod_{j=1}^{m-k} \theta(\tau_{z_j}, v'_j)^{q_j} d \times \prod_{i=1}^{q_0} v_i d \times \prod_{j=1}^{m-k} v'_j.
\]

We know that by (2.1.a), as \( N \to \infty \),

(2.4)
\[
\prod_{i=1}^{q_0} g^{(N)}(s_i, v_i) \prod_{j=1}^{m-k} g^{(N)}(\tau_{z_j}, v'_j)^{q_j} \\
\to \prod_{i=1}^{q_0} \theta(s_i, v_i) \prod_{j=1}^{m-k} \theta(\tau_{z_j}, v'_j)^{q_j} \quad \text{a.e.}
\]
Since for every \( N \in \mathbb{N} \), \(|\theta^{(N)}(s, u)| \leq |H(s, u)|\) for \( \eta \times m\text{-a.e.}(s, u)\),

\[(2.5)\]

\[
\left| \prod_{i=1}^{q_0} \theta^{(N)}(s_i, v_i) \prod_{j=1}^{m-k} (\theta^{(N)}(\tau_{z_j}, v_j'))^{q_j} - \prod_{i=1}^{q_0} \theta(s_i, v_i) \prod_{j=1}^{m-k} (\theta(\tau_{z_j}, v_j'))^{q_j} \right| 
\leq 2 \prod_{i=1}^{q_0} H(s_i, v_i) \prod_{j=1}^{m-k} H(\tau_{z_j}, v_j')^{q_j}.
\]

Then, \( \prod_{i=1}^{q_0} H(s_i, v_i) \prod_{j=1}^{m-k} H(\tau_{z_j}, v_j')^{q_j} \) is \( \times v_i \times v_j'\text{-integrable.} \) In view of (2.4) and (2.5), the dominated convergence theorem gives \( L(N) \rightarrow 0 \) as \( N \rightarrow \infty \).

Now, we claim that

\[(2.6)\]

\[
\int_{\Delta_{z_1, \ldots, z_{m-k}}^{q_0, \ldots, q_{m-k+1}}} L(N)d \times |\mu|(s_i) \rightarrow 0 \quad \text{as} \quad N \rightarrow \infty.
\]

For a.e.\((s_1, \ldots, s_{q_0}) \in \Delta_{z_1, \ldots, z_{m-k}}^{q_0, \ldots, q_{m-k+1}}, \) by (2.5) and the Fubini theorem

\[
(2.7)\]

\[
|L(N)| 
\leq 2|s_1 \cdots (\tau_{z_1} - s_{j_1}) (s_{j_1+1} - \tau_{z_1}) \cdots (t - s_{q_0})|^{-\frac{1}{2}} 
\leq 2|s_1 \cdots (\tau_{z_1} - s_{j_1}) (s_{j_1+1} - \tau_{z_1}) \cdots (t - s_{q_0})|^{-\frac{1}{2}} 
\leq 2 \prod_{j=1}^{m-k} (\|H(\tau_{z_j}, \cdot)\|_2^{q_j} - 1 \cdot \|H(\tau_{z_j}, \cdot)\|_1) \prod_{i=1}^{q_0} \|H(s_i, \cdot)\|_1.
\]

Then, by Hölder’s inequality, the right hand side of the inequality in (2.7) is \( \times \prod_{i=1}^{q_0} \mu\text{-integrable.} \) Hence, by the dominated convergence theorem, (2.6) is established. Therefore, \( J_{n}^q(F_{n}^{(N)}) \rightarrow J_{n}^q(F_n) \) as \( N \rightarrow \infty \) in the operator norm. Thus, we have proved the theorem. \( \square \)
REMARK 2.2. For each \( n \in \mathbb{N} \), define \( B_n^*(|q|) \) by

