A CERTAIN SUBGROUP OF THE WEYL GROUP OF SOME KAC-MOODY ALGEBRAS

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ABSTRACT. In this paper, we construct the minimal set of generators which generate the subgroup $T$ of the Weyl group of Kac-Moody algebra.

1. Notation and some basic facts about root systems of Kac-Moody algebras

We first recall some of the basic definitions in Kac-Moody theory.

An $n \times n$ integral matrix $A = (a_{ij})_{i,j=1}^n$ is called a generalized Cartan matrix (GCM) if

$$
\begin{align*}
    a_{ii} &= 2, & i = 1, 2, \ldots, n, \\
    a_{ij} &\leq 0 & i \neq j, \\
    a_{ij} &= 0 & \text{implies } a_{ji} = 0.
\end{align*}
$$

A realization of $A$ is a triple $(\mathfrak{h}, \Pi, \Pi^\vee)$, where $\mathfrak{h}$ is a complex vector space, $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subset \mathfrak{h}^*$ and $\Pi^\vee = \{\alpha_1^\vee, \alpha_2^\vee, \ldots, \alpha_n^\vee\} \subset \mathfrak{h}$ are indexed subsets in $\mathfrak{h}^*$ and $\mathfrak{h}$ respectively, satisfying the following three conditions;

$$
\begin{align*}
    \Pi \text{ and } \Pi^\vee \text{ are linearly independent} \\
    \alpha_j(\alpha_i^\vee) &= a_{ij} & (i,j = 1, 2, \ldots, n) \\
    \dim \mathfrak{h} &= 2n - l, & \text{where } l = \text{rank } A.
\end{align*}
$$

An $n \times n$ matrix $A = (a_{ij})_{i,j=1}^n$ is called symmetrizable if there exists an invertible diagonal matrix $D$ and a symmetrix matrix $B = (b_{i,j})$ such that $DA = B$. 

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The Kac-Moody algebra \( \mathfrak{g} = \mathfrak{g}(A) \) with the generalized Cartan matrix \( A \) is the Lie algebra generated by the elements \( e_i, f_i \) \( (i = 1, 2, \ldots, n) \) and \( \mathfrak{h} \) with the following defining relations:

\[
\begin{align*}
[h, h'] &= 0 \quad \text{for } h, h' \in \mathfrak{h}, \\
[h, e_i] &= \alpha_i(h) e_i, \\
[h, f_i] &= -\alpha_i(h) f_i \quad (i = 1, 2, \ldots, n; h \in \mathfrak{h}), \\
[e_i, f_j] &= \delta_{ij} \alpha_i^\vee \quad \text{for } i, j = 1, 2, \ldots, n, \\
(ad e_i)^{1-\alpha_{ij}}(e_j) &= (ad f_i)^{1-\alpha_{ij}}(f_j) = 0 \quad \text{for } i \neq j.
\end{align*}
\]

(1.3)

The elements of \( \Pi \) (resp. \( \Pi^\vee \)) are called the simple roots (resp. simple coroots) of \( \mathfrak{g} \).

For each \( i \in \{1, 2, \ldots, n\} \), let \( r_i \in \text{Aut}(\mathfrak{h}^\ast) \) be the simple reflection on \( \mathfrak{h}^\ast \) defined by

\[
r_i(\lambda) = \lambda - \lambda(\alpha_i^\vee) \alpha_i.
\]

The subgroup \( W \) of \( GL(\mathfrak{h}^\ast) \) generated by the \( r_i \)'s \( (i = 1, 2, \ldots, n) \) is called the Weyl group of \( \mathfrak{g} \).

We adopt the following notation: for a real column vector \( {}^t(u_1, u_2, \ldots, u_n) \), we write \( u > 0 \) if all \( u_i > 0 \) and \( u \geq 0 \) if all \( u_i \geq 0 \).

**Theorem 1.1.** [1] Let \( A \) be a real \( n \times n \) generalized Cartan matrix. Then one and only one possibilities holds for both \( A \) and \( {}^tA \):

(Fin) \( \det A \neq 0 \); there exists \( u > 0 \) such that \( Au > 0 \); \( Au \geq 0 \) implies \( v > 0 \) or \( v = 0 \).

