EXTREMAL STRUCTURE OF \( B(X^*) \)

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ABSTRACT. In this note we consider some basic facts concerning abstract \( M \) spaces and investigate extremal structure of the unit ball of bounded linear functionals on \( \sigma \)-complete abstract \( M \) spaces.

1. Introduction

The original definition of an abstract \( L_1 \) or \( M \) space is given by S. Kakutani. The representation theorems of Kakutani are followed by several results which give joint characterizations of the abstract \( L_p \) spaces and \( M \) spaces among general Banach lattices.

A Banach lattice \( X(\text{BL}, \text{for short}) \) for which \( \|x + y\| = \max(\|x\|, \|y\|) \), whenever \( x, y \in X \) and \( x \wedge y = 0 \), is called an abstract \( M \) space. Let \( 1 \leq p < \infty \). A \( BL X \) for which \( \|x + y\|^p = \|x\|^p + \|y\|^p \), whenever \( x, y \in X \) and \( x \wedge y = 0 \), is called an abstract \( L_p \) space. It is obvious that every \( L_p(\mu) \) space is an abstract \( L_p \) space if \( p < \infty \) or an abstract \( M \) space if \( p = \infty \). The converse is also true if \( p < \infty \).

If \( \{x_\alpha\}_{\alpha \in A} \) is a set in a \( BL \), we denote by \( \bigvee_{\alpha \in A} x_\alpha \) or by \( \sup_{\alpha \in A} \{x_\alpha\}_{\alpha \in A} \) the (unique) element \( x \in X \) which has the following properties: (1) \( x \geq x_\alpha \) for all \( \alpha \in A \) and (2) whenever \( z \in X \) satisfies \( z \geq x_\alpha \) for all \( \alpha \in A \) then \( z \geq x \). Unless the set \( A \) is finite, \( \bigvee_{\alpha \in A} x_\alpha \) need not always exist in a \( BL [5] \).

For an element \( x \) in a \( BL X \) we put \( x^+ = x \vee 0 \) and \( x^- = -(x \wedge 0) = (-x) \vee 0 \). Obviously, \( x = x^+ - x^- \) and \( |x| = x^+ + x^- \). Especially, if \( x = u - v \), \( u \geq 0 \), \( v \geq 0 \) in \( X \), then \( u = x^+ + u \wedge v \) and \( v = x^- + u \wedge v \). Also, if \( u \wedge v = 0 \), then \( u = x^+ \) and \( v = x^- \) [6].

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The dual $X^*$ of a BL $X$ is also a BL provided that its positive cone is defined by $x^* \geq 0$ in $X^*$ if and only if $x^*(x) \geq 0$, for every $x \geq 0$ in $X$. For any $x^*, y^* \in X^*$ and every $x \geq 0$ in $X$, we have

$$(x^* \lor y^*)(x) = \sup \{x^*(u) + y^*(x - u); 0 \leq u \leq x\}$$

and

$$(x^* \land y^*)(x) = \inf \{x^*(u) + y^*(x - u); 0 \leq u \leq x\}.$$  

For a BL $X$, if $X$ is an abstract $M$ space, then $X^*$ is an abstract $L$ space and if $X$ is an abstract $L$ space, then $X^*$ is an abstract $M$ space, respectively. Also, if $X$ is a BL, then $X^*$ is a space of regular functionals [3]. Obviously, for a BL $X$ and $x^* \in X^*$, $x^*(x) = \sup \{|x^*(y)| : |y| \leq x\}$ [4].

A BL $X$ is said to be $\sigma$-complete if every order bounded set(sequence) in $X$ has a sup, and a BL $X$ is said to be bounded $\sigma$-complete, provided that any norm bounded and order monotone sequence in $X$ is order convergent. Obviously, bounded $\sigma$-complete BL is $\sigma$-complete, but the inverse does not hold [5].