\[
(2.8) \quad B_n^*(|q|) = \left( \frac{|q|}{2\pi} \right)^{\frac{m+1}{2}} \left[ \left( \frac{|q|}{2\pi} \right) \right]^{\frac{1}{2}} \left[ \min_{1 \leq j \leq m} (\tau_j - \tau_{j-1}) \right]^{-\frac{m+1}{2}} \\
\left[ \Gamma \left( 1 - \frac{r'}{2} \right) \right]^{-\frac{m+1}{r'}} \left[ \Gamma \left( 1 - \frac{r}{2} \right) \right]^{\frac{m+1}{r'}} \sum_{j=1}^{m} w_j (\|H(\tau_j, \cdot)\|_{\infty} \vee \|H(\tau_j, \cdot)\|_1) \\
+ \left( \sum_{j=1}^{m} (\tau_{j} - \tau_{j-1})^{-\frac{r'}{2}} \right)^{\frac{1}{r'}} \left( \frac{|q|}{2\pi} \right)^{\frac{1}{2}} \|H\|_{1} \|\mu\|_{\infty} \left[ \Gamma \left( 1 - \frac{r'}{2} \right) \right]^{\frac{1}{r'}} \right]^{n}.
\]

Then \( \|J_q^{an}(F_n^{(N)})\| \leq B_n^*(|q|) \) and \( \|J_q^{an}(F_n)\| \leq B_n^*(|q|) \).

Let \( \lambda_0 > 0 \) be given and \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) be an analytic function in \( \mathbb{C}_{\lambda_0}^+ \) such that \( \sum_{n=0}^{\infty} |a_n| B_n(|q|) < \infty \) for all real \( q \) with \( 0 < |q| < \lambda_0 \). Let

\[
(2.9) \quad F(y) = f \left( \int_{(0, t)} \theta(s, y(s)) d\eta(s) \right) \quad \text{for } y \text{ in } C[0, t]
\]

and

\[
(2.10) \quad F^{(N)}(y) = f \left( \int_{(0, t)} \theta^{(N)}(s, y(s)) d\eta(s) \right) \quad \text{for } y \text{ in } C[0, t].
\]

THEOREM 2.3. Let the hypotheses of Theorem 2.1 be satisfied. Then for each real \( q > 0 \), \( J_q^{an}(F) \) and \( J_q^{an}(F^{(N)}) \), \( N = 1, 2, \ldots \), exist where \( F \) and \( F^{(N)} \) \( N = 1, 2, \ldots \), are given by (2.9) and (2.10), respectively. Moreover, as \( N \to \infty \)

\[
(2.11) \quad J_q^{an}(F^{(N)}) \to J_q^{an}(F) \quad \text{in the operator norm}.
\]
Proof. By Theorem 1.2, for each real $q > 0$, $J^q_n(F)$ and $J^q_n(F^{(N)})$, $N = 1, 2, \cdots$, exist. And further they can be represented as

$$J^q_n(F) = \sum_{n=0}^{\infty} a_n J^q_{n}(F_n)$$

and

$$J^q_n(F^{(N)}) = \sum_{n=0}^{\infty} a_n J^q_{n}(F^{(N)}_n) \quad \text{for all } N \in \mathbb{N}.$$ 

Since $\| J^q_n(F) - J^q_n(F^{(N)}) \| \leq 2 \sum_{n=0}^{\infty} |a_n| B_n^*(q)$ by Remark 2.2, we have

$$\lim_{N \to \infty} J^q_n(F^{(N)})$$

$$= \lim_{N \to \infty} \sum_{n=0}^{\infty} a_n J^q_{n}(F^{(N)}_n)$$

$$= \sum_{n=0}^{\infty} \lim_{N \to \infty} a_n J^q_{n}(F^{(N)}_n)$$

$$= \sum_{n=0}^{\infty} a_n J^q_{n}(F_n)$$

$$= J^q_n(F) \quad \text{in the operator norm.}$$

Thus, the proof of Theorem 2.3 is complete. \qed

Now, we consider the stability for the operator-valued Feynman integral of functionals with respect to wave functions.