(Aff) \( \text{corank } A = 1 \); there exists \( u > 0 \) such that \( Au = 0 \); \( Au \geq 0 \) implies \( Au = 0 \).

(Ind) \( \text{there exists } u > 0 \text{ such that } Au < 0 \); \( Au \geq 0 \), \( v \geq 0 \) implies \( v = 0 \).

Referring to cases (Fin), (Aff) or (Ind), we will say that \( A \) is of finite, affine or indefinite type, respectively.

Let \( A = (a_{ij})_{i,j=1}^n \) be a generalized Cartan matrix. We associate to \( A \) a graph \( S(A) \), called the Dynkin diagram of \( A \) as follows. If \( a_{ij}a_{ji} \leq 4 \) and \( |a_{ij}| \geq |a_{ji}| \), then the vertices \( i \) and \( j \) are connected by \( |a_{ij}| \) lines, and these lines are equipped with an arrow pointing toward \( i \) if \( |a_{ij}| > 1 \). If \( a_{ij}a_{ji} > 4 \), the vertices \( i \) and \( j \) are connected by a bold-faced line equipped with an ordered pair of integers \( (|a_{ij}|, |a_{ji}|) \).

An indecomposable generalized Cartan matrix \( A \) is said to be of strictly hyperbolic type (resp. hyperbolic type) if it is of indefinite type and connected proper subdiagram of \( S(A) \) is of finite (resp. finite or affine) type.
Suppose that $A$ is symmetrizable generalized Cartan matrix. Then the non-degenerate symmetric bilinear form $(\cdot, \cdot)$ can be defined on $\mathfrak{h}^*$ and $A$ can be expressed as

\[ A = \left( \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} \right)_{i,j=1}^n \]

which is the same as the usual expression of the generalized Cartan matrix [3].

2. Structure of the Weyl group of some Kac-Moody algebra

We know the Weyl group $W$ is a Coxter group generated by $r_i, \ldots, r_n$ and satisfies the following relations

\[ r_i^2 = 1 \quad (r_ir_j)^{m_{ij}} = 1 \quad (i \neq j) \]

where $m_{ij} \in [2, \infty)$ are given in terms of the generalized Cartan matrix by following table;

<table>
<thead>
<tr>
<th>$a_{ij}$ $a_{ij}$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>$\geq 4$</th>
<th>$\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_{ij}$</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>6</td>
<td></td>
<td>$\infty$</td>
</tr>
</tbody>
</table>

Definition 2.1 A Coxter group generated by $\{r_i | i \in I\}$ is called a free Coxter group, if the order of $r_ir_j$ is infinite for all $i \neq j \in I$.

Lemma 2.2. If $(\alpha_i, \alpha_i)(\alpha_j, \alpha_j) \leq (\alpha_i, \alpha_j)^2$ for $i, j = 1, \ldots, n$, then $W$ is a free Coxter group generated by $r_1, \ldots, r_n$.

Proof. If $i \neq j$, then

\[
a_{ij}a_{ji} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} \frac{2(\alpha_j, \alpha_i)}{(\alpha_j, \alpha_j)} = \frac{4(\alpha_i, \alpha_j)^2}{(\alpha_i, \alpha_i)(\alpha_j, \alpha_j)} \geq \frac{4(\alpha_i, \alpha_j)^2}{(\alpha_i, \alpha_j)^2} = 4.
\]

The above table shows $r_ir_j$ has infinite order.
From now on, we always assume that \( A = (a_{ij})^{n}_{i,j=0} \) is an \((n + 1) \times (n + 1)\) indecomposable symmetrizable generalized Cartan matrix, and \( S(A) \) is the Dynkin diagram corresponding to \( A \). Let \( W = \langle r_0, r_1, \ldots, r_n \rangle \) be the Weyl group of \( A \). Denote \( \hat{W} = \langle r_1, \ldots, r_n \rangle \). Set \( T = \{ r_{\beta_0} | \beta \in \hat{W} \} \).