Since every $x^* \in X^*$ can be decomposed as a difference of two non-negative elements, it follows that every norm bounded monotone sequence $\{x_n\}_{n=1}^{\infty}$ in $X$ is weak Cauchy. If, in addition, $x_n \rightharpoonup x$ for some $x \in X$ then $\|x_n - x\| \to 0$ as $n \to \infty$. This is a consequence of the fact that weak convergence to $x$ implies the existence of convex combinations of the $x_n$'s which tend strongly to $x$.

For a Banach space $X$, we always denote by $B(X)$ and $S(X)$ the unit ball and the unit sphere of $X$ respectively. $x \in S(X)$ is called an extreme point of $B(X)$ if for any given $y, z \in B(X)$ and $x = \lambda y + (1 - \lambda)z$ for some $0 < \lambda < 1$, then $x = y = z$. The set of all extreme points of $B(X)$ is denoted by $\partial B(X)$. In this note we will investigate the extreme points of the unit ball of a dual space.

Now we show some propositions which will be needed in the sequel.

2. Main theorem

A BL $X$ is an abstract $L_p$ space if and only if for any $x, y \in X, x, y \geq 0$ implies $\|x + y\| = \|x\| + \|y\|$. Moreover, $\|u\| = \|x^+\| + \|u \land v\|$, $\|v\| = \|x^-\| + \|u \land v\|$, where $x, u, v \in X, x = u - v$ and $u \geq 0, v \geq 0$ [6]. Hence, we have the following result.
**Lemma 1.** Let a BL $X$ be an abstract $L_p$ space and $x \in X$. Then $x = x^+ - x^-$ is unique in the sense that if $x = u - v, u \geq 0, v \geq 0$ and $\|u\| + \|v\| = \|x\|$, then $u = x^+$ and $v = x^-$. 

**Proposition 2.** If a BL $X$ is bounded σ-complete and $B(X)$ is order closed, then there exists $x \in S(X)$ such that $x^*(x) = \|x^*\|$ for every $x^*(\geq 0) \in X^*$, that is, $x^*$ is norm attainable.

**Proof.** Let $x_n$ be a positive element in $S(X)$ such that $x^*(x_n) \to \|x^*\|$. Since $X$ is bounded σ-complete and $B(X)$ is order closed, $y = \bigvee_{n} x_n$ exists in $X$ and $\|y\| = 1$. Hence, $y \geq x_n \geq 0$ and $x^* \geq 0$ implies $\|x^*\| \geq x^*(y) \geq x^*(x_n) \to \|x^*\|$. 

Note that the conclusion of Proposition 2 may not be true if an abstract $M$ space $X$ is not bounded σ-complete. For instance, let $X = c_0$ and $x^* = (c_n) \in l_1$ with infinitely $c_n \neq 0$. Then there does not exist $x \in S(X)$ such that $x^*(x) = \|x^*\|$.

If a BL $X$ is bounded σ-complete and $B(X)$ is not order closed, then the conclusion of Proposition 2 is not true in general.

For a subset $Y$ of a BL $X$, we define

$$Y^\perp = \{x \in X : |x| \wedge |y| = 0 \text{ whenever } y \in Y\}, \quad x^\perp = \{x\}^\perp.$$ 

If $x \in X = Y + Y^\perp$, then $x$ can be uniquely decomposed into $x = y + z$, where $y \in Y$ and $z \in Y^\perp$. In this case, we write $x|_Y = y$ and $x^*|_Y(x) = x^*(y)$ for $x^* \in X^*$.

**Proposition 3.** If an abstract $M$ space $X$ is σ-complete and $x^* \in X^*$, then for any $\varepsilon > 0$, there exists a subspace $Y$ of $X$ such that $X = Y + Y^\perp$ and $\|x^*|_Y\| < \varepsilon$, $\|x^*|_Y\| < \varepsilon$.