THEOREM 2.4. Let $\{\psi^{(N)}\}$ be a sequence in $L_1(\mathbb{R})$ and $\| \psi^{(N)} - \psi \|_1 \to 0$ as $N \to \infty$. Then for $N \in \mathbb{N}$, $J^q_n(F)\psi$ and $J^q_n(F^{(N)})\psi^{(N)}$ exist in $C_0(\mathbb{R})$ for real $q$ with $0 < |q| < \lambda_0$. Moreover,

$$\| J^q_n(F^{(N)})\psi^{(N)} - J^q_n(F)\psi \|_\infty \to 0 \quad \text{as } N \to \infty.$$  

Proof. By Theorem 1.2, $J^q_n(F)\psi$ and $J^q_n(F^{(N)})\psi^{(N)}$ exist in $C_0(\mathbb{R})$ for $q > 0$. By Theorem 2.3 and $\| J^q_n(F^{(N)})\psi^{(N)} - J^q_n(F)\psi \|_\infty \leq \| J^q_n(F^{(N)})\| \| \psi^{(N)} - \psi \|_1 + \| J^q_n(F^{(N)}) - J^q_n(F)\| \| \psi \|_1$, we prove the theorem. \qed
THEOREM 2.5. Suppose that \( \{ q_N \} \) is a sequence of real numbers which converges to a nonzero real number \( q \) with \( 0 < |q| < \lambda_0 \). Then as \( N \to \infty \),

\[
J_{q_N}(F) \to J_q^\alpha(F) \quad \text{in the operator norm}.
\]

Proof. Let \( q \) be in \( \mathbb{R} \) with \( 0 < |q| < \lambda_0 \) and \( \psi \in L_1(\mathbb{R}) \). Then, from (1.6) and (1.12) we have

\[
\left( J_{q_n}(F) \psi \right)(\xi) - \left( J_q^\alpha(F) \psi \right)(\xi)
\]

\[
= \sum_{n=0}^{\infty} a_n \sum_{k=0}^{\infty} \sum_{1 \leq z_1 < \cdots < z_{m-k} \leq m_{q_0+q_1+\cdots+q_{m-k}} = n} \sum_{j_1 = 0}^{\infty} \cdots \sum_{j_{m-k+1} = 0}^{\infty} \int_{\mathbb{R}^{m-k+1}} \frac{n! w_{z_1}^{q_1} \cdots w_{z_{m-k}}^{q_{m-k}}}{q_1! \cdots q_{m-k}!} \left[ s_1 \cdots (t - s_{q_0}) \right]^{-\frac{1}{2}}
\]

\[
\left\{ \left[ \frac{-i q_N}{2\pi} \right]^{q_0+m-k+1} - \left[ \frac{-i q}{2\pi} \right]^{q_0+m-k+1} \right\}
\]

\[
\int_{\mathbb{R}^{q_0+m-k+1}} \prod_{i=1}^{q_0} \theta(s_i, v_i) \prod_{j=1}^{m-k} \left( \theta(\tau_{z_j}, v_j) \right)^{q_j} \psi(v_{m-k+1})
\]

\[
\exp \left( \frac{i q_N}{2} \left( \frac{(v_1 - \xi)^2}{s_1} + \cdots + \frac{(v_{m-k+1} - v_{q_0})^2}{(t - s_{q_0})} \right) \right) d \times_{i=1}^{q_0} v_i d \times_{j=1}^{m-k} v_j
\]

\[
+ \left( \frac{-i q}{2\pi} \right)^{q_0+m-k+1} \int_{\mathbb{R}^{q_0+m-k+1}} \prod_{i=1}^{q_0} \theta(s_i, v_i) \prod_{j=1}^{m-k} \left( \theta(\tau_{z_j}, v_j) \right)^{q_j} \psi(v_{m-k+1})
\]

\[
\left[ \exp \left( \frac{i q_N}{2} \left( \frac{(v_1 - \xi)^2}{s_1} + \cdots + \frac{(v_{m-k+1} - v_{q_0})^2}{(t - s_{q_0})} \right) \right) \right.
\]

\[
\left. - \exp \left( \frac{i q}{2} \left( \frac{(v_1 - \xi)^2}{s_1} + \cdots + \frac{(v_{m-k+1} - v_{q_0})^2}{(t - s_{q_0})^2} \right) \right) \right]
\]

\[
d \times_{i=1}^{q_0} v_i d \times_{j=1}^{m-k} v_j \left\{ \mu(s_i) \right\}.
\]