Recall that for each real root \( \alpha \) we have defined a reflection \( r_\alpha \) by
\[
 r_\alpha(\lambda) = \lambda - \lambda(\alpha^\vee)\alpha \quad (\lambda \in \mathfrak{h}^*).
\]
Then \( r_{\beta_0} = \beta r_0 \beta^{-1} \).

In this paper, we shall normalize \( (, \) so that \( (\alpha_0, \alpha_0) = 1 \).

**Lemma 2.3.** \([4]\) Let \( r_{i_1} r_{i_2} \cdots r_{i_s} = 1 \), \( r_{ij} r_{ij+1} \neq r_{ij+1} r_{ij} \), where \( s \geq 2 \) and \( s \) is minimal for such expressions. Then \( s = 2m \geq 4 \) and \( r_{i_1} = r_{i_3} = \cdots = r_{i_{2m-1}} \), \( r_{i_2} = r_{i_4} = \cdots = r_{i_{2m}} \). Furthermore, \( m = 3, 4 \) or 6.

**Theorem 2.4.** Let \( P = \{ i \mid (\alpha_i, \alpha_i) \leq (\alpha_0, \alpha_i)^2 \} \) and \( Q = \{ i \mid (\alpha_0, \alpha_i) = 0 \} \). If \( P \cup Q = \{0, 1, \ldots, n\} \), then there exists a minimal subset \( I \) of \( \hat{W} \) such that \( \langle r_{\beta_0}, \beta \in I \rangle = T \).

**Proof.** Set \( \hat{W} = \{ \omega \in \hat{W} \mid \omega \alpha_0 = \alpha_0 \} \). Clearly \( \hat{W} = \langle r_i | r_i \alpha_0 = \alpha_0 \rangle \) and \( \hat{W} \) is a subgroup of \( \hat{W} \). Construct a set \( I \) by choosing exactly one element from each left coset of \( \hat{W}/\hat{W} \).

First, we show that \( \langle r_{\beta_0}, \beta \in \hat{W} \rangle \subset \langle r_{\beta_0}, \beta \in I \rangle \). By the construction, for each \( \omega \in \hat{W} \), there exists only one \( \omega' \in I \) such that \( \omega \hat{W} = \omega' \hat{W} \). This implies \( \omega' \omega^{-1} \in \hat{W} \) and hence \( \omega \alpha_0 = \omega' \alpha_0 \). Therefore \( r_{\omega \alpha_0} = r_{\omega' \alpha_0} \). Next, we shall show that \( I \) has no proper subset \( J \) such that \( \langle r_{\beta_0}, \beta \in J \rangle = T \). Suppose \( J \subsetneq I \) and \( \langle r_{\beta_0}, \beta \in I \rangle = T \).

Then there exists \( \beta_0 \in I \) with \( \beta_0 \notin J \). Since \( \langle r_{\beta_0}, \beta \in J \rangle = T \), there exist \( \beta_1, \ldots, \beta_t \) such that \( r_{\beta_0} \beta_0 = r_{\beta_1} \beta_0 \beta_2 \beta_{t-1} \beta_0 r_0 \beta_0^{-1} \beta_1 = 1 \).

We claim that \( \beta^{-1}_i \beta_{i+1} r_0 \neq r_0 \beta^{-1}_i \beta_{i+1} \) for all \( 1 \leq i \leq t - 1 \) and \( \beta^{-1}_i \beta_0 r_0 \neq r_0 \beta^{-1}_i \beta_0 \). Suppose not then \( \beta_{i+1} r_0 \beta^{-1}_i \beta_{i+1} = \beta_i r_0 \beta^{-1}_i \beta_0 \) for some \( i \), and hence \( r_{\beta_i \alpha_0} = r_{\beta_i \alpha_0} \), which contradicts the minimality of \( t \). Similarly, suppose that \( \beta^{-1}_i \beta_0 r_0 = r_0 \beta^{-1}_i \beta_0 \). Then \( \beta^{-1}_i \beta_0 \beta_0 \alpha_0 = r_0 \beta^{-1}_i \beta_0 \alpha_0 \).