**Proof.** Let $x$ be in $S(X)$ such that $x^*(x) > \|x^*\| - \varepsilon$, and put $Y = (x^-)^\perp$. Then $x^+ \in Y, x^- \in Y^\perp$, and by [5] $X = Y + Y^\perp$. Moreover, by properties of an abstract $M$ space $X^*$,

$$\|x^*|_Y\| + \|x^*|_Y\| + \|x^*|_Y\| + \|x^*|_Y\| = \|x^*\| < x^*(x) + \varepsilon$$

$$= x^+|_Y(x) + x^+|_Y(x) - x^-|_Y(x) - x^-|_Y(x) + \varepsilon.$$
Since $x^+|_{Y^\perp}(x) \leq 0$ and $x^-|_{Y}(x) \geq 0$ it follows that

\[
\|x^+|_{Y^\perp}\| + \|x^-|_{Y}\|
\]

\[
= \|x^+\| - \|x^+|_{Y}\| + \|x^-\| - \|x^-|_{Y^\perp}\|
\]

\[
\leq \|x^+\| - x^+|_{Y}(x) + \|x^-\| - x^-|_{Y^\perp}(x)
\]

\[
< x^+|_{Y^\perp}(x) - x^-|_{Y}(x) + \varepsilon \leq \varepsilon.
\]

**Lemma 4.** Let an abstract $M$ space $X$ be bounded $\sigma$-complete and $B(X)$ order closed. Then $x^* \in X^*$ is norm attainable if and only if there exists a subspace $Y$ of $X$ satisfying $x^+ = x^*|_{Y}$, $x^- = -x^*|_{Y^\perp}$.

**Proof.** Suppose that $x^* \in X^*$ is norm attainable. Then, by Proposition 2, there exist $x, y(\geq 0) \in S(X)$ such that $x^+|_{Y} = \|x^+\|$ and $x^-|_{Y}(y) = \|x^-\|$. Since $x^+ = x^*|_{Y}$ and $x^- = -x^*|_{Y^\perp}$, we may assume $x \in Y$ and $y \in Y^\perp$ (otherwise we replace $x, y$ by $x|_{Y}$, $y|_{Y^\perp}$ respectively). Now, we put $u = x - y$. Then $\|u\| = \|x - y\| = \max\\{\|x\|, \|y\|\} = 1$ and thus, by properties of an abstract $M$ space $X^*$, we get that

\[
\|x^*\| = \|x^+\| + \|x^-\| = x^+|_{Y}(x) + x^-|_{Y^\perp}(y)
\]

\[
= x^*|_{Y}(x) + x^*|_{Y^\perp}(-y) = x^*(u).
\]

Conversely, choose $x \in S(X)$ such that $x^*(x) = \|x^*\|$, and define $Y = (x^-)^\perp$. Then $X = Y + Y^\perp$ and $x^+ \in Y$, $x^- \in Y^\perp$. Observe that $\|x^*\| = \|x^*|_{Y}\| + \|x^*|_{Y^\perp}\|$; to prove $x^+ = x^*|_{Y}$ and $x^- = -x^*|_{Y^\perp}$, it suffices to show $x^*|_{Y} \geq 0$ and $-x^*|_{Y^\perp} \geq 0$ thanks to Lemma 1. Indeed, if $x^*|_{Y}(y) < 0$ for some $y(\geq 0) \in S(X)$, then we may assume $y \in Y$. Therefore, $z = -x^- - y$ satisfies $\|z\| = \max\\{\|x^-\|, \|y\|\} = 1$ and thus,

\[
\|x^-\| \geq x^-|_{Y^\perp}(-z) = x^*(z) - x^+|_{Y}(z) \geq x^*(z)
\]

\[
= x^*|_{Y^\perp}(-x^-) - x^*|_{Y}(y) > x^*|_{Y^\perp}(-x^-) = -x^*|_{Y^\perp}(x).
\]

Since $\|x^+\| \geq x^*(x|_{Y}) = x^*|_{Y}(x)$, this clearly leads to a contradiction that

\[
\|x^*\| = \|x^+\| + \|x^-\| > x^*|_{Y}(x) - x^*|_{Y^\perp}(x) = x^*(x) = \|x^*\|.
\]

A similar argument would show that $-x^*|_{Y^\perp} \geq 0$.