Let \( \delta = \frac{1}{2} \min\{ |q|, \lambda_0 - |q| \} \). By the hypotheses, there exists a posi-
tive integer $M_0$ such that if $N \geq M_0$, we have

\begin{align}
(2.16) & \quad \left| \frac{-i q N}{2 \pi} \frac{q_0 + m - k + 1}{2} - \frac{-i q}{2 \pi} \frac{q_0 + m - k + 1}{2} \right| \\
& \leq \left( \frac{1}{2 \pi} \right) \frac{q_0 + m - k + 1}{2} \left[ |q N| \frac{q_0 + m - k + 1}{2} + |q| \frac{q_0 + m - k + 1}{2} \right] \\
& \leq 2 \left( |q| + \delta \right) \frac{q_0 + m - k + 1}{2}.
\end{align}

For each $n \in \mathbb{N}$,

\begin{align}
(2.17) & \quad \| J_{q N}^n (F_n) - J_q^n (F_n) \| \\
& \leq \sum_{k=0}^{m} \sum_{1 \leq z_1 < \cdots < z_{m-k} \leq m \ q_0 + q_1 + \cdots + q_{m-k} = n} \sum_{j_1 + \cdots + j_{m-k+1} = q_0} \\
& \int_{\Delta_{q_0, j_1, \ldots, j_{m-k+1}}} \left[ s_1 \cdots (t - s_{q_0}) \right]^{-\frac{1}{2}} \\
& \left| \frac{-i q N}{2 \pi} \frac{q_0 + m - k + 1}{2} - \frac{-i q}{2 \pi} \frac{q_0 + m - k + 1}{2} \right| \\
& \int_{\mathbb{R}^{q_0 + m - k}} \prod_{i=1}^{q_0} |\theta(s_i, v_i)| \prod_{j=1}^{m-k} |\theta(\tau_{z_j}, v_j')|^{q_j} d v_i d v_j' \\
& + \left| \frac{-i q}{2 \pi} \right| \frac{q_0 + m - k + 1}{2} \int_{\mathbb{R}^{q_0 + m - k}} \prod_{i=1}^{q_0} |\theta(s_i, v_i)| \prod_{j=1}^{m-k} |\theta(\tau_{z_j}, v_j')|^{q_j} \\
& \left| \exp \left( \frac{i q N}{2} \frac{(v_1 - \xi)^2}{s_1} - \cdots - \frac{(v_{m-k+1} - v_{q_0})^2}{(t - s_{q_0})} \right) \right| \\
& - \exp \left( \frac{i q}{2} \frac{(v_1 - \xi)^2}{s_1} - \cdots - \frac{(v_{m-k+1} - v_{q_0})^2}{(t - s_{q_0})} \right) \right| \\
& d v_i d v_j' \right\} d \int_{\mathbb{R}} |\mu(s_i)|.
Then, for $N$ such that $N > M_0$,

\begin{equation}
\int_{\Delta_{q_0;j_1,\ldots,j_{m-k+1}}} [s_1 \cdots (\tau_{s_1} - s_{j_1})(s_{j_1+1} - \tau_{s_1}) \cdots (t - s_{q_0})]^{-\frac{1}{2}} \left\{ \left( -\frac{iqN}{2\pi} \right)^{q_0+m-k+1} - \left( -\frac{iq}{2\pi} \right)^{q_0+m-k+1} \right\} \\
\int_{\mathbb{R}^{q_0+m-k}} \prod_{i=1}^{q_0} |\theta(s_i, v_i)| \prod_{j=1}^{m-k} |\theta(\tau_{s_j}, v_j')| q_j \, d\times_{i=1}^{q_0} v_i \, d \times_{j=1}^{m-k} v_j' \\
+ \left( \frac{|q|}{2\pi} \right)^{\frac{q_0+m-k+1}{2}} \int_{\mathbb{R}^{q_0+m-k}} \prod_{i=1}^{q_0} |\theta(s_i, v_i)| \prod_{j=1}^{m-k} |\theta(\tau_{s_j}, v_j')| q_j \\
\exp \left( \frac{iqN}{2} \left( \frac{(v_1 - \xi)^2}{s_1} - \cdots - \frac{(v_{m-k+1} - v_{q_0})^2}{(t - s_{q_0})} \right) \right) \\
\exp \left( \frac{iq}{2} \left( \frac{(v_1 - \xi)^2}{s_1} - \cdots - \frac{(v_{m-k+1} - v_{q_0})^2}{(t - s_{q_0})} \right) \right) \\
\times_{i=1}^{q_0} d \times_{j=1}^{m-k} v_i \\
\times_{i=1}^{q_0} |\mu|(s_i)
\end{equation}