This implies \( -\beta^{-1}_i \beta_0 \alpha_0 = r_0 \beta^{-1}_i \beta_0 \alpha_0 \), and hence \( \beta^{-1}_i \beta_0 \alpha_0 = \alpha_0 \). This contradicts the fact that \( \beta_0, \beta_t \in I \) and \( \beta_0 \neq \beta_t \). By Lemma 2.3, \( (r_0 r_i)^k = 1 \) for
some $i$ where $k = 3, 4, 6$. On the other hand, $i \in P \cup Q$ implies $(\alpha_0, \alpha_i) = 0$ or $(\alpha_i, \alpha_i) \leq (\alpha_0, \alpha_i)^2$ and hence $(r_0 r_i)^2 = 1$ or $r_0 r_i$ has an infinite order. We come to a contradiction.

**Corollary 2.5.** Let $I$ be the subset of $\hat{W}$ which is constructed in the Proof of Theorem 2.4. Then there exists a one-to-one correspondence between $\hat{W}_0$ and $I$.

**Proof.** Let $\omega \in \hat{W}$ be as above. For each $\omega \in \hat{W}$, there exists exactly one element $\omega' \in I$ such that $\omega \hat{W} = \omega' \hat{W}$. Define a map $\phi : \hat{W}_0 \to I$ by $\phi(\omega_0) = \omega'$. Clearly $\phi$ is onto. We only need to prove that $\phi$ is one-to-one. For $\omega_1, \omega_2 \in \hat{W}$, suppose $\phi(\omega_1 \alpha_0) = \phi(\omega_2 \alpha_0)$.

Then $\omega_1 \hat{W} = \omega_2 \hat{W}$. Thus $\omega_2^{-1} \omega_1 \in \hat{W}$, and hence $\omega_1 \alpha_0 = \omega_2 \alpha_0$.

**Theorem 2.6.** Let $P$ and $Q$ be the same sets as in Theorem 2.4. If $P \cup Q = \{0, 1, \ldots, n\}$, then $T$ is a free Coxeter group which is normal in $W$.

**Proof.** Clearly $T$ is a normal subgroup of $W$. We need to show that $T$ is a free Coxeter group. Let's enumerate all elements of $I$ by $I = \{\beta_1, \ldots, \beta_m\}$. We claim that $(\beta_i \alpha_0, \beta_j \alpha_0) \leq -1$ for $i \neq j$. In fact,

\[
(\beta_i \alpha_0, \beta_j \alpha_0) = (\beta_j^{-1} \beta_i \alpha_0, \alpha_0) \\
= (\alpha_0 + k_1 \alpha_1 + \cdots + k_n \alpha_n, \alpha_0) \quad \text{where} \quad k_i \in \mathbb{Z}_{\geq 0} \\
= (\alpha_0, \alpha_0) + k_1 (\alpha_1, \alpha_0) + \cdots + k_n (\alpha_n, \alpha_0) \\
= 1 + \frac{k_1}{2} a_{01} + \cdots + \frac{k_n}{2} a_{0n} \\
\leq 1 - \frac{1}{2} a_{i0} a_{0i} \quad \text{for some} \quad i \in P \quad [3] \\
= 1 - \frac{2(\alpha_0, \alpha_i)^2}{(\alpha_i, \alpha_i)} \\
\leq 1 - 2 = -1.
\]

Since $(\beta_i \alpha_0, \beta_i \alpha_0) = (\beta_j \alpha_0, \beta_j \alpha_0) = (\alpha_0, \alpha_0)$, $(\beta_i \alpha_0, \beta_i \alpha_0)(\beta_j \alpha_0, \beta_j \alpha_0) = (\alpha_0, \alpha_0)^2 = 1 \leq (\beta_i \alpha_0, \beta_j \alpha_0)^2$. Hence $\langle r_{\beta_i \alpha_0} | \beta_i \in I \rangle$ is a free Coxeter group by Lemma 2.2.