Now we investigate the extreme points of the unit ball of a dual space. The sequence $\{x_n\}$ converges weakly to zero in a Banach space $X$ if and only if $\{x_n\}$ is bounded, and $x^*(x_n) \to 0$ for every $x^* \in \partial B(X^*)$. 
Theorem 5. Let an abstract $M$ space $X$ be $\sigma$-complete and $x^* \in S(X^*)$. Then $x^* \in \partial B(X^*)$ if and only if $x^*(x)x^*(y) = 0$ for all $x, y \in X$ such that $x \wedge y = 0$.

Proof. Sufficiency. First we show $\|x^*^+\|\|x^*-\| = 0$. In fact, by Proposition 3, for any $\varepsilon > 0$, there exist orthogonal subspaces $Y, Z$, of $X$ such that $X = Y + Z$ and $\|x^*\|_Y < \varepsilon$, $\|x^*\|_Z < \varepsilon$. Choose $x \in S(X)$ satisfying $x^*(x) > \|x^*\| - \varepsilon$, and let $x = u + v$, where $u \in Y$ and $v \in Z$. Then $x^*(u)x^*(v) = 0$ since $u \wedge v = 0$. If $x^*(v) = 0$, then

$$\|x^*\| - \varepsilon < x^*(x) = x^*|_Y (u) - x^*|_Y (u) \leq \|x^*|_Y \| + \|x^*-\|_Y < \|x^*\| + \varepsilon.$$ 

Let $\varepsilon \to 0$. Then $\|x^*\| = \|x^*\| - \|x^*\| = 0$. Similarly, assume that $x^*(u) = 0$. Then $\|x^*\| = 0$. Hence, without loss of generality, we may assume $x^* = x^*^+$.

Let $y^*, z^* \in S(X^*)$ satisfy $2x^* = y^* + z^*$. Then $2x^* = (y^* + z^*^+ - (y^* - z^*^-)$ and by properties of an abstract $M$ space $X^*$,

$$\|2x^*\| \leq \|y^*\| + \|z^*\| + \|y^*\| + \|z^*\| = \|y^*\| + \|z^*\| = 2 = \|2x^*\|.$$ 

Thus, by Lemma 1, we have $y^* + z^* = 2x^*$ and $y^* = z^* = 0$.

Now we show that $y^* = z^* = x^*$, i.e., $x^* \in \partial B(X^*)$. To this end we notice that $y^*(y) = z^*(y) = 0$ whenever $x^*(y) = 0$ (by [7], this means $x^* = ay^* = bz^*$, but $x^*, y^*, z^* \in S(X^*)$ and $2x^* = y^* + z^*$, so $a = b = 1$). First we assume $y \geq 0$; then from $y^*(y) \geq 0$, $z^*(y) \geq 0$, and $y^*(y) + z^*(y) = 2x^*(y) = 0$ we have $y^*(y) = z^*(y) = 0$. For the general case, since $x^*(y) = 0$ and by the condition given in the theorem, $x^*(y^+)^x^*(y^-) = 0$, we have $x^*(y^+) = x^*(y^-) = 0$. Hence, $y^*(y) = z^*(y) = 0$ follows from the first case.

Necessity. Assume first that there exist $x, y \in X$ such that $x \wedge y = 0$ but $x^*(x) > 0$ and $x^*(y) > 0$. Then we put $Y = y^\perp$, and then by [5] $X = Y + Y^\perp$. Now, let $y^* = x^*|_Y$ and $z^* = x^*|_{Y^\perp}$. Then $\|y^*\| > 0$, $\|z^*\| > 0$ since $x \in Y, y \in Y^\perp$. Therefore, since

$$x^* = \|y^*\| \frac{y^*}{\|y^*\|} + \|z^*\| \frac{z^*}{\|z^*\|}$$

and $\|y^*\| + \|z^*\| = \|x^*\| = 1$ according to Lemma 1 and the intrinsic $M$ space properties, we get that $x^* \in \partial B(X^*)$, which is desired result.
REFERENCES


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