\begin{align*}
&\leq \int_{\Delta_{q_0;j_1,\ldots,j_{m-k+1}}} [s_1 \cdots (\tau_{s_1} - s_{j_1})(s_{j_1+1} - \tau_{s_1}) \cdots (t - s_{q_0})]^{-\frac{1}{2}} \left[ 2\left( \frac{|q| + \delta}{2\pi} \right)^{q_0+m-k+1} + \left( \frac{|q|}{2\pi} \right)^{q_0+m-k+1} \right] \\
&\prod_{i=1}^{q_0} \|\theta(s_i, \cdot)\|_1 \|\theta(\tau_{s_1}, \cdot)\|_{q_0-1} \|\theta(\tau_{s_j}, \cdot)\|_1 \left\| \frac{d|\mu|}{dm} \right\|_{q_0} \prod_{i=1}^{q_0} d \times_{i=1}^{q_0} s_i \\
&= \left[ 2\left( \frac{|q| + \delta}{2\pi} \right)^{q_0+m-k+1} + \left( \frac{|q|}{2\pi} \right)^{q_0+m-k+1} \right] \left\| \frac{d|\mu|}{dm} \right\|_{q_0} \\
&\prod_{j=1}^{m-k} \|\theta(\tau_{s_j}, \cdot)\|_{q_0-1} \|\theta(\tau_{s_j}, \cdot)\|_1 \left( \frac{d|\mu|}{dm} \right)_{\infty} \\
&\prod_{i=1}^{q_0} \|\theta(s_i, \cdot)\|_1 \\
&[s_1 \cdots (\tau_{s_1} - s_{j_1})(s_{j_1+1} - \tau_{s_1}) \cdots (t - s_{q_0})]^{-\frac{1}{2}} \|\theta(s_i, \cdot)\|_1 \times_{i=1}^{q_0} s_i.
\end{align*}

Since $[s_1 \cdots (\tau_{s_1} - s_{j_1})(s_{j_1+1} - \tau_{s_1}) \cdots (t - s_{q_0})]^{-\frac{1}{2}} \|\theta(s_i, \cdot)\|_1$ is $\times_{i=1}^{q_0} s_i$. 
integrable and \( q_N \to q \), by the H"older's inequality and the dominated convergence theorem, \( J_{q_N}^n(F^{(N)}) \) converges to \( J_q^N(F_n) \) in the operator norm as \( N \to \infty \).

Therefore, since \( \|J_q^n(F) - J_q^n(F^{(N)})\| \leq 2 \sum_{n=0}^{\infty} a_n B_n^*(q) < \infty \)

\[
\lim_{N \to \infty} J_{q_N}^n(F) = \lim_{N \to \infty} \sum_{n=0}^{\infty} a_n J_{q_N}^n(F_n) \\
= \sum_{n=0}^{\infty} a_n \lim_{N \to \infty} J_{q_N}^n(F_n) = \sum_{n=0}^{\infty} a_n J_q^n(F_n) \\
= J_q^n(F) \quad \text{in the operator norm.} \]

\[\square\]

**Corollary 2.6.** Suppose that the hypotheses of Theorem 2.1 and Theorem 2.5 are satisfied. Then as \( N \to \infty \)

\[
J_{q_N}^n(F^{(N)}) \to J_q^n(F) \text{ in the operator norm.} \]

**Proof.** We may assume that \( |q_n| < \lambda_0 \) for sufficiently large \( N \),

\[
\|J_{q_N}^n(F^{(N)}) - J_q^n(F)\| \leq \|J_{q_N}^n(F^{(N)}) - J_q^n(F)\| + \|J_q^n(F) - J_q^n(F)\|. 
\]