**3. Hyperbolic case**
Lemma 3.1. [3]. Let $A = (a_{ij})_{i,j=0}^n$ be a generalized Cartan matrix of hyperbolic type.

Suppose $a_{ij} \neq 0$ and let $S_2$ be the subdiagram of $S(A)$ consisting of vertices $i$ and $j$. Then the following properties are satisfied:

(a) If $n = 2$, then $S_2$ is one of the following diagrams:

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|---|---|</p>
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(b) If $n = 3$, then $S_2$ is one of the first three diagrams above.

(c) If $n \geq 4$, then $S_2$ is one of the first two diagrams above.

(d) If $A$ is of strictly hyperbolic type, then $n \leq 4$.

Corollary 3.2. Let $A = (a_{ij})_{i,j=0}^n$ be a generalized Cartan matrix of hyperbolic type and $P$, $Q$ be the same sets as in Theorem 2.4. If $P \cup Q = \{0, 1, \ldots, n\}$, then rank $A \leq 3$ and $T$ is also a free Coxeter group.

Proof. It follows immediately from Theorem 2.4 and Lemma 2.2.

Theorem 3.3. Let $A$, $P$, $Q$ be as in Corollary 3.2. If $P \cup Q = \{0, 1, \ldots, n\}$ and $S(A)$ has no cycle, then the following properties are satisfied:

(a) If $A$ is of strictly hyperbolic type of rank 3, then $|I| = |\hat{W}|/2$.

(b) If $A$ is of hyperbolic type of rank 3 which is not strictly hyperbolic, then $I$ is infinite.

(c) If $A$ is of hyperbolic type of rank 2, then $T$ is generated by the set $\{r_0, r_1 r_0 r_1\}$.

Proof. Suppose that rank $A = 3$. Then we may assume that $P = \{0, 1\}$, $Q = \{2\}$. Then $\hat{W} = \{1, r_2\}$. Suppose $A$ is of strictly hyperbolic type. Then $\hat{W}$ is finite, and hence

$$|I| = |\hat{W}/\hat{W}| = |\hat{W}|/|\hat{W}| = |\hat{W}|/2.$$

Suppose that $A$ is of affine type. Then $\hat{W} = \{r_1(r_1 r_2)^m, (r_1 r_2)^m | m \in \mathbb{Z}\}$. Thus $\hat{W}$ is infinite and $|\hat{W}| = 2$. Therefore $I$ is infinite. Suppose rank $A = 2$. Then $\hat{W} = \{1, r_1\}$ and hence $T = \langle r_0, r_1 r_0 r_1 \rangle$.

Example. Let $A = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$. Then $\hat{W} = \{1, r_1, r_1 r_2, r_1 r_2 r_1, r_2, r_2 r_1\}$, $\hat{W} = \langle r_2 \rangle$ and $I = \{1, r_1, r_2 r_1\}$. $\hat{W} \alpha_0 = \{\alpha_0, r_1 \alpha_0, r_2 r_1 \alpha_0\}$. $T = \langle r_0, r_1 \alpha_0, r_2 r_1 \alpha_0 \rangle$. 

Theorem 3.4. Let $A, P, Q$ be as in Theorem 3.3. If $P \cup Q = \{0, 1, \ldots, n\}$ and $S(A)$ has a cycle, then there exists a one-to-one correspondence between $\hat{W}\alpha_0$ and $\hat{W}$.

Proof. Since $S(A)$ has a cycle, rank $A = 3$ and $P = \{0, 1, 2\}$, $Q = \phi$. Hence $\hat{W} = \{1\}$. If $\omega_1\alpha_0 = \omega_2\alpha_0$ for $\omega_1, \omega_2 \in \hat{W}$, then $\omega_2^{-1}\omega_1\alpha_0 = \alpha_0$ and hence $\omega_2^{-1}\omega_1 \in \hat{W} = \{1\}$. Therefore $\omega_1 = \omega_2$.

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