Since \( \|J_{q_N}^n(F^{(N)}) - J_q^n(F)\| \to 0 \) as \( N \to \infty \) for each \( q_N \) and \( \|J_q^n(F) - J_q^n(F)\| \to 0 \) as \( N \to \infty \), thus \( J_{q_N}^n(F^{(N)}) \to J_q^n(F) \) as \( N \to \infty \) in the operator norm. \( \square \)

**Corollary 2.7.** Suppose that the hypotheses of Theorem 2.1, Theorem 2.3 and Theorem 2.4 hold. Then as \( N \to \infty \),

\[
\|J_{q_N}^n(F^{(N)})\psi_N - J_q^n(F)\psi\|_\infty \to 0. 
\]

Finally, we treat the stability theorem for the operator-valued Feynman integral with respect to measures.
Theorem 2.8. Let \( \theta \) be a continuous function bounded by \( c \) and let \( \eta \) and \( \eta_N, N = 1, 2, \ldots \) be in \( \tilde{M}(0,t) \). Assume that

\[
\eta_N \to \eta \quad \text{weakly.}
\]

(2.22)

Let \( F \) be defined as in (2.9) and \( F_N \) be defined as in (2.9) with \( \eta \) replaced by \( \eta_N \). Then

\[
I^\eta_{\lambda}(F_N) \to I^\eta_{\lambda}(F) \quad \text{in the operator norm},
\]

uniformly in \( \lambda \) on all compact subset of \( \mathbb{C}_\lambda^+ \).

Proof. For \( \psi \in L_1(\mathbb{R}), \xi \in \mathbb{R} \) and \( \lambda > 0 \),

\[
(I_{\lambda}(F_{\psi})\psi)(\xi) = \int_{C[0,t]} f \left( \int_{(0,t)} \theta(s, \lambda^{-\frac{1}{2}} y(s) + \xi) \, d\eta_N(s) \right) \psi(\lambda^{-\frac{1}{2}} y(t) + \xi) \, dm(y)
\]

and we have similar result for \( F \) by replacing \( \eta_N \) by \( \eta \) in (2.24). Given \( y \in C[0,t] \), the function \( \theta(s, \lambda^{-\frac{1}{2}} y(s) + \xi) \) is bounded by \( c \) and it is continuous as a function of \( s \). Hence, by (2.22),

\[
\int_{(0,t)} \theta(s, \lambda^{-\frac{1}{2}} y(s) + \xi) \, d\eta_N(s) \to \int_{(0,t)} \theta(s, \lambda^{-\frac{1}{2}} y(s) + \xi) \, d\eta(s).
\]

Since \( f \) is continuous,

\[
f \left( \int_{(0,t)} \theta(s, \lambda^{-\frac{1}{2}} y(s) + \xi) \, d\eta_N(s) \right) \psi(\lambda^{-\frac{1}{2}} y(t) + \xi) \to f \left( \int_{(0,t)} \theta(s, \lambda^{-\frac{1}{2}} y(s) + \xi) \, d\eta_N(s) \right) \psi(\lambda^{-\frac{1}{2}} y(t) + \xi).
\]

By the uniform boundedness principle and (2.22),

\[
\left| \int_{(0,t)} \theta(s, \lambda^{-\frac{1}{2}} y(s) + \xi) \, d\eta_N(s) \right|
\]
is bounded by $M$ with $M = c \cdot \sup_N \|\eta_N\|$. Thus

\begin{equation}
\left| f \left( \int_{(0,t)} \theta(s, \lambda^{-\frac{1}{2}} y(s) + \xi) \, d\eta_m(s) \right) \psi(\lambda^{-\frac{1}{2}} y(t) + \xi) \right|
\leq M_1 |\psi(\lambda^{-\frac{1}{2}} y(t) + \xi)|,
\end{equation}

where $M_1 = \sup_{|z| \leq M} |f(z)| < \infty$. Recall that $|\psi(\lambda^{-\frac{1}{2}} y(t) + \xi)|$ is Wiener integrable. In view of (2.26) and (2.27), the dominated convergence theorem yields

\begin{equation}
(I_\lambda(F_N)\psi)(\xi) \rightarrow (I_\lambda(F)\psi)(\xi) \quad \text{as } N \rightarrow \infty \text{ for a.e.} \, \xi \in \mathbb{R}.
\end{equation}

Thus,

\begin{equation}
I_\lambda(F_N)\psi \rightarrow I_\lambda(F)\psi \quad \text{in } C_0(\mathbb{R}).
\end{equation}

Now, $I_\lambda^{an}(F_N)$ is analytic for $\lambda \in C_{\lambda_0}^+$. By Wiener integration formula [10],

\begin{equation}
\|I_\lambda^{an}(F_N)\psi\|_\infty \leq M_1 \left( \frac{\lambda_0}{2\pi t} \right)^{\frac{1}{2}} \|\psi\|_1 \quad N = 1, 2, \ldots.
\end{equation}

Hence, by (2.29) and (2.30), Vitali Theorem [5] gives the result for $\lambda \in C_{\lambda_0}^+$. □

The conclusion of Theorem 2.8 can be reinforced provided that we assume that measures converge in the strong sense.

**Theorem 2.9.** Assume that $\eta_N \rightarrow \eta$ in norm. Then, under the hypotheses of Theorem 2.8

\begin{equation}
I_\lambda^{an}(F_N) \rightarrow I_\lambda^{an}(F)
\end{equation}

uniformly in $\lambda$ on all compact subset of $C_{\lambda_0}^+$. 
Proof. Given \( y \in C[0,t] \) and \( \xi \in \mathbb{R} \)

\[
\left| \int_{(0,t)} \theta(s, \lambda^{-\frac{1}{2}} y(s) + \xi) d\eta_N(s) - \int_{(0,t)} \theta(s, \lambda^{-\frac{1}{2}} y(s) + \xi) d\eta(s) \right| \leq \|\eta_N - \eta\| \|\theta\|_{\infty}.
\]

For \( \psi \in L_1(\mathbb{R}) \), a.e. \( \xi \in \mathbb{R} \) and \( \lambda > 0 \)

\[
\left| f \left( \int_{(0,t)} \theta(s, \lambda^{-\frac{1}{2}} y(s) + \xi) d\eta_N(s) \right) \psi(\lambda^{-\frac{1}{2}} y(t) + \xi) - f \left( \int_{(0,t)} \theta(s, \lambda^{-\frac{1}{2}} y(s) + \xi) d\eta(s) \right) \psi(\lambda^{-\frac{1}{2}} y(t) + \xi) \right| \leq T_N|\psi(\lambda^{-\frac{1}{2}} y(t) + \xi)|,
\]

where \( T_N = \sup\{|f(z_1) - f(z_2)| \mid |z_1 - z_2| \leq \|\eta_N - \eta\| \|\theta\|_{\infty}\} \). Thus, for \( \lambda > 0 \)

\[
\|I_\lambda(F_N)\psi - I_\lambda(F)\psi\|_{\infty} \leq \int_{C_0[0,t]} T_N|\psi(\lambda^{-\frac{1}{2}} y(t) + \xi)| dm(y)
\]

\[
\leq T_N \left( \frac{\lambda_0}{2\pi t} \right)^{\frac{1}{2}} \|\psi\|_1.
\]

Since \( \psi \) is arbitrary and \( T_N \to 0, I_\lambda(F_N) \to I_\lambda(F) \) in the operator norm topology. Since \( I_\lambda^{an}(F_N) \) is analytic in \( \mathcal{C}_{\lambda0}^+ \) and \( \|I_\lambda^{an}(F_N)\| \leq \|\eta_N\| \|\theta\|_{\infty} \), the Vitali theorem [5] yields (2.31). Thus, the proof of theorem is complete. \( \square \)

Acknowledgement. This paper was supported in part by Non Directed Research Fund, Korea Research Foundation, KOSEF and BSRIP, Ministry of Education, 1997. The second author was supported by Post Doctorial Fellowship, Korea Research Foundation, 1996.
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